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PROJECTION METHODS FOR RELATIVELY GENERALIZED ASYMPTOTICALLY NONEXPANSIVE MAPPINGS AND EQUILIBRIUM PROBLEMS

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Abstract. In this paper, we study common solutions of equilibrium and fixed point problems. A weak convergence theorem for common solutions is established in a uniformly smooth and uniformly convex Banach space.

Keywords: Equilibrium problem; Fixed point; Nonexpansive mapping; Relatively asymptotically nonexpansive mapping.

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1. Introduction and preliminaries

Let E be a real Banach space and let E^* be the dual space of E. Recall that the normalized duality mapping J from E to 2^{E^*} is defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth iff $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$

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exists for each $x, y \in U_E$. It is also said to be uniformly smooth iff the above limit is attained uniformly for $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. Recall that E is said to be uniformly convex iff $\lim_{n\to\infty} \|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty} \|\frac{x_n+y_n}{2}\| = 1$. It is well known that if E is uniformly smooth if and only if E^* is uniformly convex.

Recall that a Banach space E enjoys the Kadec-Klee property if for any sequence $\{x_n\} \subset E$, and $x \in E$ with $x_n \rightharpoonup x$, and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. It is well known that if E is a uniformly convex Banach spaces, then E enjoys the Kadec-Klee property [1].

Let C be a nonempty closed and convex subset of E and let F be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall that the following equilibrium problem. Find $\bar{x} \in C$ such that

$$F(\bar{x}\, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

From now on, we use $EP(F)$ to denote the solution set of equilibrium problem (1.1) and assume that F satisfies the following conditions:

(A1) $F(x, x) = 0, \forall x \in C$;

(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0, \forall x, y \in C$;

(A3)

$$\limsup_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y), \forall x, y, z \in C;$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and weakly lower semi-continuous.

Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T. Recall that T is said to be closed if for any sequence $\{x_n\} \subset C$ such that $\lim_{n\to\infty} x_n = x_0$ and $\lim_{n\to\infty} Tx_n = y_0$, then $Tx_0 = y_0$. In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \forall x, y \in E.$$

If E is a reflexive, strictly convex and smooth Banach space, then $\phi(x,y)=0$ if and only if $x=y$. Observe that, in a Hilbert space H, the equality is reduced to $\phi(x,y)=\|x-y\|^2$, $x,y\in H$. As we all know if C is a nonempty closed convex subset of a Hilbert space H and $P_C: H\to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently, it is not available in more general Banach spaces. In this connection, Alber [2] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection P_C in Hilbert spaces. Recall that the generalized projection $\Pi_C: E\to C$ is a map that assigns to an arbitrary point $x\in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x=\bar{x}$, where $\bar{x}$ is the solution to the minimization problem $\phi(\bar{x},x)=\min\limits_{y\in C}\phi(y,x)$. Existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J. In Hilbert spaces, $\Pi_C=P_C$. It is obvious from the definition of a function ϕ that $(\|x\|-\|y\|)^2\le\phi(x,y)\le(\|y\|+\|x\|)^2$, $\forall x,y\in E$,

Recall that a point p in C is said to be an asymptotic fixed point of a mapping T iff C contains a sequence $\{x_n\}$ which converges weakly to p so that $\lim_{n\to\infty}\|x_n-T^nx_n\|=0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$.

Recall that a mapping T is said to be relatively nonexpansive iff

$$\widetilde{F}(T)=F(T)\neq\emptyset,\quad \phi(p,Tx)\le\phi(p,x),\quad \forall x\in C,\forall p\in F(T).$$

Recall that a mapping T is said to be relatively asymptotically nonexpansive iff

$$\widetilde{F}(T)=F(T)\neq\emptyset,\quad \phi(p,T^nx)\le(1+\mu_n)\phi(p,x),\quad \forall x\in C,\forall p\in F(T),\forall n\ge 1,$$

where $\{\mu_n\}\subset[0,\infty)$ is a sequence such that $\mu_n\to 0$ as $n\to\infty$.

Recall that a mapping T is said to be relatively generalized asymptotically nonexpansive iff

$$\widetilde{F}(T)=F(T)\neq\emptyset,\quad \phi(p,T^nx)\le(1+\mu_n)\phi(p,x)+\nu_n,\quad \forall x\in C,\forall p\in F(T),\forall n\ge 1,$$

where $\{\mu_n\}\subset[0,\infty)$ and $\{\nu_n\}\subset[0,\infty)$ are sequences such that $\mu_n\to 0$ and $\nu_n\to 0$ as $n\to\infty$.

Remark 1.1. The class of relatively nonexpansive mappings were first considered in [3] and [4]. The class of relatively asymptotically nonexpansive mappings were first considered in [5] and [6].

Remark 1.2. The class of relatively nonexpansive mappings is reduced to the class of quasi-nonexpansive mappings, the class of relatively asymptotically nonexpansive mappings is reduced to the class of asymptotically quasi-nonexpansive mappings, and the class of relatively generalized asymptotically nonexpansive mappings is reduced to the class of generalized asymptotically quasi-nonexpansive mappings [7] in the framework of Hilbert spaces.

Recently, equilibrium problems and fixed point problems of relatively nonexpansive mappings have been extensively investigated based on iterative methods; fore more details, see [5-26] and the references therein. In this article, we investigate an equilibrium problem and a fixed point problem of a relatively generalized asymptotically nonexpansive mapping. A weak convergence theorem is established in a uniformly smooth and uniformly convex Banach space.

Lemma 1.3. [8,27] *Let C be a closed convex subset of a uniformly smooth and uniformly convex Banach space E. Let F be a bifunction from $C\times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $r>0$ and $x\in E$. Then There exists $z\in C$ such that $F(z,y)+\frac{1}{r}\langle y-z, Jz-Jx\rangle\geq 0$, $\forall y\in C$. Define a mapping $S_r: E\to C$ by $S_rx=\{z\in C: F(z,y)+\frac{1}{r}\langle y-z, Jz-Jx\rangle, \forall y\in C\}$. Then the following conclusions hold:*

(a) *S_r is single-valued;*

(b) *S_r is a firmly nonexpansive-type mapping, i.e., for all $x,y\in E$,*

$$\langle S_rx-S_ry, JS_rx-JS_ry\rangle\leq\langle S_rx-S_ry, Jx-Jy\rangle$$

(c) *$F(S_r)=EP(F)$ is closed and convex;*

(d) *S_r is relatively nonexpansive;*

(e) *$\phi(q,S_rx)+\phi(S_rx,x)\leq\phi(q,x), \quad \forall q\in F(S_r)$.*

Lemma 1.4. [2] *Let E be a reflexive, strictly convex, and smooth Banach space, C a nonempty, closed, and convex subset of E, and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 1.5. [2] *Let C be a nonempty, closed, and convex subset of a smooth Banach space E, and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 1.6. [28] *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and*

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|)$$

for all $x, y \in B_r = \{x \in E : \|x\| \leq r\}$ and $t \in [0, 1]$.

Lemma 1.7. [29] *Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be three nonnegative sequences satisfying the following condition:*

$$a_{n+1} \leq (1 + b_n)a_n + c_n, \quad \forall n \geq n_0,$$

where n_0 is some nonnegative integer. If $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, then the limit of the sequence $\{a_n\}$ exists. If, in addition, there exists a subsequence $\{\alpha_{n_i}\} \subset \{\alpha_n\}$ such that $\alpha_{n_i} \to 0$, then $\alpha_n \to 0$ as $n \to \infty$.

Lemma 1.8. [30] *Let E be a smooth and uniformly convex Banach space and let $r > 0$. Then there exists a strictly increasing, continuous, and convex function $g : [0, 2r] \to R$ such that $g(0) = 0$ and $g(\|x - y\|) \leq \phi(x, y)$ for all $x, y \in B_r$.*

Lemma 1.9. *Let E be a uniformly convex and uniformly smooth Banach space. Let C be a nonempty closed and convex subset of E. Let $T : C \to C$ be a relatively generalized asymptotically nonexpansive mapping. Then $F(T)$ is a closed convex subset of C.*

Proof. Since T is continuous, we obtain the closedness of $F(T)$. Next, we show that $F(T)$ is convex. For x, $y \in F(T)$ and $t \in (0,1)$, put $p = tx + (1-t)y$. It is sufficient to show $Tp = p$. In fact, we have

$$\begin{aligned}\phi(p, T^n x) &= \|p\|^2 - 2\langle p, JT^n x\rangle + \|T^n x\|^2 \\ &= \|p\|^2 - 2t\langle x, JT^n x\rangle - 2(1-t)\langle y, JT^n x\rangle + \|T^n x\|^2 \\ &= \|p\|^2 + t\phi(x, T^n p) + (1-t)\phi(y, T^n p) - t\|x\|^2 - (1-t)\|y\|^2 \\ &\leq \mu_n(t\|x\|^2 + (1-t)\|y\|^2 - \|p\|^2) + \nu_n.\end{aligned}$$

Taking the limit as $n \to \infty$ yields that $\lim_{n\to\infty} \phi(p, T^n x) = 0$. Since T is continuous, we find that $p \in F(T)$. This completes the proof.

2. Main results

Theorem 2.1. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty closed and convex subset of E. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $T : C \to C$ be a relatively generalized asymptotically nonexpansive mapping with the sequences $\{\mu_{n,1}\}$ and $\{\nu_{n,1}\}$. Let $S : C \to C$ be a relatively generalized asymptotically nonexpansive mapping with the sequences $\{\mu_{n,2}\}$ and $\{\nu_{n,2}\}$. Assume that $\Phi := F(T) \cap F(S) \cap EP(F)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner: $y_0 \in E$ chosen arbitrarily and*

$$\begin{cases} x_n \in C \text{ such that } F(x_n, x) + \frac{1}{r_n}\langle x - x_n, Jx_n - Jy_n\rangle \geq 0, \quad \forall x \in C, \\ y_{n+1} = J^{-1}\big(\alpha_n Jx_n + \beta_n JT^n x_n + \gamma_n JS^n x_n\big), \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are real sequences in $[0,1]$ and $\{r_n\}$ is a real number sequence in $[r, \infty)$, where $r > 0$ is some real number. Assume that J is weakly sequentially continuous and the following restrictions hold: (a) $\alpha_n + \beta_n + \gamma_n = 1$. (b) $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \nu_n < \infty$, where $\mu_n = \max\{\mu_{n,1}, \mu_{n,2}\}$ and $\nu_n = \max\{\nu_{n,1}, \nu_{n,2}\}$. (c) $\liminf_{n\to\infty} \alpha_n\beta_n > 0$, $\liminf_{n\to\infty} \alpha_n\gamma_n > 0$. Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \Phi$, where $\bar{x} = \lim_{n\to\infty} \Pi_\Phi x_n$.

Proof. Fixing $p \in \Phi$, we find that

$$\begin{aligned}
\phi(p, x_{n+1}) &\leq \phi(p, y_{n+1}) \\
&= \|p\|^2 - 2\langle p, \alpha_n Jx_n + \beta_n JT^n x_n + \gamma_n JS^n x_n \rangle \\
&\quad + \|\alpha_n Jx_n + \beta_n JT^n x_n + \gamma_n JS^n x_n\|^2 \\
&\leq \|p\|^2 - 2\alpha_n \langle p, Jx_n\rangle - 2\beta_n \langle p, JT^n x_n\rangle - 2\gamma_n \langle p, JS^n x_n\rangle \\
&\quad + \alpha_n \|x_n\|^2 + \beta_n \|T^n x_n\|^2 + \gamma_n \|S^n x_n\|^2 \\
&= \alpha_n \phi(p, x_n) + \beta_n \phi(p, T^n x_n) + \gamma_n \phi(p, S^n x_n) \\
&\leq \phi(p, x_n) + \beta_n \mu_n \phi(p, x_n) + \gamma_n \mu_n \phi(p, x_n) + \nu_n \\
&\leq (1 + \mu_n)\phi(p, x_n) + \nu_n.
\end{aligned} \tag{2.1}$$

Using Lemma 1.7 yields that $\lim_{n\to\infty} \phi(p, x_n)$ exits. This in turn implies that the sequence $\{x_n\}$ is bounded. Using Lemma 1.6, we find that

$$\begin{aligned}
\phi(p, x_{n+1}) &= \phi(p, S_{r_{n+1}} y_{n+1}) \\
&\leq \|p\|^2 - 2\langle p, \alpha_n Jx_n + \beta_n JT^n x_n + \gamma_n JS^n x_n \rangle \\
&\quad + \|\alpha_n Jx_n + \beta_n JT^n x_n + \gamma_n JS^n x_n\|^2 \\
&\leq \|p\|^2 - 2\alpha_n \langle p, Jx_n\rangle - 2\beta_n \langle p, JT^n x_n\rangle - 2\gamma_n \langle p, JS^n x_n\rangle \\
&\quad + \alpha_n \|x_n\|^2 + \beta_n \|T^n x_n\|^2 + \gamma_n \|S^n x_n\|^2 - \alpha_n \beta_n g(\|JT^n x_n - Jx_n\|) \\
&\leq \phi(p, x_n) + \beta_n \mu_n \phi(p, x_n) + \gamma_n \mu_n \phi(p, x_n) - \alpha_n \beta_n g(\|JT^n x_n - Jx_n\|) + \nu_n \\
&\leq (1 + \mu_n)\phi(p, x_n) - \alpha_n \beta_n g(\|JT^n x_n - Jx_n\|) + \nu_n.
\end{aligned}$$

Hence, we have

$$\alpha_n \beta_n g(\|JT^n x_n - Jx_n\|) \leq (1 + \mu_n)\phi(p, x_n) - \phi(p, x_{n+1}) + \nu_n.$$

It follows that $\lim_{n\to\infty} g(\|JT^n x_n - Jx_n\|) = 0$. That is, $\lim_{n\to\infty} \|JT^n x_n - Jx_n\| = 0$. Since J^{-1} is uniformly norm-to-norm continuous on bounded sets, we find that $\lim_{n\to\infty} \|T^n x_n - x_n\| = 0$. In the same way, we find that $\lim_{n\to\infty} \|S^n x_n - x_n\| = 0$. Since $\{x_n\}$ is bounded, we see that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $p \in C$. It follows that $p \in F(T) \cap F(S)$.

Next, we prove $p \in EP(F)$. Let $r = \sup_{n\geq 1}\{\|x_n\|, \|y_n\|\}$. Using Lemma 1.8 yields that there exists a continuous, strictly increasing and convex function h with $h(0) = 0$ such that $h(x, y) \leq \phi(x, y)$, $\forall x, y \in B_r$. It follows from (2.1) that

$$\begin{aligned} h(\|x_n - y_n\|) &\leq \phi(p, y_n) - \phi(p, x_n) \\ &\leq \phi(p, x_{n-1}) - \phi(p, x_n) + \mu_{n-1}\phi(p, x_{n-1}) + \nu_n. \end{aligned}$$

This implies that $\lim_{n\to\infty} h(\|x_n - y_n\|) = 0$. In view of the property of h, we get $\lim_{n\to\infty} \|x_n - y_n\| = 0$. Since J is uniformly norm-to-norm continuous on bounded sets, one has $\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0$. Since $\{r_n\}$ is a real number sequence in $[r, \infty)$, where $r > 0$ is some real number, one finds that $\lim_{n\to\infty} \frac{\|Jx_n - Jy_n\|}{r_n} = 0$. Notice that $x_n = S_{r_n} y_n$, one sees that

$$F(x_n, x) + \frac{1}{r_n}\langle x - x_n, Jx_n - Jy_n\rangle \geq 0, \quad \forall x \in C,$$

By replacing n by n_i, one finds from (A2) that

$$\begin{aligned} \|x - x_{n_i}\| \frac{\|Jx_{n_i} - Jy_{n_i}\|}{r_{n_i}} &\geq \frac{1}{r_{n_i}}\langle x - x_{n_i}, Jx_{n_i} - Jy_{n_i}\rangle \\ &\geq F(x, x_{n_i}) \end{aligned}$$

Letting $i \to \infty$ in the above inequality, one obtains from (A4) that $F(x, p) \leq 0$, $\forall x \in C$. For $0 < t < 1$ and $y \in C$, define $x_t = tx + (1 - t)p$. It follows that $x_t \in C$, which yields that $F(x_t, p) \leq 0$. It follows from the $(A1)$ and $(A4)$ that $0 = F(x_t, x_t) \leq tF(x_t, x) + (1 - t)F(x_x, p) \leq tF(x_t, x)$. That is, $F(x_t, x) \geq 0$. Letting $t \downarrow 0$, we obtain from $(A3)$ that $F(p, x) \geq 0$, $\forall x \in C$. This implies that $p \in EP(F)$. This completes the proof that $p \in F(T) \cap F(S) \cap EP(F)$. Define $z_n = \Pi_{F(T)\cap F(S)\cap EP(F)} x_n$. It follows from (2.1) that

$$\phi(z_n, x_{n+1}) \leq (1 + \mu_n)\phi(z_n, x_n) + \nu_n. \tag{2.2}$$

Using Lemma 1.4, we find that

$$\begin{aligned} \phi(z_{n+1}, x_{n+1}) &= \phi(\Pi_{F(T)\cap F(S)\cap EP(F)} x_{n+1}, x_{n+1}) \\ &\leq \phi(z_n, x_{n+1}) - \phi(z_n, \Pi_{F(T)\cap F(S)\cap EP(F)} x_{n+1}) \\ &\leq \phi(z_n, x_{n+1}) - \phi(z_n, z_{n+1}) \\ &\leq \phi(z_n, x_{n+1}). \end{aligned}$$

It follows from (2.2) that $\phi(z_{n+1}, x_{n+1}) \leq (1 + \mu_n)\phi(z_n, x_n) + \nu_n$. Hence, the sequence $\{\phi(z_n, x_n)\}$ is a convergence sequence. It follows from (2.1) that

$$\phi(p, x_{n+m}) \leq \phi(p, x_n) + L(\sum_{i=1}^{m} \mu_{n+m-i}) + \sum_{i=1}^{m}(1 + \mu_{n+m-i})\nu_{n+m-i-1} + \nu_{n+m-1}, \qquad (2.3)$$

where $L = \sup_{n\geq 1} \phi(p, x_n)$. Since $z_n \in F(T) \cap F(S) \cap EP(F)$, we find that

$$\phi(z_n, x_{n+m}) \leq \phi(z_n, x_n) + M(\sum_{i=1}^{m} \mu_{n+m-i}) + \sum_{i=1}^{m}(1 + \mu_{n+m-i})\nu_{n+m-i-1} + \nu_{n+m-1},$$

where $M = \sup_{n\geq 1} \phi(z_n, x_n)$. Since $z_{n+m} = \Pi_{F(T)\cap F(S)\cap EP(F)} x_{n+m}$, we find that

$$\begin{aligned}
&\phi(z_n, z_{n+m}) + \phi(z_{n+m}, x_{n+m}) \\
&\leq \phi(z_n, x_{n+m}) \\
&\leq \phi(z_n, x_n) + M(\sum_{i=1}^{m} \mu_{n+m-i}) + \sum_{i=1}^{m}(1 + \mu_{n+m-i})\nu_{n+m-i-1} + \nu_{n+m-1}.
\end{aligned}$$

Hence, we have

$$\phi(z_n, z_{n+m}) \leq \phi(z_n, x_n) - \phi(z_{n+m}, x_{n+m}) + M(\sum_{i=1}^{m} \mu_{n+m-i}) + \sum_{i=1}^{m}(1 + \mu_{n+m-i})\nu_{n+m-i-1} + \nu_{n+m-1}.$$

Using Lemma 1.9, we find that there exists a continuous, strictly increasing, and convex function g with

$$\begin{aligned}
&g(\|z_n - z_m\|) \\
&\leq \phi(z_n, z_m) \\
&\leq \phi(z_n, x_n) - \phi(z_{n+m}, x_{n+m}) + L(\sum_{i=1}^{m} \mu_{n+m-i}) + \sum_{i=1}^{m}(1 + \mu_{n+m-i})\nu_{n+m-i-1} + \nu_{n+m-1}.
\end{aligned}$$

This shows that $\{z_n\}$ is a Cauchy sequence. Since $F(T) \cap F(S) \cap EP(F)$ is closed, one sees that $\{z_n\}$ converges strongly to $z \in F(T) \cap F(S) \cap EP(F)$. Since $p \in F(T) \cap F(S) \cap EP(F)$, we find from Lemma 1.5 that $\langle z_{n_k} - p, Jx_{n_k} - Jz_{n_k}\rangle \geq 0$. Notice that J is weakly sequentially continuous. Letting $k \to \infty$, we find that $\langle z - p, Jp - Jz\rangle \geq 0$. It follows from the monotonicity of J, we find that $\langle z - p, Jp - Jz\rangle \leq 0$. Since the space is uniformly convex, we find that $z = p$. This completes the proof.

Remark 2.2. Theorem 2.1 mainly improves the corresponding results in [9] and [24].

If $T = S$, then Theorem 2.1 is reduced to the following.

Corollary 2.3. *Let E be a uniformly smooth and uniformly convex Banach space and let C be a nonempty closed and convex subset of E. Let F be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)-(A4). Let $T : C \to C$ be a relatively generalized asymptotically nonexpansive mapping with the sequence $\{\mu_n\}$ and ν_n. Assume that $\Phi := F(T) \cap EP(F)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner:*

$$\begin{cases} y_0 \in E \quad \textit{chosen arbitrarily}, \\ x_n \in C \textit{ such that } F(x_n, x) + \frac{1}{r_n}\langle x - x_n, Jx_n - Jy_n\rangle \geq 0, \quad \forall x \in C, \\ y_{n+1} = J^{-1}\big(\alpha_n Jx_n + (1-\alpha_n)JT^n x_n\big), \quad \forall n \geq 0, \end{cases}$$

where $\{\alpha_n\}$ is a real sequence in $[0,1]$ and $\{r_n\}$ is a real number sequence in $[r, \infty)$, where $r > 0$ is some real number. Assume that J is weakly sequentially continuous and the following restrictions hold: (a) $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \nu_n < \infty$. (b) $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Then the sequence $\{x_n\}$ converges weakly to $\bar{x} \in \Phi$, where $\bar{x} = \lim_{n\to\infty} \Pi_\Phi x_n$.

Competing interests

The author declares that they have no competing interests.

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