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ON THE CONVERGENCE OF HYBRID PROJECTION ALGORITHMS FOR A FAMILY OF ASYMPTOTICALLY STRICT QUASI-ϕ-PSEUDOCONTRACTIONS IN THE INTERMEDIATE SENSE

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Abstract. In this paper, a hybrid projection algorithm for a family of asymptotically strict quasi-ϕ-pseudocontractions in the intermediate sense is considered in the framework of Banach spaces. A strong convergence theorem of the proposed algorithm to a common fixed point of a family of asymptotically strict quasi-ϕ-pseudocontractions in the intermediate sense is proved. Our main result extends and improves many corresponding results.

Keywords: asymptotically strict quasi-ϕ-pseudocontractions in the intermediate sense; asymptotically strict quasi-ϕ-pseudocontractions; hybrid projection algorithm; fixed point; Banach space.

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1. Introduction

Throughout this paper, we always assume that E is a Banach space, E^* is the dual space of E, C is a nonempty closed convex subset of E, $T : C \to C$ be a nonlinear mapping. The symbols

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$\rightharpoonup$, $\rightarrow$, $\mathbb{N}$ and $\mathbb{R}^+$ are denoted by a weak convergence, a strong convergence, the set of positive integers, and the set of positive real numbers respectively. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|f^*\|^2\}, \quad \forall\, x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing of elements between E and E*. It is well known that if E^* is strictly convex, then J is single valued; if E^* is reflexive, and smooth, then J is single valued, and demicontinuous (see [1] for more details and the references therein).

Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. E is said to be *strictly convex* if $\|\frac{x+y}{2}\| < 1$ for all $x,\ y \in U_E$ with $x \neq y$. It is said to be *uniformly convex* if , for any $\varepsilon \in (0,2]$, there exists $\delta > 0$ such that for any $x, y \in U_E$,

$$\|x - y\| \geq \varepsilon \;\; implies \;\; \|\frac{x+y}{2}\| \leq 1 - \delta.$$

It is known that a uniformly convex Banach space is reflexive and strictly convex. E is said to be smooth provided $\lim_{n\to\infty} \frac{\|x+ty\|-\|x\|}{t}$ exists for all $x,\ y \in U_E$. It is also said be uniformly smooth if the limit is attained uniformly for all $x,\ y \in U_E$.

A Banach space E is said to enjoy *Kadec-Klee property* if for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$, as $n \to \infty$ (see [2] and the references therein for more details). It is well known that if E is a uniformly convex Banach space, then E enjoys Kadec-Klee property.

It is also well known that if C is a nonempty, closed, and convex subset of a Hilbert space H, and $P_C : H \to C$ is the metric projection from H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [3] introduced a generalized projection operator in Banach spaces which is an analogue of the metric projection in Hilbert spaces.

Let E be a smooth Banach space. Consider the Lyapunov functional $\phi : E \times E \to \mathbb{R}^+$defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall\, x,\ y \in E. \tag{1.1}$$

Notice that, in a Hilbert space H, (1.1) is reduced to $\phi(x,y) = \|x-y\|^2$ for all $x,\ y \in H$. It is obvious from the definition of ϕ that

$$(\|x\| - \|y\|)^2 \leq \phi(x,y) \leq (\|x\| + \|y\|)^2, \quad \forall\, x,\ y \in E. \tag{1.2}$$

In addition, the function ϕ has the following property:

$$\phi(y,x) = \phi(z,x) + \phi(y,z) + 2\langle z-y, Jx - Jz\rangle, \quad \forall\, x,\ y,\ z \in E, \tag{1.3}$$

and

$$\phi(x, J^{-1}(\lambda Jy + (1-\lambda)Jz)) \leq \lambda\phi(x,y) + (1-\lambda)\phi(x,z), \tag{1.4}$$

for all $\lambda \in [0,1]$ and $x,\ y,\ z \in E$.

Let C is a nonempty, closed, and convex subset of a reflexive,strictly convex, and smooth Banach space E. The *generalized projection* $\Pi_C : E \to C$ is defined by

$$\Pi_C x = arg \inf_{y\in C} \phi(y,x), \quad \forall\, x \in E.$$

In the following, we recall some definitions.

(1) A point p in C is said to be an *asymptotic fixed point* of T [4] if C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

(2) T is said to be *relatively nonexpansive* if

$$\widetilde{F}(T) = F(T) \neq \emptyset, \quad \phi(p,Tx) \leq \phi(p,x), \quad \forall\, x \in C,\ \forall\, p \in F(T).$$

The asymptotic behavior of relatively nonexpansive mappings was studied in [5-7].

(3) T is said to be *relatively asymptotically nonexpansive* if

$$\widetilde{F}(T) = F(T) \neq \emptyset, \quad \phi(p,T^n x) \leq (1+\mu_n)\phi(p,x), \quad \forall\, x \in C,\ \forall\, p \in F(T),\ \forall\, n \geq 1,$$

where $\{\mu_n\} \subset [0,\infty)$ is a sequence such that $\mu_n \to 1$ as $n \to \infty$. The class of relatively asymptotically nonexpansive mappings was first considered in Su and Qin [8]; see also, Agarwal, Cho, and Qin [9], and Qin et al. [10].

(4) T is said to be *quasi-ϕ-nonexpanisve* if

$$F(T) \neq \emptyset, \quad \phi(p,Tx) \leq \phi(p,x), \quad \forall\, x \in C, \,\forall\, p \in F(T).$$

Remark 1.1. In some articles, The class of quasi-ϕ-nonexpanisve mappings is called hemi-relatively nonexpansive mapping; see [11-13].

(5) T is said to be *asymptotically quasi-ϕ-nonexpansive* if there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ such that

$$F(T) \neq \emptyset, \quad \phi(p,T^n x) \leq (1+\mu_n)\phi(p,x), \quad \forall x \in C, \forall p \in F(T), \forall n \geq 1.$$

The class of asymptotically quasi-ϕ-nonexpansive mappings was considered in Qin, Cho, and Kang [14], and Zhou, Gao, and Tan [15]; see also [16, 17].

Remark 1.2. The class of quasi-ϕ-nonexpansive mappings and the class of asymptotically quasi-ϕ-nonexpansive mappings are more general than the class of relatively nonexpansive mapppings and the class of relatively asymptotically nonexpansive mappings. Notice that quasi-ϕ-nonexpansive mappings and asymptotically quasi-ϕ-nonexpansive do not require $F(T) = \widetilde{F}(T)$.

(6) T is said to be a *strict quasi-ϕ-pseudocontraction* if $F(T) \neq \emptyset$ and there exists a constant $\kappa \in [0,1)$ such that

$$\phi(p,Tx) \leq \phi(p,x) + \kappa\phi(x,Tx), \quad \forall\, x \in C, \; p \in F(T).$$

(7) T is said to be an *asymptotically strict quasi-ϕ-pseudocontraction* if $F(T) \neq \emptyset$ and there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as $n \to \infty$ and a constant $\kappa \in [0,1)$ such that

$$\phi(p,T^n) \leq (1+\mu_n)\phi(p,x) + \kappa\phi(x,T^n x), \quad \forall x \in C, \forall p \in F(T), \forall n \geq 1.$$

The class of asymptotically strict quasi-ϕ-pseudocontractions was first considered in Qin et al. [18].

(8) T is said to be an *asymptotically strict quasi-ϕ-pseudocontraction in the intermediate sense* if $F(T) \neq \emptyset$ and there exists a sequence $\{\mu_n\} \subset [0,\infty)$ with $\mu_n \to 0$ as$n \to \infty$ and a

constant $\kappa \in [0,1)$ such that

$$\limsup_{n\to\infty} \sup_{p\in F(T),x\in C} (\phi(p,T^n x) - (1+\mu_n)\phi(p,x) - \kappa\phi(x,T^n x)) \le 0. \tag{1.5}$$

Put

$$\xi_n = \max\{0, \sup_{p\in F(T),x\in C} (\phi(p,T^n x) - (1+\mu_n)\phi(p,x) - \kappa\phi(x,T^n x))\}.$$

It follows that $\xi_n \to 0$ as $n \to \infty$. Then, (1.5) is reduced to the following:

$$\phi(p,T^n x) \le (1+\mu_n)\phi(p,x) + \kappa\phi(x,T^n x) + \xi_n, \quad \forall p \in F(T), \forall x \in C, \forall n \ge 1.$$

The class of asymptotically strict quasi-ϕ-pseudocontractions in the intermediate sense was first considered in Qin et al. [19].

Remark 1.3. If $\kappa = 0$ and $\mu_n \equiv 0$, then we call T an asymptotically quasi-ϕ-nonexpansive mapping in the intermediate sense.

Remark 1.4. (i) If $\kappa = 0$ in (7), then the class of asymptotically strict quasi-ϕ-pseudocontractions is reduced to the class of asymptotically quasi-ϕ-nonexpansive mappings.

(ii) If $\xi_n \equiv 0$ in (8), then the class of asymptotically strict quasi-ϕ-pseudocontractions in the intermediate sense is reduced to the class of asymptotically strict quasi-ϕ-pseudocontractions.

(9) T is said to be asymptotically regular on C if, for any bounded subset K of C,

$$\lim_{n\to\infty} \sup_{x\in K}\{\|T^{n+1}x - T^n x\|\} = 0.$$

Fixed point theory as an important branch of nonlinear analysis theory has been of great concern by many authors. In order to construct an iterative process to approximate the fixed point of a contractive mapping, the Picard iterative algorithm was proposed. It is known that T, where T stands for a contractive mapping, enjoys a unique fixed point, and the sequence generated by the Picard iterative algorithm can converge to the unique fixed point. However, for more general nonexpansive mappings, the Picard iterative algorithm fails to converge to fixed points of nonexpansive mapppings even when they enjoy fixed points. The Krasnoselskii-Mann iterative algorithm and the Ishikawa iterative algorithm have been studied for approximating fixed points of nonexpansive mappings and their extensions. However, both the Krasnoselskii-Mann iterative algorithm and the Ishikawa iterative algorithms are weak convergence for nonexpansive mappings only; see [20] and [21] for the classic weak convergence theorems. In many disciplines, including economics, image recovery, quantum physics and control theory, strong convergence (norm convergence) is often much more desirable than weak convergence. Projection methods, which were first introduced by Haugazeau [22], have been considered for the approximation of fixed points of nonexpansive mappings and their extensions. The advantage of projection methods is that strong convergence of iterative sequences can be guaranteed without any compact assumptions. In recent years, many authors have studied on iterative algorithm of fixed points of nonlinear mappings by projection methods; see [11-19, 23].

In this paper, a family of asymptotically strict quasi-ϕ-pseudocontractions in the intermediate sense is considered in a reflexive, strictly convex, and smooth Banach space. Based on a simple hybrid projection algorithm, a theorem of strong convergence for common fixed points is obtained. The results presented in this paper mainly improve the known corresponding results announced in the literature sources listed in this work.

2. Preliminaries

In order to prove our main results, we also need the following lemmas.

Lemma 2.1. ([24]) *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \leq 0, \quad \forall y \in C.$$

Lemma 2.2. ([24]) *Let E be a reflexive, strictly convex, and smooth Banach space, C a nonempty closed convex subset of E and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.3. ([24]) *Let E be a reflexive, strictly convex, and smooth Banach space. Then we have the following:*

$$\phi(x,y)=0 \;\Leftrightarrow\; x=y, \quad \forall x,y\in E.$$

Lemma 2.4. *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property, let C be a nonempty, closed, and convex subset of E. Let $T: C\to C$ is a closed asymptotically strict quasi-ϕ-pseudocontraction in the intermediate sense with a sequence $\{\mu_n\}\subset[0,\infty)$ such that $\mu_n\to 0$ as $n\to\infty$. Then, $F(T)$ is closed and convex.*

Proof. First, we show that $F(T)$ is closed. Let $\{p_n\}$ be a sequence in $F(T)$ such that $p_n\to p$ as $n\to\infty$. We see that $p\in F(T)$. Indeed, from the definition of T, we have

$$\phi(p_n,T^np)\le(1+\mu_n)\phi(p_n,p)+\kappa\phi(p,T^np)+\xi_n.$$

On the other hand, we have from (1.3) that

$$\phi(p_n,T^np)=\phi(p_n,p)+\phi(p,T^np)+2\langle p_n-p,Jp-JT^np\rangle.$$

It follow that

$$\phi(p_n,p)+\phi(p,T^np)+2\langle p_n-p,Jp-JT^np\rangle\le(1+\mu_n)\phi(p_n,p)+\kappa\phi(p,T^np)+\xi_n,$$

which implies that

$$\phi(p,T^np)\le\frac{\mu_n}{1-\kappa}\phi(p_n,p)+\frac{2}{1-\kappa}\langle p-p_n,Jp-JT^np\rangle+\frac{\xi_n}{1-\kappa}.$$

From $\lim_{n\to\infty}p_n=p$, $\lim_{n\to\infty}\mu_n=\lim_{n\to\infty}\xi_n=p$, and the above inequality, it follows that

$$\lim_{n\to\infty}\phi(p,T^np)=0.$$

In view of (1.2), we have

$$\lim_{n\to\infty}\|T^np\|=\|p\|. \tag{2.1}$$

It follows that

$$\lim_{n\to\infty}\|J(T^np)\|=\|Jp\|. \tag{2.2}$$

Since E^* is reflexive, we may, without loss of generality, assume that $J(T^n p) \rightharpoonup e^* \in E^*$. In view of the reflexivity of E, we have $J(E) = E^*$. This shows that there exists an element $e \in E$ such that $Je = e^*$. It follows that

$$\begin{aligned}\phi(p, T^n p) &= \|p\|^2 - 2\langle p, J(T^n p)\rangle + \|T^n p\|^2 \\ &= \|p\|^2 - 2\langle p, J(T^n p)\rangle + \|J(T^n p)\|^2\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both side of the above equality, we obtain that

$$\begin{aligned}0 &\geq \|p\|^2 - 2\langle p, e^*\rangle + \|e^*\|^2 \\ &= \|p\|^2 - 2\langle p, Je\rangle + \|Je\|^2 \\ &= \|p\|^2 - 2\langle p, Je\rangle + \|e\|^2 \\ &= \phi(p, e).\end{aligned}$$

This implies from Lemma 2.3 that $p = e$, that is, $Jp = e^*$. It follows that $J(T^n p) \rightharpoonup Jp \in E^*$. In view of the Kadec-Klee proerty of E^*, we obtain from (2.2) that

$$\lim_{n\to\infty} \|J(T^n p) - Jp\| = 0.$$

Since $J^{-1} : E^* \to E$ is demicontinuous, we see that $T^n p \rightharpoonup p$. By virtue of the Kadec-Klee property of E, we see from (2.1) that $T^n p \to p$ as $n \to \infty$, which implies that

$$TT^n p = T^{n+1} p \to p, \quad as\ n \to \infty.$$

In view of the closedness of T, we can obtain that $p \in F(T)$. This shows that $F(T)$ is closed.

Next, we show that $F(T)$ is convex. Let $p_1, p_2 \in F(T)$, and $p_t = tp_1 + (1-t)p_2$, where $t \in (0,1)$. We see that $p_t \in F(T)$. Indeed, we see from the definition of T that

$$\phi(p_1, T^n p_t) \leq (1+\mu_n)\phi(p_1, p_t) + \kappa\phi(p_t, T^n p_t) + \xi_n. \tag{2.3}$$

and

$$\phi(p_2, T^n p_t) \leq (1+\mu_n)\phi(p_2, p_t) + \kappa\phi(p_t, T^n p_t) + \xi_n. \tag{2.4}$$

In view of (1.3), we have that

$$\phi(p_1, T^n p_t) = \phi(p_1, p_t) + \phi(p_t, T^n p_t) + 2\langle p_1 - p_t, Jp_t - J(T^n p_t)\rangle, \tag{2.5}$$

and

$$\phi(p_2,T^n p_t)=\phi(p_2,p_t)+\phi(p_t,T^n p_t)+2\langle p_2-p_t,Jp_t-J(T^n p_t)\rangle, \tag{2.6}$$

It follows from (2.3), (2.4), (2.5), and (2.6) that

$$\phi(p_t,T^n p_t)\le \frac{\mu_n}{1-\kappa}\phi(p_1,p_t)+\frac{2}{1-\kappa}\langle p_t-p_1,Jp_t-J(T^n p_t)\rangle+\frac{\xi_n}{1-\kappa} \tag{2.7}$$

and

$$\phi(p_t,T^n p_t)\le \frac{\mu_n}{1-\kappa}\phi(p_1,p_t)+\frac{2}{1-\kappa}\langle p_t-p_1,Jp_t-J(T^n p_t)\rangle+\frac{\xi_n}{1-\kappa} \tag{2.8}$$

Multiplying t and $(1-t)$ on both sides (2.7) and (2.8) respectively yields that

$$\phi(p_t,T^n p_t)\le \frac{t\mu_n}{1-\kappa}\phi(p_1,P_t)+\frac{(1-t)\mu_n}{1-\kappa}\phi(p_2,P_t)+\frac{\xi_n}{1-\kappa}.$$

It follows that

$$\lim_{n\to\infty}\phi(p_t,T^n p_t)=0.$$

By using (1.2), we arrive at

$$\lim_{n\to\infty}\|T^n p_t\|=\|p_t\|, \tag{2.9}$$

which implies that

$$\lim_{n\to\infty}\|J(T^n p_t)\|=\|Jp_t\|, \tag{2.10}$$

Since E^* is reflexive, we may assume that $J(T^n p_t)\rightharpoonup w^*\in E^*$. By virtue of the reflexivity of E, we have $J(E)=E^*$. This show that there exists an element $w\in E$ such that $Jw=w^*$. It follows that

$$\begin{aligned}\phi(p_t,T^n p_t)&=\|p_t\|^2-2\langle p_t,J(T^n p_t)\rangle+\|T^n p_t\|^2\\&=\|p_t\|^2-2\langle p_t,J(T^n p_t)\rangle+\|J(T^n p_t)\|^2\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both side of the above equality, we obtain that

$$\begin{aligned}0&\ge\|p_t\|^2-2\langle p_t,w^*\rangle+\|w^*\|^2\\&=\|p_t\|^2-2\langle p_t,Jw\rangle+\|Jw\|^2\\&=\|p_t\|^2-2\langle p_t,Jw\rangle+\|w\|^2\\&=\phi(p_t,w).\end{aligned}$$

This implies from Lemma 2.3 that $p_t = w$, that is, $Jp_t = w^*$. It follows that $J(T^n p_t) \rightharpoonup Jp_t \in E^*$. In view of the Kadec-Klee proerty of E^*, we obtain from (2.10) that

$$\lim_{n\to\infty} \|J(T^n p_t) - Jp_t\| = 0.$$

Since $J^{-1} : E^* \to E$ is demicontinuous, we see that $T^n p_t \rightharpoonup p_t$. In view of the Kadec-Klee property of E, we see from (2.9) that $T^n p_t \to p_t$ as $n \to \infty$. This implies that

$$TT^n p_t = T^{n+1} p_t \to p_t, \quad as\ n \to \infty.$$

In view of the closedness of T, we can obtain that $p_t \in F(T)$. This shows that $F(T)$ is convex. This completes the proof that $F(T)$ is closed and convex. ■

3. Main results

In this section, we will give our main results in this paper.

Theorem 3.1. *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property. Let C be a nonempty bounded and closed convex subset of E. Let $\wedge$ be an index set and $T_i : C \to C$, where $i \in \wedge$, be an asymptotically strict quasi-ϕ-pseudocontraction in the intermediate sense with a sequence $\{\mu_{(n,i)}\} \subset [0,\infty)$ such that $\mu_{(n,i)} \to 0$ as $n \to \infty$. Assume that $\bigcap_{i\in\bigwedge} F(T_i) \neq \emptyset$. For each $i \in \wedge$, assume that T_i is uniformly asymptotically regular on C, and $F(T_i)$ is bounded. Let $\{x_n\}$ be a sequence generated in the following iterative scheme:*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{(1,i)} = C, \\ C_1 = \bigcap_{i\in\wedge} C_{(1,i)}, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \phi(x_n, T_i^n x_n) \le \frac{2}{1-\kappa_i}\langle x_n - z, Jx_n - J(T_i^n x_n)\rangle + \theta_{(n,i)}\}, \\ C_{n+1} = \bigcap_{i\in\wedge} C_{(n+1,i)}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 0, \end{cases} \tag{3.1}$$

where $\theta_{(n,i)} = \mu_{(n,i)}\frac{M_{(n,i)}}{1-\kappa_i} + \frac{\xi_{(n,i)}}{1-\kappa_i}$, $M_{(n,i)} = \sup\{\phi(p,x_n) : p \in F(T_i)\}$ *and*

$$\xi_{(n,i)} = \max\{0, \sup_{p\in F(T_i), x\in C}(\phi(p,T_i^n x) - (1+\mu_{(n,i)})\phi(p,x) - \kappa_i\phi(x,T_i^n x))\}.$$

Then the sequence $\{x_n\}$ *converges strongly to a point* $\bar{x} = \Pi_{\bigcap_{i\in\wedge}F(T_i)}x_0$.

Proof. We split the proof into six steps.

Step 1. Show that $\Pi_{\bigcap_{i\in\wedge}F(T_i)}x_0$ **is well defined for every** $x_0 \in E$**.**

By Lemma 2.4, we know that $F(T_i)$ is a closed and convex subset of C for each $i \in \wedge$. Therefore, by using the property of intersection, we get $\bigcap_{i\in\wedge}F(T_i)$ is also closed and convex. $\Pi_{\bigcap_{i\in\wedge}F(T_i)}x_0$ is well defined for every $x_0 \in E$.

Step 2. Show that C_n **is closed and convex for each** $n \geq 1$**.**

From the structure of C_n in (3.1), it is obvious that C_n is closed for each $n \geq 1$. Therefore, we only show that C_n is convex for each $n \geq 1$. It suffices to show that, for each fixed but arbitrary $i \in \wedge$, $C_{(n,i)}$ is convex for each $n \geq 1$. This can be proved by induction on n. For $n = 1$, it is obvious that $C_{(1,i)} = C$ is convex. Suppose that $C_{(n,i)}$ is convex for some $n \in \mathbb{N}$. Next, we show that $C_{(n+1,i)}$ is also convex for the same n. Let $a_1,\ a_2 \in C_{(n+1,i)}$ and $a_t = ta_1 + (1-t)a_2$, where $t \in (0,1)$. It follows that

$$\phi(x_n,T_i^n x_n) \leq \frac{2}{1-\kappa_i}\langle x_n - a_1, Jx_n - J(T_i^n x_n)\rangle + \mu_{(n,i)}\frac{M_{(n,i)}}{1-\kappa_n} + \frac{\xi_{(n,i)}}{1-\kappa_i},$$

and

$$\phi(x_n,T_i^n x_n) \leq \frac{2}{1-\kappa_i}\langle x_n - a_2, Jx_n - J(T_i^n x_n)\rangle + \mu_{(n,i)}\frac{M_{(n,i)}}{1-\kappa_n} + \frac{\xi_{(n,i)}}{1-\kappa_i},$$

where $a_1,\ a_2 \in C_{(n,i)}$. Multiplying t and $(1-t)$ on both sides of the above two inequalities respectively yields that

$$\begin{aligned}\phi(x_n,T_i^n x_n) &\leq \frac{2}{1-\kappa_i}\langle x_n - a, Jx_n - J(T_i^n x_n)\rangle + \mu_{(n,i)}\frac{M_{(n,i)}}{1-\kappa_n} + \frac{\xi_{(n,i)}}{1-\kappa_i}\\ &= \frac{2}{1-\kappa_i}\langle x_n - a, Jx_n - J(T_i^n x_n)\rangle + \theta_{(n,i)},\end{aligned}$$

where $a \in C_{(n,i)}$. It follows that $C_{(n+1,i)}$ is convex for the same n. This, in turn, implies that $C_n = \bigcap_{i\in\wedge}C_{(n,i)}$ is convex for each $n \geq 1$. Hence, $C_n = \bigcap_{i\in\wedge}C_{(n,i)}$ is closed and convex for each $n \geq 1$.

Step 3. Show that $\bigcap_{i\in\wedge} F(T_i) \subset C_n$ for $n \geq 1$.

It is obvious that $F(T_i) \subset C = C_1$ for each $i \in \wedge$. Suppose that $F(T_i) \subset C_n$ for some $n \in \mathbb{N}$. For any $w \in F(T_i) \subset C_n$, we see that

$$\phi(w, T_i^n x_n) \leq (1+\mu_{(n,i)})\phi(w, x_n) + \kappa_i \phi(x_n, T_i^n x_n) + \xi_{(n,i)}. \tag{3.2}$$

On the other hand, we obtain from (1.3) that

$$\phi(w, T_i^n x_n) = \phi(w, x_n) + \phi(x_n, T_i^n x_n) - 2\langle w - x_n, Jx_n - J(T_i^n x_n)\rangle. \tag{3.3}$$

Combining (3.2) with (3.3), we arrive at

$$\begin{aligned}
\phi(x_n, T_i^n x_n) &\leq \frac{\mu_{(n,i)}}{1-\kappa_i}\phi(w, x_n) + \frac{2}{1-\kappa_i}\langle x_n - w, Jx_n - J(T_i^n x_n)\rangle + \frac{\xi_{(n,i)}}{1-\kappa_i} \\
&\leq \mu_{(n,i)}\frac{M_{(n,i)}}{1-\kappa_i} + \frac{2}{1-\kappa_i}\langle x_n - w, Jx_n - J(T_i^n x_n)\rangle + \frac{\xi_{(n,i)}}{1-\kappa_i} \\
&= \frac{2}{1-\kappa_i}\langle x_n - w, Jx_n - J(T_i^n x_n)\rangle + \theta_{(n,i)},
\end{aligned}$$

which implies that $z \in C_{(n+1,i)}$ for the same n. By the mathematical induction principle, $F(T_i) \subset C_n$ for each $i \in \wedge$. This implies that $\bigcap_{i\in\wedge} F(T_i) \subset C_n$.

Step 4. Show that $x_n \to \bar{x} \in E$, as $n \to \infty$.

Since $x_n = \Pi_{C_n} x_0$, we see that

$$\langle x_n - z, Jx_0 - Jx_n\rangle \geq 0, \quad \forall\, z \in C_n.$$

It follows from $\bigcap_{i\in\wedge} F(T_i) \subset C_n$ that

$$\langle x_n - p, Jx_0 - Jx_n\rangle \geq 0, \quad \forall\, p \in \bigcap_{i\in\wedge} F(T_i). \tag{3.4}$$

In view of $\bigcap_{i\in\wedge} F(T_i) \subset C_n$ and Lemma 2.2, we obtain that

$$\begin{aligned}
\phi(x_n, x_0) &\leq \phi(\Pi_{C_n} x_0, x_0) \\
&\leq \phi(\Pi_{\bigcap_{i\in\wedge} F(T_i)} x_0, x_0) - \phi(\Pi_{\bigcap_{i\in\wedge} F(T_i)} x_0, x_n) \\
&\leq \phi(\Pi_{\bigcap_{i\in\wedge} F(T_i)} x_0, x_0).
\end{aligned}$$

This implies that the sequence $\phi(x_n, x_0)$ is bounded. It follows from (1.2) that the sequence $\{x_n\}$ is also bounded. Since the space is reflexive, we may assume that $x_n \rightharpoonup \bar{x}$, as $n \to \infty$. Since

C_n is closed, and convex, we see that $\bar{x} \in C_n$. On the other hand, we see from the weakly lower semicontinuity of the norm that

$$\begin{aligned}\phi(\bar{x},x_0) &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jx_0\rangle + \|x_0\|^2 \\ &\le \liminf_{n\to\infty}(\|x_n\|^2 - 2\langle x_n, Jx_0\rangle + \|x_0\|^2) \\ &= \liminf_{n\to\infty}\phi(x_n.x_0) \\ &\le \limsup_{n\to\infty}\phi(x_n.x_0) \\ &\le \phi(\bar{x},x_0),\end{aligned}$$

which implies that $\phi(x_n,x_0) \to \phi(\bar{x},x_0)$ as $n\to\infty$. Hence, $\|x_n\| \to \|\bar{x}\|$ as $n\to\infty$. In view of the Kadec-Klee property of E, we see that $x_n \to \bar{x}$ as $n\to\infty$.

Step 5. Show that $\bar{x} \in \bigcap_{i\in\wedge} F(T_i)$**.**

Notice that $x_n = \Pi_{C_n}x_0$. It follows that

$$\begin{aligned}\phi(x_{n+1},x_n) &= \phi(x_{n+1},\Pi_{C_n}x_0) \\ &\le \phi(x_{n+1},x_0) - \phi(\Pi_{C_n}x_0,x_0) \\ &= \phi(x_{n+1},x_0) - \phi(x_n,x_0).\end{aligned} \tag{3.5}$$

Since $\lim_{n\to\infty}\phi(x_n,x_0) = \phi(\bar{x},x_0)$ and (3.5), we have

$$\lim_{n\to\infty}\phi(x_{n+1},x_n) = 0. \tag{3.6}$$

In view of $x_{n+1} = \Pi_{C_{n+1}}x_0 \in C_{n+1}$, we obtain that

$$\phi(x_n,T_i^n x_n) \le \frac{2}{1-\kappa_i}\langle x_n - x_{n+1}, Jx_n - JT_i^n x_n\rangle + \theta_{(n,i)}, \quad \forall\, i\in\wedge.$$

From $\theta_{(n,i)} \to 0$ and $x_n \to \bar{x}$ as $n\to\infty$, we have

$$\lim_{n\to\infty}\phi(x_n,T_i^n x_n) = 0, \quad \forall\, i\in\wedge.$$

By using (1.2), we see that

$$\lim_{n\to\infty}(\|x_n\| - \|T_i^n x_n\|) = 0, \quad \forall\, i\in\wedge. \tag{3.7}$$

On the other hand, we have from $x_n \to \bar{x}$ that

$$\lim_{n\to\infty} \|T_i^n x_n\| = \lim_{n\to\infty} [(\|T_i^n x_n\| - \|x_n\|) + \|x_n\|] = \|\bar{x}\|, \quad \forall\, i \in \wedge. \tag{3.8}$$

It follows that

$$\lim_{n\to\infty} \|J(T_i^n x_n)\| = \|J\bar{x}\|, \quad \forall\, i \in \wedge. \tag{3.9}$$

This implies that $\{J(T_i^n x_n)\}$ is bounded for each $i \in \wedge$. Since E and E^* are reflexive, we may assume that $J(T_i^n x_n) \rightharpoonup y_i^* \in E*$ for each $i \in \wedge$. In view of the reflexivity of E, we see that there exists an element $y_i \in E$ such that $Jy_i = y_i^*$ for each $i \in \wedge$. It follows that

$$\begin{aligned}\phi(x_n, T_i^n x_n) &= \|x_n\|^2 - 2\langle x_n, J(T_i^n x_n)\rangle + \|T_i^n x_n\|^2 \\ &= \|x_n\|^2 - 2\langle x_n, J(T_i^n x_n)\rangle + \|J(T_i^n x_n)\|^2.\end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides of the above equality yields that

$$\begin{aligned}0 &\geq \|\bar{x}\|^2 - 2\langle \bar{x}, y_i^*)\rangle + \|y_i^*\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jy_i\rangle + \|Jy_i\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jy_i\rangle + \|y_i\|^2 \\ &= \phi(\bar{x}, y_i),\end{aligned}$$

That is, for each $i \in \wedge$, $\bar{x} = y_i$, which in turn implies that $y_i^* = J\bar{x}$. It follows that $J(T_i^n x_n) \rightharpoonup J\bar{x} \in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain from (3.9) that $\lim_{n\to\infty} J(T_i^n x_n) = J\bar{x}$. Since $J^{-1} : E^* \to E$ is demicontinuous, we get $T_i^n x_n \rightharpoonup \bar{x}$ for each $i \in \wedge$. This implies, from (3.8) and the Kadec-Klee property of E, that $\lim_{n\to\infty} T_i^n x_n = \bar{x}$. Notice that

$$\|T_i^{n+1} x_n - \bar{x}\| \leq \|T_i^{n+1} x_n - T_i^n x_n\| + \|T_i^n x_n - \bar{x}\|, \quad \forall\, i \in \wedge.$$

This implies from the asymptotic regularity of T_i that

$$\lim_{n\to\infty} \|T_i^{n+1} x_n - \bar{x}\| = 0, \quad \forall\, i \in \wedge,$$

that is, for each $i \in \wedge$, $T_i T_i^n x_n - \bar{x} \to 0$ as $n \to \infty$. In view of the closedness of T_i, we have $T_i \bar{x} = \bar{x}$ for each $i \in \wedge$, that is, $\bar{x} \in \bigcap_{i\in\wedge} F(T_i)$.

Step 6. Show that $\bar{x} = \Pi_{\bigcap_{i\in\wedge} F(T_i)} x_0$.

Indeed, by taking $n \to \infty$ in (3.4), we arrive at

$$\langle \bar{x} - p, Jx_0 - J\bar{x} \rangle \geq 0, \quad \forall\, p \in \bigcap_{i \in \wedge} F(T_i).$$

It follows from Lemma 2.1 that $\bar{x} = \Pi_{\bigcap_{i\in\wedge} F(T_i)} x_0$. This completes the proof. ■

For a single mapping, we obtain from Theorem 3.1 the following:

Corollary 3.2. *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property. Let C be a nonempty bounded and closed convex subset of E. Let $\wedge$ be an index set and $T : C \to C$ be an asymptotically strict quasi-ϕ-pseudocontraction in the intermediate sense with a sequence $\{\mu_n\} \subset [0, \infty)$ such that $\mu_n \to 0$ as $n \to \infty$. Assume that $F(T) \neq \emptyset$. Assume that T is uniformly asymptotically regular on C, and $F(T)$ is bounded. Let $\{x_n\}$ be a sequence generated in the following iterative scheme:*

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_1 = C, \\ C_{n+1} = \{z \in C_n : \phi(x_n, T^n x_n) \leq \frac{2}{1-\kappa}\langle x_n - z, Jx_n - J(T^n x_n)\rangle + \theta_n\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \geq 0, \end{cases}$$

where $\theta_n = \mu_n \frac{M_n}{1-\kappa} + \frac{\xi_n}{1-\kappa}$, $M_n = \sup\{\phi(p, x_n) : p \in F(T)\}$ and

$$\xi_n = \max\{0, \sup_{p \in F(T), x \in C} (\phi(p, T^n x) - (1+\mu_n)\phi(p, x) - \kappa\phi(x, T^n x))\}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $\bar{x} = \Pi_{F(T)} x_0$.

Since every asymptotically strict quasi-ϕ-pseudocontraction in the intermediate sense is an asymptotically strict quasi-ϕ-pseudocontraction, we immediately obtain the following corollary:

Corollary 3.3. *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property. Let C be a nonempty bounded and closed convex subset of E. Let $\wedge$ be an index set and $T_i : C \to C$, where $i \in \wedge$, be an asymptotically strict quasi-ϕ-pseudocontraction with a sequence $\{\mu_{(n,i)}\} \subset [0, \infty)$ such that $\mu_{(n,i)} \to 0$ as $n \to \infty$. Assume that $\bigcap_{i\in\bigwedge} F(T_i) \neq \emptyset$. For each $i \in \wedge$, assume that T_i is uniformly asymptotically regular on C, and*

$F(T_i)$ is bounded. Let $\{x_n\}$ be a sequence generated in the following iterative scheme:

$$\begin{cases} x_0 \in E \text{ chosen arbitrarily}, \\ C_{(1,i)} = C, \\ C_1 = \bigcap_{i\in\wedge} C_{(1,i)}, \\ C_{(n+1,i)} = \{z \in C_{(n,i)} : \\ \qquad \phi(x_n, T_i^n x_n) \le \frac{2}{1-\kappa_i}\langle x_n - z, Jx_n - J(T_i^n x_n)\rangle + \mu_{(n,i)}\frac{M_{(n,i)}}{1-\kappa_i}\}, \\ C_{n+1} = \bigcap_{i\in\wedge} C_{(n+1,i)}, \\ x_{n+1} = \Pi_{C_{n+1}} x_0, \quad \forall n \ge 0, \end{cases}$$

where $M_{(n,i)} = \sup\{\phi(p,x_n) : p \in F(T_i)\}$. Then the sequence $\{x_n\}$ converges strongly to a point $\bar{x} = \Pi_{\bigcap_{i\in\wedge} F(T_i)} x_0$.

Remark 3.4. Comparing with Theorem 2.1 in Qing-Nian Zhang [23], we have the following:

(a) improve the mapping from asymptotically strict quasi-ϕ-pseudocontractions to asymptotically strict quasi-ϕ-pseudocontractions in the intermediate sense;

(b) improve the mapping from a single mapping to a family of mappings.

Remark 3.5. Comparing with Theorem 3.1 in Qin et al. [19], we improve the space from a uniformly convex and smooth Banach space to a reflexive, strictly convex, and smooth Banach space.

Conflict of Interests

The authors declare that there is no conflict of interests.

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