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ITERATIVE APPROXIMATION OF SOLUTIONS OF MONOTONE QUASI-VARIATIONAL INEQUALITIES VIA NONLINEAR MAPPINGS

QINGNIAN ZHANG

School of Mathematics and Information Science,

North China University of Water Resources and Electric Power, Zhengzhou, China

Abstract. In this paper, we investigate the monotone quasivariational inequality via fixed point problems of nonlinear mapping in Hilbert spaces. Convergence of algorithms for finding some common element of the set of fixed points of strictly pseudocontractive mappings and the set of solutions of the monotone quasivariational inequalities.

Keywords: variational inequality; nonexpansive mapping; fixed point; quasivariational inequality.

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1. Introduction-Preliminaries

Recently, monotone quasi-variational inequalities have been studied as an effective and powerful tool for studying a wide class of real world problems which arise in economics, finance, image reconstruction, ecology, transportation, and network; see [1-21] and the references therein. It is well known that quasi-variational inequality problems include many important problems in nonlinear analysis and optimization such as the Nash equilibrium problems, fixed point problems, complementarity problems, vector optimization problems, saddle point problems and

E-mail address: wszhangqn@yeah.net

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game theory. For the solutions of quasi-variational inequality problems, there are several algorithms to solve the problem; see the literature.

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let K be a closed convex subset of H and let $A, g : H \to H$ be two nonlinear mappings and $J : H \to P(H)$ a multivalued mapping, where $P(H)$ denotes the power set of H. Then the following problems is said to be the monotone quasivariational inequality (MQVI) problem: determine an element $u \in H$ such that $g(u) \in J(u)$ and

$$\langle Au, v - g(u)\rangle \geq 0, \quad \forall v \in J(u), \tag{1.1}$$

which was considered by Verma [6].

For $g = I$, the identity on H, the MQVI problem (1.1) reduces to: determine an element $u \in H$ such that $u \in J(u)$ and

$$\langle Au, v - u\rangle \geq 0, \quad \forall v \in J(u). \tag{1.2}$$

Recall the following definitions:

(1) T is called r-strongly monotone if there exists a constant $v > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq r\|x - y\|^2, \quad \forall x, y \in H.$$

This implies that $\|Tx - Ty\| \geq r\|x - y\|$, $\forall x, y \in K$, that is, A is r-expansive and, when $r = 1$, it is expansive.

(2) A is said to be μ-cocoercive if there exists a constant $\mu > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \mu\|Tx - Ty\|^2, \quad \forall x, y \in H.$$

Clearly, every μ-cocoercive mapping A is $\frac{1}{\mu}$-Lipschitz continuous.

(3) A is called relaxed γ-cocoerceive if there exists a constant $u > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq (-\gamma)\|Tx - Ty\|^2, \quad \forall x, y \in H.$$

(4) $A : H \to H$ is called Lipschitz continuous if there is a constant $L > 0$ such that for all $x, y \in H$, we have

$$\|Tx - Ty\| \leq L\|x - y\|.$$

(5) A is said to be relaxed (γ, r)-cocoercive if there exist two constants $u, v > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq (-\gamma)\|Tx - Ty\|^2 + r\|x - y\|^2, \quad \forall x, y \in H.$$

For $u = 0$, A is v-strongly monotone. This class of mappings is more general that the class of strongly monotone mappings. It is easy to see that we have the following implication:

r-strongly monotonicity $\Rightarrow$ relaxed (γ, r)-cocoercivity.

(6) $S : H \to H$ is said to be nonexpansive if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in H.$$

S is said to be strictly pseudocontractive if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in H.$$

It is clear that strictly pseudocontractive mappings include nonexpansive mapping as a special case.

We now recall some well-known concepts and results:

Lemma 1.1. *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n, \quad \forall n \geq n_0,$$

where n_0 *is some nonnegative integer,* $\{\lambda_n\}$ *is a sequence in* $(0, 1)$ *with* $\sum_{n=1}^{\infty} \lambda_n = \infty$, $b_n = o(\lambda_n)$, *then* $\lim_{n\to\infty} a_n = 0$.

Lemma 1.2. *For any* $z \in H$, $u \in K$ *satisfies the inequality:*

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in K,$$

if and only if $u = P_K z$.

Lemma 1.3. [6] *Let* H *be a real Hilbert space and* $A, g : H \to H$ *be any mappings. Suppose that* $J : H \to P(H)$ *is a multivalued mapping such that* $J(u) = m(u) + K$ *for any* $u \in H$, *where* $m : H \to H$ *is a singlevalued mapping and* K *is a closed convex subset of* H. *Then the following statements are equivalent:*

(1) *An element* $u \in H$ *is a solution of the MQVI problem* (1.1).

(2) $g(u) = m(u) + P_K[g(u) - m(u) - \rho Au]$.

It follows from (2) that

$$u = u - g(u) + m(u) + P_K[g(u) - m(u) - \rho(Au)],$$

where $\rho > 0$ is a constant. If u is some common element of the set of fixed points of the nonexpansive mapping S and the set of solutions of the (MQVI) problem (1.1), then we have

$$\begin{aligned} u &= u - g(u) + m(u) + P_K[g(u) - m(u) - \rho Au] \\ &= S\{u - g(u) + m(u) + P_K[g(u) - m(u) - \rho(Au)]\}. \end{aligned} \tag{1.3}$$

Next, we shall denote the solution of the MQVI problem (1.1) by Ω and assume that $F(S) \cap \Omega \neq \emptyset$.

2. Algorithms

For $J(u) = m(u) + K$, where $m : H \to H$ is a single-valued mapping and K is a closed convex subset of H, we have the following algorithms.

Algorithm 2.1. For any $u_0 \in H$ and $g(u_0) \in K$, the sequence $\{u_n\}$ is defined by an iterative scheme

$$\begin{aligned} u_{n+1} = (1-\alpha_n)u_n + \alpha_n\big(\alpha I + (1-\alpha)S\big)\{&u_n - g(u_n) + m(u_n) \\ &+ P_K[g(u_n) - m(u_n) - \rho Au_n]\}, \quad \forall n \geq 0, \end{aligned}$$

where $\rho > 0$ is a constant and S is a κ-strictly pseudocontractive mapping.

If $S = I$, the algorithm (2.1) reduces to the following algorithm:

Algorithm 2.2. For any $u_0 \in H$ and $g(u_0) \in K$, the sequence $\{u_n\}$ is defined by an iterative scheme

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\{u_n - g(u_n) + m(u_n) + P_K[g(u_n) - m(u_n) - \rho Au_n]\}, \forall n \geq 0,$$

where $\rho > 0$ is a constant.

If $S = g = I$ and $\{\alpha_n\} \equiv 1$, the algorithm (2.1) reduces to the following:

Algorithm 2.3. For any $u_0 \in H$ and $g(u_0) \in K$, the sequence $\{u_n\}$ is defined by an iterative scheme

$$u_{n+1} = m(u_n) + P_K[u_n - m(u_n) - \rho Au_n], \quad \forall n \geq 0,$$

where $\rho > 0$ is a constant.

If $J(u) \equiv K$, where K is a closed convex subset of H, the algorithm (2.1) reduces to the following:

Algorithm 2.4. For any $u_0 \in H$ and $g(u_0) \in K$, the sequence $\{u_n\}$ is defined by an iterative scheme

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n S\{u_n - g(u_n) + P_K[g(u_n) - \rho(Au_n)]\}, \quad \forall n \geq 0,$$

where $\rho > 0$ is a constant and S is a nonexpansive mapping.

3. Main results

Theorem 3.1. *Let K be a closed convex subset of a Hilbert space H and $A : H \to H$ be a relaxed (γ_1, r_1)-cocoerceive and μ_1-Lipschitz continuous mapping, $g : H \to H$ be a relaxed (γ_2, r_2)-cocoerceive and μ_2-Lipschitz continuous mapping and S is a κ-strictly pseudocontractive mapping. Suppose that $J : H \to P(H)$ is a multi-valued such that $J(u) = m(u) + K$ for any $u \in H$, where $m : H \to H$ is a single-valued Lipschitz mapping with the constant $\lambda > 0$. Assume that $\alpha \in [\kappa, 1)$, $\{\alpha_n\} \subset [0,1]$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\theta_1 + 2\theta_2 + 2\lambda < 1$, where $\theta_1 = \sqrt{1 + \rho^2\mu_1^2 - 2\rho r_1 + 2\rho\gamma_1\mu_1^2}$ and $\theta_2 = \sqrt{1 + \mu_2^2 - 2r_2 + 2\gamma_2\mu_2^2}$. Then the sequence $\{u_n\}$ generated by (2.1) converges strongly to $u^* \in F(S) \cap \Omega$.*

Proof. Let $T =: \alpha I + (1-\alpha)S$. In view of Zhou [12], we see that T is nonexpansive with $F(T) = F(S)$. Fix $u^* \in F(S) \cap \Omega$. It follows that

$$u^* = (1-\alpha_n)u^* + \alpha_n T\{u^* - g(u^*) + m(u^*) + P_K[g(u^*) - m(u^*) - \rho Au^*]\}.$$

It follows that

$$\begin{aligned}\|u_n-u^*\| &\le (1-\alpha_n)\|u_n-u^*\|+\alpha_n\|u_n-g(u_n)+m(u_n)\\ &\quad +P_K[g(u_n)-m(u_n)-\rho(Au_n)]\\ &\quad -\{u^*-g(u^*)+m(u^*)+P_K[g(u^*)-m(u^*)-\rho Au^*]\}\|\\ &\le (1-\alpha_n)\|u_n-u^*\|+\alpha_n\|u_n-u^*-(g(u_n)-g(u^*))\|\\ &\quad +\alpha_n\|m(u_n)-m(u^*)\|+\alpha_n\|g(u_n)-m(u_n)-\rho Au_n\\ &\quad -[g(u^*)-m(u^*)-\rho Au^*]\}\|\\ &\le (1-\alpha_n)\|u_n-u^*\|+2\alpha_n\|u_n-u^*-(g(u_n)-g(u^*))\|\\ &\quad +2\alpha_n\|m(u_n)-m(u^*)\|+\alpha_n\|u_n-u^*-\rho(Au_n-Au^*)\|\end{aligned} \tag{3.1}$$

Next, we consider the second, the third and the forth terms of rightside of (3.1), respectively. By the assumption that A is relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitz continuous, we obtain

$$\begin{aligned}&\|u_n-u^*-\rho(Au_n-Au^*)\|^2\\ &\le \|u_n-u^*\|^2-2\rho[-\gamma_1\|Au_n-Au^*\|^2\\ &\quad +r_1\|u_n-u^*\|^2]+\rho^2\|Au_n-Au^*\|^2\\ &\le \|u_n-u^*\|^2+2\rho\mu_1^2\gamma_1\|u_n-u^*\|^2\\ &\quad -2\rho r_1\|u_n-u^*\|^2+\rho^2\mu_1^2\|u_n-u^*\|^2\\ &= (1+2\rho\mu_1^2\gamma_1-2\rho r_1+\rho^2\mu_1^2)\|u_n-u^*\|^2\\ &= \theta_1^2\|u_n-u^*\|^2\end{aligned} \tag{3.2}$$

where

$$\theta_1=\sqrt{1+\rho^2\mu_1^2-2\rho r_1+2\rho\gamma_1\mu_1^2}.$$

Observing $m: H \to H$ is Lipschitz continuous with the Lipschitz continuity constant $\lambda > 0$, we have

$$\|m(u_n)-m(u^*)\| \le \lambda\|u_n-u^*\|. \tag{3.3}$$

Again, by the assumption that g is relaxed (γ_2, r_2)-cocoercive and μ_2-Lipschitz continuous, we obtain

$$
\begin{aligned}
& \|u_n - u^* - g(u_n) - g(u^*)\|^2 \\
\leq & \|u_n - u^*\|^2 - 2[-\gamma_2 \|g(u_n) - g(u^*)\|^2 \\
& + r_2 \|u_n - u^*\|^2] + \|g(u_n) - g(u^*)\|^2 \\
\leq & \|u_n - u^*\|^2 + 2\mu_2^2 \gamma_2 \|u_n - u^*\|^2 \\
& - 2r_2 \|u_n - u^*\|^2 + \mu_2^2 \|u_n - u^*\|^2 \\
= & (1 + 2\mu_2^2 \gamma_2 - 2r_2 + \mu_2^2) \|u_n - u^*\|^2 \\
= & \theta_2^2 \|u_n - u^*\|^2,
\end{aligned}
\tag{3.4}
$$

where

$$\theta_2 = \sqrt{1 + \mu_2^2 - 2r_2 + 2\gamma_2 \mu_2^2}.$$

Substitute (3.2), (3.3) and (3.4) into (3.1) yields that $\|u_n - u^*\| \leq [1 - \alpha_n(1 - 2\theta_2 - 2\lambda - \theta_1)]\|u_n - u^*\|$. We therefore obtain the desired conclusion immediately. This completes the proof.

Corollary 3.2. *Let K be a closed convex subset of a Hilbert space H and $A : H \to H$ be a relaxed (γ_1, r_1)-cocoerceive and μ_1-Lipschitz continuous mapping, $g : H \to H$ be a relaxed (γ_2, r_2)-cocoerceive and μ_2-Lipschitz continuous mapping and S is a nonexpansive mapping. Suppose that $J : H \to P(H)$ is a multi-valued such that $J(u) = m(u) + K$ for any $u \in H$, where $m : H \to H$ is a single-valued Lipschitz mapping with the constant $\lambda > 0$. Assume that $\{\alpha_n\} \subset [0,1]$ is chosen such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ and $\theta_1 + 2\theta_2 + 2\lambda < 1$, where $\theta_1 = \sqrt{1 + \rho^2\mu_1^2 - 2\rho r_1 + 2\rho\gamma_1\mu_1^2}$ and $\theta_2 = \sqrt{1 + \mu_2^2 - 2r_2 + 2\gamma_2\mu_2^2}$. Then the sequence $\{u_n\}$ generated by (2.1) converges strongly to $u^* \in F(S) \cap \Omega$.*

Conflict of Interests

The author declare that there is no conflict of interests.

REFERENCES

[1] Y.J. Cho, X. Qin, M. Shang, Y. Su, Generalized nonlinear variational inclusions involving (A,η)-monotone mappings in Hilbert spaces, 2007 (2007) 29653.

[2] M.R. Bai, Y.Y. Tang,Y.P. Liu, Generalized implicit quasivariational inequalities with relaxed Lipschitz and relaxed monotone mappings, Appl. Math. Lett. 12 (1999) 69-76.

[3] P. Cubioti, Discontinuous quasivariational-like inequalities, Comput. Math. Aappl. 29 (1995) 9-12.

[4] S.S. Chang, Fuzzy quasivariational inclusions in Banach spaces, Appl. Math. Comput. 145 (2003) 805-819. Shih-sen Chang

[5] A.H Siddiqi, Q.H Ansari An algorithm for a class of quasivariational inequalities, J. Math. Anal. Appl. 145 (1990) 413-418.

[6] R.U. Verma, A class of projection-contraction methods applied to monotone variational inequalities, Appl. Math. Lett. 13 (2000) 55-62.

[7] R.U. Verma, A new class of iterative algorithms for approximation-solvability of nonlinear variational inequalities, Comput. Math. Appl. 41 (2001) 505-512.

[8] R.U. Verma, Projection methods, algorithms, and a new system of nonlinear variational inequalities Comput. Math. Appl. 41 (2001) 1025-1031.

[9] M.A. Noor, Wiener-Hopf equations and variational inequalities, J. Optim. Theory Appl. 79 (1993) 197-206.

[10] M.A. Noor, Sensitivity analysis for quasi variational inequalities, J. Optim. Theory Appl. 95 (1997) 399-407.

[11] M.A. Noor, Some developments in general variational inequalities, Appl. Math. Comput. 152 (2004) 199-277.

[12] X. Qin, M. Shang, Generalized variational inequalities involving relaxed monotone mappings and nonexpansive mappings, J. Inequal. Appl. 2007, 2007:020457.

[13] H. Zhou, Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert spaces, Nonlinear Anal. 69 (2008) 456-462.

[14] A.Y. Al-Bayati, R.Z. Al-Kawaz, A new hybrid WC-FR conjugate gradient-algorithm with modified secant condition for unconstrained optimization, J. Math. Comput. Sci. 2 (2012) 937-966.

[15] J. Shen, L.P. Pang, An approximate bundle method for solving variational inequalities, Commn. Optim. Theory, 1 (2012) 1-18.

[16] H.S. Abdel-Salam, K. Al-Khaled, Variational iteration method for solving optimization problems, J. Math. Comput. Sci. 2 (2012) 1475-1497.

[17] H. Zegeye, N. Shahzad, Strong convergence theorem for a common point of solution of variational inequality and fixed point problem, Adv. Fixed Point Theory, 2 (2012) 374-397.

[18] B.O. Osu, O.U. Solomon, A stochastic algorithm for the valuation of financial derivatives using the hyperbolic distributional variates, Math. Fianc. Lett. 1 (2012) 43-56.

[19] Z.M. Wang, W.D. Lou, A new iterative algorithm of common solutions to quasi-variational inclusion and fixed point problems, J. Math. Comput. Sci. 3 (2013) 57-72.

[20] V.A. Khan, K. Ebadullah, I-convergent difference sequence spaces defined by a sequence of moduli, J. Math. Comput. Sci. 2 (2012) 265-273.

[21] J. Ye, J. Huang, Strong convergence theorems for fixed point problems and generalized equilibrium problems of three relatively quasi-nonexpansive mappings in Banach spaces, J. Math. Comput. Sci. 1 (2011) 1-18.