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STRONG CONVERGENCE THEOREMS FOR TWO HEMI-RELATIVELY NONEXPANSIVE MULTI-VALUED MAPPINGS IN BANACH SPACES

ZI-MING WANG*, QI ZHAO

Department of Foundation, Shandong Yingcai University, Jinan 250104, China

Abstract. In this paper, a simple projection algorithm is introduced for finding a common fixed point of two hemi-relatively nonexpansive multi-valued mappings. Furthermore, strong convergence theorem is established in a Banach space.

Keywords: hemi-relatively nonexpansive multi-valued mappings; fixed point; Banach space.

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1. Introduction

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$, where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single valued and if E is uniformly smooth then J is uniformly

*Corresponding author

E-mail addresses: wangziming@ymail.com (Z.M. Wang), zaho231200@163.com (Q. Zhao)

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continuous on bounded subsets of E. Moreover, if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is single valued, one-to-one, surjective, and it is the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*}$; see, [1,2]. We note that in a Hilbert space H, J is the identity operator.

A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x,\ y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\|\lim_{n\to\infty} \frac{x_n+y_n}{2}\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{1.1}$$

exists for each $x,\ y \in U_E$. It is also said to be uniformly smooth if the limit (1.1) is attained uniformly for $x,\ y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also known that if E is uniformly smooth if and only if E^* is uniformly convex.

Let E be a smooth Banach space. The Lyapunov functional $\phi : E \times E \to \mathbb{R}$ defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{1.2}$$

for $x,\ y \in E$, is studied by Alber [3], Reich [4] and Kimimura and Takahashi [5]. It is obvious from the definition of the function ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2$$

for $x,\ y \in E$. Observe that in a Hilbert space H, (1.2) reduces to $\phi(x,y) = \|x-y\|^2$, for $x,\ y \in H$.

Let E be a reflexive, strictly convex and smooth Banach space and let C be a nonempty closed and convex subset of E. The *generalized projection mapping*, introduced by Alber [3], is a mapping $\Pi_C : E \to C$, that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min\{\phi(y,x) : y \in K\}.$$

Let C be a nonempty closed convex subset of E and T a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point p is said to be an asymptotic fixed point of T [6]

if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widehat{F}(T)$. A point p is said to be a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of strong asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. In the following, we shall recall some definitions.

Definition 1.1. (1) A mapping $T : C \times C$ is said to be relatively nonexpansive if $\widehat{F}(T) = F(T) \neq \varnothing$ and

$$\phi(p, Tx) \leq \phi(p, x), \forall\, x \in C,\ p \in F(T).$$

(2) A mapping $T : C \times C$ is said to be weak relatively nonexpansive if $\widetilde{F}(T) = F(T) \neq \varnothing$ and

$$\phi(p, Tx) \leq \phi(p, x), \forall\, x \in C,\ p \in F(T).$$

(3) A mapping $T : C \times C$ is said to be hemi-relatively nonexpansive if $F(T) \neq \varnothing$ and

$$\phi(p, Tx) \leq \phi(p, x), \forall\, x \in C,\ p \in F(T).$$

The asymptotic behavior of relatively nonexpansive mappings was studied in [6-8]. It is obvious that the class of hemi-relatively nonexpansive mappings is more general than the class of relatively nonexpansive mappings and the class of weak relatively nonexpansive mappings. In fact, for any mapping $T : C \times C$, we have $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$.

Let C be a nonempty closed convex subset of a smooth Banach space E. Let $N(C)$ be the family of nonempty subsets of C.

Definition 1.2. (1) A point $p \in C$ is said to be an asymptotic fixed point of multi-valued mapping T if C contains a sequence $\{x_n\}$ which converges weakly to p such that

$$\lim_{n\to\infty} d(x_n, T(x_n)) := \lim_{n\to\infty} \inf_{y \in T(x_n)} \|x_n - y\| = 0.$$

The set of asymptotic fixed points of multi-valued mapping T will be also denoted by $\widehat{F}(T)$.
(2) A point $p \in C$ is said to be a strong asymptotic fixed point of multi-valued mapping T if C contains a sequence $\{x_n\}$ which converges strongly to p such that

$$\lim_{n\to\infty} d(x_n, T(x_n)) := \lim_{n\to\infty} \inf_{y \in T(x_n)} \|x_n - y\| = 0.$$

The set of strong asymptotic fixed points of multi-valued mapping T will be also denoted by $\widetilde{F}(T)$.

(3) A multi-valued mapping $T: C \to N(C)$ is said to be relatively nonexpansive, if $\widehat{F}(T) = F(T) \neq \varnothing$ and

$$\phi(p,w) \leq \phi(p,x), \forall\, x \in C,\ w \in Tx,\ p \in F(T).$$

(4) A multi-valued mapping $T: C \to N(C)$ is said to be weak relatively nonexpansive, if $\widetilde{F}(T) = F(T) \neq \varnothing$ and

$$\phi(p,w) \leq \phi(p,x), \forall\, x \in C,\ w \in Tx,\ p \in F(T).$$

(5) A multi-valued mapping $T: C \to N(C)$ is said to be hemi-relatively nonexpansive, if $F(T) \neq \varnothing$ and

$$\phi(p,w) \leq \phi(p,x), \forall\, x \in C,\ w \in Tx,\ p \in F(T).$$

Remark 1.3. The class of hemi-relatively nonexpanisve multi-valued mappings is more general than the class of (weak) relatively nonexpansive multi-valued mappings which requires the restriction: $(\widetilde{F}(T) = F(T))$ $\widehat{F}(T) = F(T)$.

In 2005, Matsushita and Takahashi [9] proved weak and strong convergence theorems to approximate a fixed point of a single relatively nonexpansive mapping in a uniformly convex and uniformly smooth Banach space E. In 2007, Plubtieng and Ungchittrakool [10] improved Matsushita and Takahashi [9]'s result from a single relatively nonexpansive mapping to two relatively nonexpansive mappings, and proved two strong convergence theorems for finding a common fixed point of two relatively nonexpansive mappings in a Banach space by using the hybrid method in mathematical programming. Very recently, Zhang, Su and Cheng [11] introduced a simple projection algorithm for a countable family of weak relatively nonexpansive mappings and proved strong convergence theorems in Banach spaces. In fact, a number of authors improved Matsushita and Takahashi's result in different directions; see, for example, [12-15] and the references therein. In 2011, Homaeipour and Razani [16] proved weak and strong convergence theorems for a single relatively nonexpansive multi-valued mapping in a uniformly convex and uniformly smooth Banach space E.

Motivated and inspired by the results mentioned above, we construct an iterative scheme which converges strongly to a common element in the common fixed point set of two closed and hemi-relatively nonexpansive multi-valued mappings. The main result in this paper extends and improves corresponding results in Matsushita and Takahashi [9], Plubtieng and Ungchittrakool [10], Homaeipour and Razani [16] and many others.

2. Preliminaries

We also need the following lemmas for the proof of our main results.

Lemma 2.1. [9] *Let E be a strictly convex and smooth Banach space, then $\phi(x,y)=0$ if and only if $x=y$.*

Lemma 2.2. [3] *Let C be a convex subset of a smooth real Banach space E. Let $x\in E$ and $x_0\in C$. Then $x_0=\Pi_C x$ if and only if*

$$\langle z-x_0, Jx_0-Jx\rangle \geq 0, \quad \forall z\in C.$$

Lemma 2.3. [3] *Let C be a nonempty, closed and convex subset of a reflexive, strictly convex and smooth real Banach space E and let $x\in E$. Then, for each $y\in C$,*

$$\phi(y,\Pi_C x)+\phi(\Pi_C x,x)\leq \phi(y,x).$$

Lemma 2.4. [17] *Let E be a uniformly convex and smooth real Banach space and let $\{x_n\}$, $\{y_n\}$ be two sequences of E. If $\phi(x_n,y_n)\to 0$ and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n-y_n\|\to 0$.*

Lemma 2.5. *Let E be a strictly convex and smooth Banach space, let C be a closed convex subset of E. Suppose $T: C\to N(C)$ is a hemi-relatively nonexpansive multi-valued mapping. Then $F(T)$ is closed and convex.*

Proof. First, we show that $F(T)$ is closed. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n\to p$ as $n\to\infty$. Since the multi-valued operator T is hemi-relatively nonexpansive, one has $\phi(p_n,\widetilde{p})\leq \phi(p_n,p)$ for all $\widetilde{p}\in Tp$ and for all $n\in\mathbb{N}$. Therefore,

$$\phi(p,\widetilde{p})=\lim_{n\to\infty}\phi(p_n,\widetilde{p})\leq \lim_{n\to\infty}\phi(p_n,p)=\phi(p,p).$$

Applying Lemma 2.1, one gets $p = \widetilde{p}$. Hence $Tp = \{p\}$. Therefore, $p \in F(T)$.

Next, we show that $F(T)$ is convex. To this end, for arbitrary $p_1,\ p_2 \in F(T)$, $t \in (0,1)$, putting $p = tp_1 + (1-t)p_2$ we prove that $Tp = \{p\}$. Let $q \in Tp$, we have

$$\begin{aligned}
\phi(p,q) &= \|p\|^2 - 2\langle p, Jq\rangle + \|q\|^2 \\
&= \|p\|^2 - 2\langle tp_1 + (1-t)p_2, Jq\rangle + \|q\|^2 \\
&= \|p\|^2 - 2t\langle p_1, Jq\rangle - (1-t)\langle p_2, Jq\rangle + \|q\|^2 \\
&= \|p\|^2 - 2t\langle p_1, Jq\rangle - (1-t)\langle p_2, Jq\rangle + \|q\|^2 \\
&= \|p\|^2 + t\phi(p_1,q) + (1-t)\phi(p_2,q) - t\|p_1\|^2 - (1-t)\|p_2\|^2 \\
&\le \|p\|^2 + t\phi(p_1,p) + (1-t)\phi(p_2,p) - t\|p_1\|^2 - (1-t)\|p_2\|^2 \\
&= \|p\|^2 - 2\langle tp_1 + (1-t)p_2, Jp\rangle + \|p\|^2 \\
&= \|p\|^2 - 2\langle p, Jp\rangle + \|p\|^2 \\
&= 0.
\end{aligned}$$

Using Lemma 2.1 again, we also obtain $p = q$. Hence, $T(p) = \{p\}$, that is, $p \in F(T)$. Therefore, $F(T)$ is convex. ■

Recall that a multi-valued mapping $T : C \to N(C)$ is said to be closed if, for any sequence $\{x_n\} \subset C$ with $x_n \to x$ and $w_n \in T(x_n)$ with $w_n \to y$, then $y \in Tx$.

3. Main results

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex Banach space E. Let S, $T : C \to N(C)$ be two closed and hemi-relatively nonexpansive multi-valued mappings such that $\mathscr{F} = F(S) \cap F(T) \neq \varnothing$. For a point $x_0 \in C$ chosen arbitrarily, let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases}
C_0 = C, \\
C_{n+1} = \{z \in C_n : \phi(z,u_n) \le \phi(z,v_n) \le \phi(z,x_n)\}, \\
x_{n+1} = \Pi_{C_{n+1}} x_0,
\end{cases} \tag{3.1}$$

where the sequences $u_n \in Sv_n$ and $v_n \in Tx_n$. Then the sequence $\{x_n\}$ converges strongly to a point $q = \Pi_{\mathscr{F}} x_0$, where $\Pi_{\mathscr{F}}$ is the generalized projection from C onto $\mathscr{F}$.

Proof. Firstly, we show that C_{n+1} is closed and convex. The closedness of C_{n+1} is obvious, so we only show that C_{n+1} is convex. Since

$$\phi(z,u_n) \leq \phi(z,v_n)$$

is equivalent to

$$\|u_n\|^2 - \|v_n\|^2 - 2\langle z, Ju_n - Jv_n\rangle \geq 0.$$

Analogously,

$$\phi(z,v_n) \leq \phi(z,x_n)$$

is equivalent to

$$\|v_n\|^2 - \|x_n\|^2 - 2\langle z, Jv_n - Jx_n\rangle \geq 0.$$

Hence, C_{n+1} is convex. Therefore, C_{n+1} is closed and convex for all $n \in \mathbb{N} \cup \{0\}$.

Now, we prove that $F(S) \bigcap F(T) \subset C_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Let $w \in F(S) \bigcap F(T)$, from the definition of S, we have

$$\phi(w,u_n) \leq \phi(w,v_n). \tag{3.2}$$

Similarly, from the definition of T, we have

$$\phi(w,v_n) \leq \phi(w,x_n). \tag{3.3}$$

Therefore, by combining (3.2) with (3.3), $w \in C_{n+1}$. This implies that $F(S) \bigcap F(T) \subset C_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$.

On the other hand, noticing $x_n = \Pi_{C_n} x_0$, from Lemma 2.2, one sees

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall\, z \in C_n.$$

Since $F \subset C_n$ for all $n \subset \mathbb{N} \cup \{0\}$, we arrive at

$$\langle x_n - p, Jx_0 - Jx_n\rangle \geq 0, \quad \forall\, z \in C_n.$$

From Lemma 2.3, one has

$$\phi(x_n,x_0) = \phi(\Pi_{C_n} x_0, x_0) \leq \phi(p,x_0) - \phi(p,x_n) \leq \phi(p,x_0)$$

for each $p \in F \subset C_n$. Therefore, the sequence $\{\phi(x_n, x_0)\}$ is bounded. In the meantime, noticing that $x_n = \Pi_{C_n} x_0$ and $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, one has

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0), \forall\, n \in \mathbb{N} \cup \{0\}.$$

This implies that the sequence $\{\phi(x_n, x_0)\}$ is nondecreasing. It follows that the limit of $\{\phi(x_n, x_0)\}$ exists. By the construction of C_n, we have that $x_m = \Pi_{C_m} x_0 \in C_m \subset C_n$ for any positive integer $m \geq n$. It follows that

$$\phi(x_m, x_n) = \phi(x_m, \Pi_{C_n} x_0) \leq \phi(x_m, x_0) - \phi(x_n, x_0). \tag{3.4}$$

Letting $m,\ n \to \infty$ in (3.4), due to the existence of the limit of $\{\phi(x_n, x_0)\}$, one has $\phi(x_m, x_n) \to 0$. It follows from Lemma 2.4 that $x_n - x_m \to 0$ as $m,\ n \to \infty$. Hence, $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a point $p \in C$ such that $x_n \to p$ as $n \to \infty$.

Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1} \subset C_n$, we have from the definition of C_{n+1} that

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, v_n) \leq \phi(x_{n+1}, x_n).$$

From the inequality above, we have

$$\phi(x_{n+1}, v_n) \leq \phi(x_{n+1}, x_n), \quad \forall\, n \in \mathbb{N} \cup \{0\}, \tag{3.5}$$

and

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall\, n \in \mathbb{N} \cup \{0\}, \tag{3.6}$$

On the other hand, taking $m = n+1$ in (3.4), we have

$$\lim_{n \to \infty} \phi(x_{n+1}, x_n) = 0. \tag{3.7}$$

From (3.5), (3.6) and (3.7), we obtain

$$\lim_{n \to \infty} \phi(x_{n+1}, v_n) = 0, \tag{3.8}$$

and

$$\lim_{n \to \infty} \phi(x_{n+1}, u_n) = 0, \tag{3.9}$$

By using Lemma 2.4, the inequalities (3.7),(3.8) and (3.9) imply that

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0, \tag{3.10}$$

$$\lim_{n\to\infty} \|x_{n+1} - v_n\| = 0, \tag{3.11}$$

and

$$\lim_{n\to\infty} \|x_{n+1} - u_n\| = 0. \tag{3.12}$$

Noticing that

$$\|x_n - v_n\| = \|x_n - x_{n+1} + x_{n+1} - v_n\|$$

$$= \|x_n - x_{n+1}\| + \|x_{n+1} - v_n\|$$

It follows from (3.10) and (3.11) that

$$\lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{3.13}$$

Since $x_n \to p$ as $n \to \infty$, the inequality (3.13) implies that $v_n \to p$ as $n \to \infty$. In view of the closedness of T, it yields that $p \in Tp$, i.e.,

$$p \in F(T). \tag{3.14}$$

On the other hand, noticing that

$$\|v_n - u_n\| = \|v_n - x_{n+1} + x_{n+1} - u_n\|$$

$$= \|v_n - x_{n+1}\| + \|x_{n+1} - u_n\|.$$

It follows from (3.11) and (3.12) that

$$\lim_{n\to\infty} \|v_n - u_n\| = 0. \tag{3.15}$$

Since $v_n \to p$ as $n \to \infty$, the inequality (3.15) follows that $u_n \to p$ as $n \to \infty$. In view of $u_n \in Sv_n$ and the closeness of S, it yields that $p \in Sp$, i.e.,

$$p \in F(S). \tag{3.16}$$

The inequalities (3.14) and (3.16) imply that $p \in \mathscr{F}$.

Finally, we prove that $x_n \to \infty q = \Pi_{\mathscr{F}} x_0$ as $n \to \infty$. Noticing that $q = \Pi_{\mathscr{F}} x_0$, we have $q \in \mathscr{F} \subset C_n$. And since $x_n = \Pi_{C_n} x_0$, we have

$$\phi(x_n, x_0) \leq \phi(q, x_0), \quad \forall\, n \geq 0.$$

This implies that

$$\phi(p, x_0) \leq \lim_{n\to\infty} \phi(x_n, x_0) \leq \phi(q, x_0).$$

From the definition of $\Pi_{\mathscr{F}}x_0$ and the inequality above, we have $p = q$. Therefore,$x_n \to q = \Pi_{\mathscr{F}}x_0$. This completes the proof of Theorem 3.1. ■

If $T = I$ in (3.1), we can obtain the following corollary immediately.

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex Banach space E. Let $S : C \to N(C)$ be a closed and hemi-relatively nonexpansive multi-valued mapping such that $\mathscr{F} = F(S) \cap F(T) \neq \varnothing$. For a point $x_0 \in C$ chosen arbitrarily, let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} C_0 = C, \\ C_{n+1} = \{z \in C_n : \phi(z, u_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where the sequence $u_n \in Sx_n$. Then the sequence $\{x_n\}$ converges strongly to a point $q = \Pi_{F(S)}x_0$, where $\Pi_{F(S)}$ is the generalized projection from C onto $F(S)$.

If the two hemi-relatively nonexpansive mappings S, T are single-valued in (3.1), the following corollary can be obtained from Theorem 3.1.

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a uniformly smooth and uniformly convex Banach space E. Let S, $T : C \to N(C)$ be two closed and hemi-relatively nonexpansive single-valued mappings such that $\mathscr{F} = F(S) \cap F(T) \neq \varnothing$. For a point $x_0 \in C$ chosen arbitrarily, let $\{x_n\}$ be a sequence generated by the following iterative algorithm:*

$$\begin{cases} C_0 = C, \\ C_{n+1} = \{z \in C_n : \phi(z, Sy_n) \leq \phi(z, Tx_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \end{cases}$$

where the sequence $y_n \in Tx_n$. Then the sequence $\{x_n\}$ converges strongly to a point $q = \Pi_{\mathscr{F}}x_0$, where $\Pi_{\mathscr{F}}$ is the generalized projection from C onto $\mathscr{F}$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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