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FIXED POINT THEOREMS FOR F-SUZUKI-WEAK CONTRACTIONS ON COMPLETE CONE METRIC SPACES

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Abstract. In this paper, we introduced a new concept of contraction called F-Suzuki-weak contraction and prove a fixed point theorem concerning F-Suzuki-weak contractions in complete cone metric spaces over a solid cone. Our results improve the corresponding results announced by many others.

Keywords: Fixed point; Cone metric space; F-Suzuki-weak contraction.

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1. Introduction

In 2007, Huang and Zhang [8] introduced cone metric spaces as a generalization of metric spaces and extended the Banach contraction principle and some other fixed point results in cone metric spaces over a normal solid cone, being unaware about the existence of such spaces under the name of $K-$metric spaces and $K-$normed spaces. However they weant further and defined the convergence of a sequence and the Cauchy sequences via interior points of the cone. Later, many authors have studied cone metric spaces over solid cone and fixed point theorems in such spaces (see [1, 2, 5, 7, 10, 12, 14, 15, 17, 19, 20, 22, 23, 26, 27, 29] and the references therein).

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In 1962, Edelstein [6] considered some fixed and periodic point results of contractive mappings in metric spaces. In 2009, Suzuki [25], proved a generalization of Edelstein's theorem as follows:

Theorem 1.1. [25] *Let (X,d) be a compact metric space and let $T: X \to X$. Assume that*

$$\frac{1}{2}d(x,Tx) < d(x,y) \Rightarrow d(Tx,Ty) < d(x,y),$$

holds for all $x,y \in X$. Then T has a unique fixed point in X.

Recently, Wardowski [27] has introduced the concept of an F-contraction as follows

Definition 1.2. Let $\mathscr{F}$ be the family of all functions $F\colon (0,\infty) \to \mathbb{R}$ such that

$(F1)$ F is strictly increasing, i.e. for all $x,y \in \mathbb{R}_+$ such that $x<y$, $F(x) < F(y)$;

$(F2)$ For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers

$$\lim_{n\to\infty} \alpha_n = 0 \Leftrightarrow \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

$(F3)$ There exist $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

Let (X,d) be a metric space. A mapping $T: X \to X$ is said to be an F-contraction on (X,d) if there exists $F \in \mathscr{F}$ and $\tau > 0$ such that

$$\forall x,y \in X,\ \ [d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y))]. \tag{1}$$

Obviously $F(\alpha) = \ln(\alpha)$ and $F(\alpha) = \frac{-1}{\sqrt{\alpha}}$, for $\alpha > 0$ satisfy in the above conditions. For more functions in $\mathscr{F}$ the reader is referred to [24] and [27].

Remark 1.3. From $(F1)$ and (1) it is easy to conclude that every $F-$contraction is necessarily continuous.

Wardowski [27] stated a modified version of Banach contraction principle as follows:

Theorem 1.4. [27] *Let (X,d) be a complete metric space and let $T: X \to X$ be an F-contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.*

Remark 1.5. Define $F_B\colon \mathbb{R}_+ \to \mathbb{R}$ by $F_B(\alpha) = \ln\alpha$, then $F_B \in \mathscr{F}$. Note that, with $F = F_B$ the F-contraction reduces into a Banach contraction. Therefore, the Banach contractions are a particular case of F-contractions. While, there exist F-contractions which are not Banach contractions (see, [27]).

Very recently, Wardowski and Van Dung [28] have introduced the notion of an F-weak contraction and prove a fixed point theorem for F-weak contractions, which generalizes some results known from the literature. They give some examples to show that their result is a proper extension for Theorem 2.1 of [27]. Wardowski and Van Dung [28] have introduced the concept of an F-weak contraction as follows:

Definition 1.6. Let (X,d) be a metric space. A mapping $T : X \to X$ is said to be an F-weak contraction on (X,d) if there exists $F \in \mathscr{F}$ and $\tau > 0$ such that for all $x,y \in X$ satisfying $d(Tx,Ty) > 0$, the following holds:

$$\tau + F(d(Tx,Ty)) \le F\left(\max\{(d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}\right). \tag{2}$$

By using the notion of F-weak contraction, Wardowski and Van Dung have proved the following fixed point theorem which generalizes the result of Wardowski.

Theorem 1.7. *Let (X,d) be a complete metric space and let $T : X \to X$ be an F-weak contraction. If T or F is continuous, then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n\in\mathbb{N}}$ converges to x^*.*

Inspired by the results of Edelstein [6], Piri and Kumam [18], Suzuki [25], Kadelburg et.al, [4], Wardowski [27] and Wardowski and Van Dung [28], we introduce the notion of F-Suzuki-weak contractions on cone metric spaces and present some new results on complete cone metric spaces. Our results extend several known fixed point results.

2. Preliminaries

First we state some basic definitions and properties about the cone metric spaces.

Definition 2.1. [8] Let E be a real Banach space with norm $\|\cdot\|$ and P be a subset of E. Then P is called a cone if and only if

(a) P is closed, nonempty, and, $P \neq \{\theta\}$, where θ is the zero vector in E;

(b) if $a,b \in \mathbb{R}$, $a,b \ge 0$, and $x,y \in P$, then $ax+by \in P$;

(c) if $x \in P$ and $-x \in P$, then $x = \theta$.

A cone P is called trivial if $P = \{\theta\}$. A nontrivial cone P is said to be a solid cone if P° is nonempty (where P° designate the interior of P). Obviously $\theta \notin P^\circ$.

Given a cone P in a Banach space E, we define a partial ordering $\preceq$ with respect to P by

$$x \preceq y \Leftrightarrow y - x \in P.$$

We also write $x \prec y$ whenever $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in P^\circ$. The cone P is called normal if there is a number $K > 0$, such that for all $x, y \in E$, we have

$$\theta \preceq x \preceq y \Rightarrow \| x \| \leq K \| y \| .$$

The least positive number K satisfying this inequality is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. Equivalently the cone P is called regular if every decreasing sequence which is bounded from below is convergent. Regular cones are normal and there exist normal cones which are not regular. Throughout this paper we assume that E is a Banach space and P a cone in E. Although, some properties of the real cone (i.e., the cone $P = [0, \infty)$ in the Banach space $\mathbb{R}$ with usual norm) are not satisfied in non-normal cones (see, e.g., [9, 11]); but some properties of solid cones which hold good and will be used frequently in this paper are as follows:

(a) if $x \preceq y$ and $y \ll z$, then $x \ll z$;

(b) if $x \ll y$ and $y \preceq z$, then $x \ll z$;

(c) if $x \ll y$ and $y \ll z$, then $x \ll z$;

(d) if $\theta \preceq x \ll c$ for each $c \in P^\circ$, then $x = \theta$;

(e) if $x \preceq y + c$ for each $c \in P^\circ$, then $x \preceq y$;

(f) if $x \preceq kx$ for each $k \in [0, 1)$, then $x = \theta$;

(g) if $x \ll y$ and $k \in (0, \infty)$, then $kx \ll ky$.

Definition 2.2. [8] A cone metric space is an ordered pair (X, d), where X is any set and $d : X \times X \to E$ is a mapping satisfying

(d_1) $d(x, y) \in P$, that is, $\theta \preceq d(x, y)$, for all $x, y \in X$, and $d(x, y) = \theta$ if and only if $x = y$;

(d_2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d_3) $d(x,y) \preceq d(x,z)+d(z,y)$ for all $x,y,z \in X$.

Definition 2.3. [8] Let (X,d) be a cone metric space, $x \in X$ and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. Then

(i) $\{x_n\}_{n=1}^{\infty}$ converges to x if for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n,x) \ll c$ for all $n \geq N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x, (n \to \infty)$;

(ii) $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence whenever for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n,x_m) \ll c$ for all $n,m \geq N$;

(iii) (X,d) is a complete cone metric space if every Cauchy sequence in X converges in X.

Lemma 2.4. *Let (X,d) be a cone metric space, P a normal cone with normal constant K. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in X and $x,y \in X$.*

(i) $x_n \to x, (n \to \infty)$ *if and only if* $\lim_{n\to\infty} d(x_n,x) = 0$*;*

(ii) *if* $x_n \to x, (n \to \infty)$ *and* $x_n \to y, (n \to \infty)$ *then x=y;*

(iii) $\{x_n\}_{n=1}^{\infty}$ *is a cauchy sequence if and only if* $\lim_{n\to\infty} d(x_n,x_m) = 0$*;*

(iv) *if* $x_n \to x, (n \to \infty)$ *and* $y_n \to y, (n \to \infty)$ *then* $d(x_n,y_n) \to d(x,y), (n \to \infty)$.

Let $F : P-\{\theta\} \to \mathbb{R}$ be a mapping satisfying the following conditions:

$(CF1)$ for all $x,y \in P$ such that $x \ll y$, $F(x) \leq F(y)$;

$(CF2)$ for every sequence $\{v_n\}$ in P°

$$\lim_{n\to\infty} v_n = \theta \Longleftrightarrow \lim_{n\to\infty} F(v_n) = -\infty;$$

$(CF3)$ F is continuous.

We denote the set of all functions F satisfying above three properties by $\mathfrak{F}$.

Example 2.5. *Let $E = C^1_{\mathbb{R}}[0,1]$ endowed with the norm $\| f \| = \| f \|_\infty + \| f' \|_\infty$ and let $P = \{f \in V : f(t) \geq 0, \forall t \in [0,1]\}$. Then, P is a non-normal solid cone in E (see* [3]*). Define $F : P-\{\theta\} \to [0,\infty)$ such that $F(f) = \ln \| f \|_\infty$. Then $F \in \mathfrak{F}$.*

Let (X,d) be a cone metric space with a solid cone P and $F \in \mathfrak{F}$. In further discussion, we use the expressions containing the term $F(d(p))$, where $p = d(x,y)$ for some $x,y \in X$. For simplicity, it is assumed that $p \in P$ cannot procured the value θ in such expressions.

3. Main results

Definition 3.1. Let (X,d) be a cone metric space with a solid cone P. A mapping $T : X \to X$ is said to be an F-Suzuki-weak contraction if for all $x, y \in X$, $d(x,y)$, $d(x,Tx)$, $d(y,Ty)$ and $\frac{d(x,Ty)+d(y,Tx)}{2}$ are comparable and there exist a real number $\tau > 0$ and $F \in \mathfrak{F}$ such that

$$\left(\frac{1}{2}d(x,Tx) - d(x,y)\right) \notin P^\circ \implies [\tau + F(d(Tx,Ty)) \leq$$
$$F(\max\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\})]. \tag{3}$$

For the mapping $T : X \to X$ we denote the set of all fixed points of T by Fix(T), i.e., Fix(T)= $\{x \in X : Tx = x\}$.

Theorem 3.2. *Let (X,d) be a complete cone metric space, P a normal cone with normal constant K and $T : X \to X$ be an F-Suzuki-weak contraction. Then T has a unique fixed point $x^* \in X$ and for every $x \in X$ the sequence $\{T^n x\}_{n=1}^{\infty}$ converges to x^*.*

Proof. Chose $x \in X$ and set

$$x_1 = Tx,\ x_2 = Tx_1 = T^2x,\ \cdots,\ x_{n+1} = Tx_n = T^n x, \cdots. \tag{4}$$

Note that, if $Tx_n = x_n$ for any $n \in \mathbb{N}$, then $x_n \in$ Fix(T). If there exists $n \in \mathbb{N}$ such that

$$\left(\frac{1}{2}d(x_n,Tx_n) - d(x_n,Tx_n)\right) \in P^\circ,$$

i.e., $-\frac{1}{2}d(x_n,Tx_n) \in P^\circ$, then by the definition of cone metric and the properties of cone P we have $d(x_n,Tx_n) = \theta$, i.e., $Tx_n = x_n$ and so $x_n \in$ Fix(T). Therefore, in further discussion we assume that

$$Tx_n \neq x_n,\ \ \left(\frac{1}{2}d(x_n,Tx_n) - d(x_n,Tx_n)\right) \notin P^\circ \quad \forall n \in \mathbb{N}. \tag{5}$$

Then using (3), the following holds for every $n \in \mathbb{N}$:

$$\begin{aligned}
\tau + F(d(Tx_n, T^2x_n)) &\leq F(\max\{(d(x_n, Tx_n), d(x_n, Tx_n), d(Tx_n, T^2x_n), \\
&\quad \frac{d(x_n, T^2x_n) + d(Tx_n, Tx_n)}{2}\}) \\
&= F(\max\{(d(x_n, Tx_n), d(Tx_n, T^2x_n), \frac{d(x_n, T^2x_n)}{2}\}) \\
&\leq F(\max\{(d(x_n, Tx_n), d(Tx_n, T^2x_n), \frac{d(x_n, Tx_n) + d(Tx_n, T^2x_n)}{2}\}) \\
&= F(\max\{(d(x_n, Tx_n), d(Tx_n, T^2x_n)\}). \qquad (6)
\end{aligned}$$

If there exists $n \in \mathbb{N}$ such that

$$\max\{(d(x_n, Tx_n), d(Tx_n, T^2x_n)\} = d(Tx_n, T^2x_n),$$

then (6) becomes

$$\tau + F(d(Tx_n, T^2x_n)) \leq F(d(Tx_n, T^2x_n)).$$

It is a contradiction. Therefore

$$\max\{d(x_n, Tx_n), d(Tx_n, T^2x_n)\} = d(x_n, Tx_n),$$

for all $n \in \mathbb{N}$. Thus, from (6), we have

$$F(d(Tx_n, T^2x_n)) \leq F(d(x_n, Tx_n)) - \tau, \ \ \forall n \in \mathbb{N}. \qquad (7)$$

i.e.,

$$F(d(x_{n+1}, Tx_{n+1})) \leq F(d(x_n, Tx_n)) - \tau, \ \ \forall n \in \mathbb{N}.$$

Repeating this process, we get

$$\begin{aligned}
F(d(x_n, Tx_n)) &\leq F(d(x_{n-1}, Tx_{n-1})) - \tau \\
&\leq F(d(x_{n-2}, Tx_{n-2})) - 2\tau \\
&\leq F(d(x_{n-3}, Tx_{n-3})) - 3\tau \\
&\ \ \vdots \\
&\leq F(d(x_0, Tx_0)) - n\tau. \qquad (8)
\end{aligned}$$

From (8), we obtain $\lim_{n\to\infty} F(d(x_n, Tx_n)) = -\infty$, which together with $(CF2)$ gives

$$\lim_{n\to\infty} d(x_n, Tx_n) = \theta. \tag{9}$$

We now show that, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Arguing by contradiction, we assume that there exist $c \in P^\circ$, sequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that

$$p(n) > q(n) > n,\ \ c - d(x_{p(n)}, x_{q(n)}) \notin P^\circ, c - d(x_{p(n)-1}, x_{q(n)}) \in P^\circ, \forall n \in \mathbb{N}. \tag{10}$$

Regarding (10) and the triangular inequality, we find

$$\begin{aligned} d(x_{p(n)}, x_{q(n)}) &\preceq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) \\ &\ll d(x_{p(n)}, x_{p(n)-1}) + c \\ &= d(x_{p(n)-1}, Tx_{p(n)-1}) + c, \end{aligned}$$

and equivalently, we get

$$c + d(x_{p(n)-1}, Tx_{p(n)-1}) - d(x_{p(n)}, x_{q(n)}) \in P^\circ.$$

We can choose $\delta > 0$ such that

$$c + d(x_{p(n)-1}, Tx_{p(n)-1}) - d(x_{p(n)}, x_{q(n)}) - x \in P^\circ, \quad \forall x \in E, \| x \| < \delta. \tag{11}$$

From (9), there exists $N_1 \in \mathbb{N}$, such that

$$\| d(x_{p(n)-1}, Tx_{p(n)-1}) \| < \delta, \quad \forall n \geq N_1. \tag{12}$$

It follows from (11) and (12) that

$$\begin{aligned} c - d(x_{p(n)}, x_{q(n)}) = c &+ d(x_{p(n)-1}, Tx_{p(n)-1}) - d(x_{p(n)}, x_{q(n)}) \\ &- d(x_{p(n)-1}, Tx_{p(n)-1}) \in P^\circ,\ \forall n \geq N_1, \end{aligned}$$

This is a contradiction to (10). So $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. By completeness of (X, d), $\{x_n\}_{n=1}^{\infty}$ converges to some point $x^* \in X$. So from lemma 2.4, we have

$$\lim_{n\to\infty} d(x_n, x^*) = \theta. \tag{13}$$

Now, we claim that,

$$\begin{aligned} &(I) \quad \frac{1}{2}d(x_n,Tx_n)-d(x_n,x^*)\notin P^\circ \\ &\qquad\qquad or \\ &(II) \quad \frac{1}{2}d(Tx_n,T^2x_n)-d(Tx_n,x^*)\notin P^\circ, \quad \forall n\in\mathbb{N}. \end{aligned} \tag{14}$$

Arguing by contradiction, we assume that there exists $n\in\mathbb{N}$ such that

$$\frac{1}{2}d(x_n,Tx_n)-d(x_n,x^*)\in P^\circ \text{ and } \frac{1}{2}d(Tx_n,T^2x_n)-d(Tx_n,x^*)\in P^\circ,$$

or equivalently

$$d(x_n,x^*)\ll \frac{1}{2}d(x_n,Tx_n) \text{ and } d(Tx_n,x^*)\ll \frac{1}{2}d(Tx_n,T^2x_n). \tag{15}$$

Regarding (15) and the triangular inequality, we find

$$\begin{aligned} d(x_n,Tx_n) &\preceq d(x_n,x^*)+d(x^*,Tx_n) \\ &\ll \frac{1}{2}d(x_n,Tx_n)+\frac{1}{2}d(Tx_n,T^2x_n), \end{aligned}$$

which implies that

$$d(x_n,Tx_n)\ll d(Tx_n,T^2x_n).$$

Hence from $(CF1)$, we get

$$F(d(x_n,Tx_n))\leq F(d(Tx_n,T^2x_n)). \tag{16}$$

Which is a contradiction to (7). So, the proof of (14) is complete. Suppose part (I) of (14) is true and $x^*\notin \mathrm{Fix}(T)$, then from (3), we have

$$\begin{aligned} \tau+F(d(Tx_n,Tx^*)) &\leq F(\max\{(d(x_n,x^*),d(x_n,Tx_n),d(x^*,Tx^*), \\ &\qquad \frac{d(x_n,Tx^*)+d(x^*,Tx_n)}{2}\}) \\ &\leq F(\max\{d(x_n,x^*),d(x_n,Tx_n),d(x^*,Tx^*), \\ &\qquad \frac{d(x_n,x^*)+d(x^*,Tx^*)+d(x^*,x_n)+d(x_n,Tx_n)}{2}\}). \end{aligned} \tag{17}$$

Since $x_n \to x^*, (n \to \infty)$. So from Lemma 2.4, we get

$$\lim_{n\to\infty} d(Tx_n, Tx^*) = \lim_{n\to\infty} d(x_{n+1}, Tx^*) = d(x^*, Tx^*). \tag{18}$$

Letting $n \to \infty$ in the inequality (17) and using (9), (13), (18) and $(CF3)$, we obtain

$$\tau + F(d(x^*, Tx^*)) \leq F(d(x^*, Tx^*)),$$

which is a contradiction. Therefore $x^* = Tx^*$.

Now suppose part (II) of (14) is true, and $x^* \notin \mathrm{Fix}(T)$, then from (3) and (1), we have

$$\begin{aligned}
\tau + d(T^2x_n, Tx^*) &\leq F(\max\{d(Tx_n, x^*), d(Tx_n, T^2x_n), d(x^*, Tx^*), \\
&\qquad \frac{d(Tx_n, Tx^*) + d(x^*, T^2x_n)}{2}\}) \\
&\leq F(\max\{d(Tx_n, x^*), d(Tx_n, T^2x_n), d(x^*, Tx^*), \\
&\qquad \frac{d(Tx_n, x^*) + d(x^*, Tx^*) + d(x^*, T^2x_n)}{2}\}) \\
&= F(\max\{d(x_{n+1}, x^*), d(x_{n+1}, Tx_{n+1}), d(x^*, Tx^*), \\
&\qquad \frac{d(x_{n+1}, x^*) + d(x^*, Tx^*) + d(x^*, x_{n+2})}{2}\}).
\end{aligned} \tag{19}$$

Since $x_n \to x^*, (n \to \infty)$. So from Lemma 2.4, we get

$$\lim_{n\to\infty} d(T^2x_n, Tx^*) = \lim_{n\to\infty} d(x_{n+2}, Tx^*) = d(x^*, Tx^*). \tag{20}$$

Letting $n \to \infty$ in the inequality (19) and using (9), (13), (20) and $(CF3)$, we obtain

$$\tau + F(d(x^*, Tx^*)) \leq F(d(x^*, Tx^*)),$$

which is a contradiction. Therefore $x^* = Tx^*$. Hence in each case we have $\mathrm{Fix}(T) \neq \emptyset$. Now let us to show that T has at most one fixed point. Indeed, if $x^*, y^* \in X$ be two distinct fixed points of T, that is, $Tx^* = x^* \neq y^* = Ty^*$. If $-d(x^*, y^*) \in P^\circ$, we have $x^* = y^*$ and so $-d(x^*, y^*) \notin P^\circ$.

Since $\frac{1}{2}d(x^*,Tx^*) = \frac{1}{2}d(x^*,x^*) = \theta$, then $\left(\frac{1}{2}d(x^*,Tx^*) - d(x^*,y^*)\right) \notin P^\circ$. Since T is an F-Suzuki-weak contraction with a constant $\tau > 0$, we have

$$\begin{aligned}
\tau + F(d(x^*,y^*)) &= \tau + F(d(Tx^*,Ty^*)) \\
&\leq F(\max\{d(x^*,y^*), d(x^*,Tx^*), d(y^*,Ty^*), \frac{d(x^*,Ty^*)+d(y^*,Tx^*)}{2}\}) \\
&= F(\max\{d(x^*,y^*), d(x^*,x^*), d(y^*,y^*), \frac{d(x^*,y^*)+d(y^*,x^*)}{2}\}) \\
&= F(\max\{d(x^*,y^*), \theta\}) \\
&= F(d(x^*,y^*))
\end{aligned}$$

which is a contradiction. Thus, the fixed point is unique.

Theorem 3.3. *Let T be a self-mapping of a complete metric space (X, d_X). Suppose there exist $\tau > 0$ such that for all $x, y \in X$*

$$\frac{1}{2}d_X(x,Tx) < d_X(x,y) \Rightarrow \tau + F(\max\{d_X(x,y), d_X(x,Tx), d_X(y,Ty), \frac{d_X(x,Ty)+d_X(y,Tx)}{2}\}),$$

where, $F : \mathbb{R}^+ = (0,\infty) \to \mathbb{R}$ is a mapping satisfying the following conditions:

$(F1')$ *F is increasing, i.e., for all $x, y \in \mathbb{R}_+$ such that $x < y$, $F(x) \leq F(y)$;*

$(F2)$ *For each sequence $\{\alpha_n\}_{n=1}^\infty$ of positive numbers*

$$\lim_{n\to\infty} \alpha_n = 0 \Leftrightarrow \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

$(F3')$ *F is continuous on $\mathbb{R}^+$.*

Then T has a unique fixed point.

Proof. By taking $E = \mathbb{R}$, $P = [0,\infty)$, $\alpha_1(t) = \| t \|$ and $\alpha_2(t) = \alpha_3(t) = 0$ in Theorem 3.2 the proof is complete.

Remark 3.4. Theorem 3.3 gives all consequence of Theorem 2.1 of [27] and Theorem 2.4 of [28] with condition $(F3')$ instead of condition $(F3)$. Also, in the proof of Theorem 3.3, we assume that F is increasing While in condition $(F1)$ of Theorem 2.1 of [27] and Theorem 2.4

of [28] F is strictly increasing. Also, the contractive condition used in Theorem 3.3 is weaker than what used in Theorem 2.1 of [27] and Theorem 2.4 of [28].

Example 3.5. *Let* $\tau \in (0, \frac{1}{2}]$, $F(\alpha) = \frac{-1}{\alpha} + \alpha \in \mathfrak{F}$, $E = \mathbb{R}$, $P = [0, \infty)$, $X = \{0\} \cup [2,3]$ *and define a metric* d *on* X *by* $d(x,y) = \mid x-y \mid$. *Let* $T : X \to X$ *be given by*

$$Tx = \begin{cases} 0, & x \in \{0\} \cup [2,3), \\ 2.5, & x = 3. \end{cases}$$

Obviously, (X,d) *is complete cone metric space. Since* T *is not continuous,* T *is not an* F*-contraction by Remark 1.3. First observe that*

$$\forall x, y \in X, \ \ [Tx \neq Ty \Leftrightarrow (x = 0 \wedge y = 3) \ \ or \ \ (x \in [2,3) \wedge y = 3)].$$

Also

$$\frac{1}{2} d(x,Tx) < d(x,y) \Rightarrow x = 0 \wedge y = 3.$$

Let $A(x,y) = \{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\}$. *For* $x = 0$ *and* $y = 3$, *we have*

$$\mid Tx - Ty \mid = 2.5 < 3 = \mid x - y \mid \leq \max A(0,3).$$

Therefore

$$\begin{aligned} \tau - \frac{1}{d(T0,T3)} + d(T0,T3) &= \tau - \frac{1}{\mid T0 - T3 \mid} + \mid T0 - T3 \mid \\ &\leq \tau - \frac{1}{\max A(0,3)} + \mid T0 - T3 \mid \\ &\leq 0.5 - \frac{1}{\max A(0,3)} + 2.5 \\ &= -\frac{1}{\max A(0,3)} + d(0,3) \\ &= -\frac{1}{\max A(0,3)} + \max A(0,3). \end{aligned}$$

Hence

$$\tau + F(d(T0,T3)) \leq F(\max\{d(0,3), d(0,T0), d(3,T3), \frac{1}{2}[d(0,T3) + d(3,T0)]\}).$$

For $x \in [2,3)$ *and* $y = 3$, *we have*

$$\frac{1}{2} d(x,Tx) = \frac{1}{2} \mid x - Tx \mid = \frac{1}{2} x \geq 1 > \mid x - 3 \mid = d(x,3).$$

Therefore

$$\tau + F(d(Tx,T3)) \leq F(\max\{d(x,3), d(x,Tx), d(3,T3), \frac{1}{2}[d(x,T3) + d(3,Tx)]\}).$$

Hence T if F-Suzuki-weak contraction..

Conflict of Interests

The author declares that there is no conflict of interests.

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