Available online at http://jfpt.scik.org
J. Fixed Point Theory, 2015, 2015:3
ISSN: 2052-5338

HYBRID IMPLICIT ITERATION METHODS FOR A FINITE OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. In this paper, we propose a hybrid implicit iteration method for a finite of asymptotically nonexpansive mappings in Banach spaces. Under Opial's condition, semicompact, condition $(\overline{C})$ and $\liminf_{n\to\infty} d(x_n, \mathfrak{F}(T)) = 0$, we prove some strong and weak convergence theorems for this family of mappings using proposed iteration method. The results presented in this paper extend and improve the corresponding results of Xu (2001), Zeng (2006) and Jiang (2014).

Keywords: Asymptotically nonexpansive mapping; Common fixed point; Opial's condition; Demiclosed principle; Semicompactness.

2010 AMS Subject Classification: 47H09, 47H10.

1. Introduction

Let E be a real Banach space and let K be a nonempty convex subset of E. A mapping $T : K \to K$ is said to be *nonexpansive* if $\|Tx - Ty\| \le \|x - y\|$ for all $x, y \in K$. A mapping $A : K \to K$ is said to be *L-Lipschitzian* if there exists a constant $L > 0$ such that $\|Ax - Ay\| \le L\|x - y\|$ for

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Received March 18, 2015

all $x,\ y\in K,\ n\geq 1$. A mapping $T:K\rightarrow K$ is said to be *asymptotically nonexpansive* if there exists a sequence $\{k_n\}\subset[1,\infty)$ with $\lim_{n\rightarrow\infty}k_n=1$ such that $\|T^nx-T^ny\|\leq k_n\|x-y\|$ for all $x,\ y\in K,\ n\geq 1$.

Let K be a nonempty convex subset of E and let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-maps of K. In 2001, Xu and Ori [1] introduced the following implicit iteration method. For any $x_0\in K$ and $\{\alpha_n\}_{n=1}^\infty\subset(0,1)$, the sequence $\{x_n\}_{n=1}^\infty$ is generates as follows:

$$\begin{aligned}
x_1 &= \alpha_1x_0+(1-\alpha_1)\,T_1x_1,\\
x_2 &= \alpha_2x_1+(1-\alpha_2)\,T_2x_2,\\
&\vdots\\
x_N &= \alpha_Nx_{N-1}+(1-\alpha_N)\,T_Nx_N\\
x_{N+1} &= \alpha_{N+1}x_N+(1-\alpha_{N+1})\,T_1x_{N+1}\\
&\vdots
\end{aligned}$$

The scheme is expressed in compact form as

$$x_n=\alpha_nx_{n-1}+(1-\alpha_n)T_nx_n,\quad \forall n\geq 1, \tag{1.1}$$

where $T_n=T_{n(mod\ N)}$ (here the mod N takes values in $\{1,2,\cdots,N\}$), they proved weak convergence theorem in Hilbert spaces.

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self-maps of a real Hilbert space H and $G:H\rightarrow H$. Suppose that there exists some constants $\kappa,\eta>0$ such that the mapping G is $\kappa-$*Lipschitzian* and $\eta-$*strongly monotone*. Let $\{\alpha_n\}_{n=1}^\infty\subset(0,1)$, $\{\lambda_n\}_{n=1}^\infty\subset[0,1)$ and take a fixed number $\mu\in\left(0,\frac{2\eta}{\kappa^2}\right)$. Zeng and Yao [2] introduced the following implicit hybrid iteration method. For an arbitrary initial point $x_0 \in H$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated as follows:

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1)\left[T_1 x_1 - \lambda_1 \mu G(T_1 x_1)\right], \\
x_2 &= \alpha_2 x_1 + (1-\alpha_2)\left[T_2 x_2 - \lambda_2 \mu G(T_2 x_2)\right], \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N)\left[T_N x_N - \lambda_N \mu G(T_N x_N)\right] \\
&\vdots
\end{aligned}$$

This scheme can be expressed in a concise form as follows

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)[T_n x_n - \lambda_n \mu G(T_n x_n)], \quad \forall n \geq 1, \tag{1.2.}$$

where $T_n = T_{n(mod\ N)}$. By using the iteration scheme (1.2), they obtained the weak and strong convergence theorems in Hilbert space. It is clear that if $\lambda_n = 0$, for all $n \geq 1$, then the implicit iteration scheme (1.2) reduces to the implicit iteration process (1.1).

Recently, Jiang *et al.* [3] extended the results of Zeng and Yao from Hilbert spaces to Banach spaces and proved weak and strong convergence theorems without the strong monotonicity condition.

In this paper, motivated and inspired by above results, an hybrid implicit iteration method for a finite of asymptotically nonexpansive mappings is introduced. Weak and strong convergence theorems are obtained. The results presented in this paper improve and extend the corresponding results of [1], [2] and [3].

The hybrid implicit iteration method.

Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E. Let $\{T_i\}_{i=1}^{N} : K \to K$ be N asymptotically nonexpansive mappings with $\mathfrak{F} = \cap_{n=1}^{N} F(T_i) \neq \emptyset$ and $A : K \to K$ be an *L-Lipschitzian mapping*. Assume that $\{\alpha_n\}$ is a real sequences in $(0,1)$,

$\{\lambda_n\} \subset [0,1)$, μ is positive fixed constant. Then defined a sequence $\{x_n\}$ by

$$\begin{aligned}
x_1 &= \alpha_1 x_0 + (1-\alpha_1)\left[T_1 x_1 - \lambda_1 \mu A\left(T_1 x_1\right)\right], \\
x_2 &= \alpha_2 x_1 + (1-\alpha_2)\left[T_2 x_2 - \lambda_2 \mu A\left(T_2 x_2\right)\right], \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + (1-\alpha_N)\left[T_N x_N - \lambda_N \mu A\left(T_N x_N\right)\right] \\
x_{N+1} &= \alpha_{N+1} x_N + (1-\alpha_{N+1})\left[T_1^2 x_{N+1} - \lambda_{N+1} \mu A\left(T_1^2 x_{N+1}\right)\right] \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + (1-\alpha_{2N})\left[T_N^2 x_{2N} - \lambda_{2N} \mu A\left(T_N^2 x_{2N}\right)\right] \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + (1-\alpha_{2N+1})\left[T_1^3 x_{2N+1} - \lambda_{2N+1} \mu A\left(T_1^3 x_{2N+1}\right)\right] \\
&\vdots \qquad (1.3)
\end{aligned}$$

which is called hybrid implicit iteration for a finite family of asymptotically nonexpansive mappings $\{T_1, T_2, \cdots, T_N\}$.

For each $n \geq 1$, it can be written as $n = (k-1)N + i$, where $i = i(n) \in \{1,2,\cdots,N\}$, $k = k(n) \geq 1$ is a positive integer and $k(n) \to \infty$ as $n \to \infty$. Therefore, we can (1.3) in the following compact form:

$$\begin{aligned}
x_n &= \alpha_n x_{n-1} + (1-\alpha_n) T^{\lambda_n} x_n \\
&= \alpha_n x_{n-1} + (1-\alpha_n)\left[T_{i(n)}^{k(n)} x_n - \lambda_n \mu A\left(T_{i(n)}^{k(n)} x_n\right)\right], \quad n \geq 1. \qquad (1.4)
\end{aligned}$$

The main purpose of this paper is to study the convergence of an iterative sequence $\{x_n\}$ defined by (1.4) to a common fixed point for a finite family of asymptotically nonexpansive mappings under Opial's condition , semicompact, condition $\left(\overline{C}\right)$ and $\liminf_{n\to\infty} d\left(x_n, \mathfrak{F}(T)\right) = 0$, respectively.

2. Preliminaries

For the sake of convenience, we first recall some definitions and conclusions.

A Banach space E is said to satisfy the *Opial's condition* [4] if, for all sequences $\{x_n\}$ in E such that $\{x_n\}$ converges weakly to some $x \in E$, the inequality $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ holds for all $y \neq x$ in E.

Definition 1.1. Let D be a closed subset of E and $T : D \to D$ be a mapping.

(i) T is said to be demiclosed at the origin if for each sequence $\{x_n\}$ in D, the conditions $x_n \to x_0$ weakly and $Tx_n \to 0$ strongly imply $Tx_0 = 0$.

(ii) T is said to be demicompact if any bounded sequence $\{x_n\}$ in D such that $\{x_n - Tx_n\}$ converges, there exists a subsequence say $\left\{x_{n_j}\right\}$ of $\{x_n\}$ that strongly to some x^* in D.

(iii) T is said to be semicompact if for any bounded sequence $\{x_n\}$ in D such that $\{x_n - Tx_n\} \to 0$ as $n \to \infty$, there exists a subsequence say $\left\{x_{n_j}\right\} \subset \{x_n\}$ such that $x_{n_j} \to x^*$ in D.

Senter and Datson [5] established a relation between condition (A) and demicompactness. They showed that the condition (A) is weaker than demicompactness for a nonexpansive mapping T defined on a bounded set. A mapping $T : K \to K$ with $F(T) = \{x \in K : Tx = x\} \neq \emptyset$ is said to satisfy condition (A) [5] if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x_n, F(T)))$ for all $x \in K$, where $d(x_n, F(T)) = \inf\{\|x - q\| : q \in F(T)\}$.

A family $\{T_i\}_{i=1}^N$ of N of N self mappings of K with $\mathfrak{F} = \cap_{n=1}^N F(T_i) \neq \emptyset$ is said to satisfy condition $(\overline{C})$ on K [6] if there exists a nondecreasing function $f : [0,1] \to [0,1]$ with $f(0) = 0$ and $f(t) > 0$ for all $t \in (0,\infty)$ and all $x \in K$ such that $\|x - T_l x\| \geq f(d(x_n, F(T)))$ for at least one T_l, $l = 1, 2, \cdots, N$; or in other words, at least one of the T_i's satisfies condition (A).

Lemma 2.1. [7] *Let $\{a_n\}$, $\{b_n\}$ and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n) a_n + b_n, \ \ \forall n \geq 1,$$

if $\sum_{n=1}^{\infty} b_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If in addition $\{a_n\}$ has a subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.2. [8] *Let E be a uniformly convex Banach space and let a, b be two constants with $0 < a < b < 1$. Suppose that $\{t_n\} \subset [a,b]$ is a real sequence and $\{x_n\}$, $\{y_n\}$ are two sequences* *in E. Then the conditions*

$$\lim_{n\to\infty}\|t_nx_n+(1-t_n)y_n\|=d,\ \limsup_{n\to\infty}\|x_n\|\leq d,\ \limsup_{n\to\infty}\|y_n\|\leq d$$

imply that $\lim_{n\to\infty}\|x_n-y_n\|=0$*, where* $d\geq 0$ *is a constant.*

Lemma 2.3. [9] *Let E be a real uniformly convex Banach space, K a nonempty closed convex subset of E, and* $T:K\to K$ *be an asymptotically nonexpansive mapping. Then* $I-T$ *is demiclosed at zero, i.e. for each sequence* $\{x_n\}$ *in K, if* $\{x_n\}$ *converges weakly to* $q\in K$ *and* $\{(I-T)x_n\}$ *converges strongly to* 0*, then* $(I-T)q=0$.

3. Main results

Theorem 3.1. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E. Let* $\{T_i\}_{i=1}^N$ *be N asymptotically nonexpansive self mappings of K with sequences* $\left\{h_n^{(i)}\right\}$, $1\leq i\leq N$ *and* $\mathfrak{F}=\cap_{n=1}^N F(T_i)\neq\emptyset$*;* $A:K\to K$ *is an L-Lischitzian mapping. Let the hybrid implicit iteration* $\{x_n\}$ *be the sequence defined by (1.4), where* $\{\alpha_n\}$ *is real sequences in* $(0,1)$ *and* $\{\lambda_n\}\subset[0,1)$ *satisfy the following conditions:*

(i) there exists constants $\tau_1,\tau_2\in(0,1)$ *such that* $\tau_1\leq(1-\alpha_n)\leq\tau_2, \forall n\geq 1$;

(ii) $\sum_{n=1}^{\infty}\lambda_n<\infty$;

(iii) $\sum_{n=1}^{\infty}(h_n-1)<\infty$*, where* $h_n=\max\left\{h_n^{(1)},\ h_n^{(2)},\ \cdots,h_n^{(N)}\right\}$.

Then

(1) $\lim_{n\to\infty}\|x_n-p\|=0$ *exists,* $\forall p\in\mathfrak{F}$;

(2) $\lim_{n\to\infty}\|T_lx_n-x_n\|=0, \forall l=1,2,\cdots,N,$

(3) $\{x_n\}$ *converges strongly to a common fixed point of* $\{T_1,T_2,\cdots,T_N\}$ *if and only if* $\liminf_{n\to\infty} d(x_n,F)=0$.

Proof. (1) Since $\mathfrak{F} = \cap_{n=1}^{N} F(T_i) \neq \emptyset$, for each $p \in \mathfrak{F}$, we have

$$\begin{aligned}
& \|x_n - p\| \\
= \; & \left\|\alpha_n x_{n-1} + (1-\alpha_n)\left[T_{i(n)}^{k(n)} x_n - \lambda_n \mu A\left(T_{i(n)}^{k(n)} x_n\right)\right] - p\right\| \\
\leq \; & \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)\left\|T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} p\right\| \\
& + (1-\alpha_n)\lambda_n \mu \left\|A\left(T_{i(n)}^{k(n)} x_n\right)\right\| \\
\leq \; & \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)\left\|T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} p\right\| \\
& + (1-\alpha_n)\lambda_n \mu \left\|A\left(T_{i(n)}^{k(n)} x_n\right) - A(p)\right\| + (1-\alpha_n)\lambda_n \mu \|A(p)\| \\
\leq \; & \alpha_n \|x_{n-1} - p\| + (1-\alpha_n) h_{k(n)} \|x_n - p\| \\
& + (1-\alpha_n)\lambda_n \mu h_{k(n)} L \|x_n - p\| + (1-\alpha_n)\lambda_n \mu \|A(p)\|. \qquad (3.1)
\end{aligned}$$

Since $h_{k(n)} \to 1$ $(n \to \infty)$, we know that $\{h_{k(n)}\}$ is bounded, and there exists $R_1 \geq 1$ such that $h_{k(n)} \leq R_1$. Let $d_n = h_{k(n)} - 1$, $\forall n \geq 1$, by condition (iii) we have $\sum_{n=1}^{\infty} d_n < \infty$. Hence, we have

$$\begin{aligned}
& \|x_n - p\| \\
\leq \; & \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)(1+d_n)\|x_n - p\| \\
& + (1-\alpha_n)\lambda_n \mu R_1 L \|x_n - p\| + (1-\alpha_n)\lambda_n \mu \|A(p)\| \\
\leq \; & \alpha_n \|x_{n-1} - p\| + (1-\alpha_n + d_n)\|x_n - p\| \\
& + \lambda_n \mu R_1 L \|x_n - p\| + \lambda_n \mu \|A(p)\|. \qquad (3.2)
\end{aligned}$$

Simplifying we have

$$\begin{aligned}
& \|x_n - p\| \\
\leq \; & \|x_{n-1} - p\| + \frac{d_n}{\alpha_n}\|x_n - p\| + \frac{\lambda_n \mu R_1 L}{\alpha_n}\|x_n - p\| + \frac{\lambda_n \mu}{\alpha_n}\|A(p)\| \\
= \; & \|x_{n-1} - p\| + \frac{d_n + \lambda_n \mu R_1 L}{\alpha_n}\|x_n - p\| + \frac{\lambda_n \mu}{\alpha_n}\|A(p)\|. \qquad (3.3)
\end{aligned}$$

By condition (i) we have $1-\tau_2 \leq \alpha_n$. Therefore, from (3.3) we get

$$\begin{aligned}
& \|x_n - p\| \\
\leq\ & \|x_{n-1} - p\| + \frac{d_n + \lambda_n \mu R_1 L}{1-\tau_2} \|x_n - p\| + \frac{\lambda_n \mu}{1-\tau_2} \|A(p)\| \\
\leq\ & \frac{1-\tau_2}{1-\tau_2 - (d_n + \lambda_n \mu R_1 L)} \|x_{n-1} - p\| \\
& + \frac{\lambda_n \mu}{1-\tau_2 - (d_n + \lambda_n \mu R_1 L)} \|A(p)\| \\
=\ & \left(1 + \frac{d_n + \lambda_n \mu R_1 L}{1-\tau_2 - (d_n + \lambda_n \mu R_1 L)}\right) \|x_{n-1} - p\| \\
& + \frac{\lambda_n \mu}{1-\tau_2 - (d_n + \lambda_n \mu R_1 L)} \|A(p)\|. \qquad (3.4)
\end{aligned}$$

Since condition (ii) and $\sum_{n=1}^{\infty} d_n < \infty$, we know that $d_n + \lambda_n \mu R_1 L \to 0$ as $n \to \infty$; therefore there exists a positive integer n_0 such that $d_n + \lambda_n \mu R_1 L \leq \frac{1-\tau_2}{2}$, for all $n \geq n_0$. It follows from (3.4) that

$$\begin{aligned}
\|x_n - p\| &\leq \left(1 + \frac{2(d_n + \lambda_n \mu R_1 L)}{1-\tau_2}\right) \|x_{n-1} - p\| + \frac{2\lambda_n \mu}{1-\tau_2} \|A(p)\| \\
&= (1+\delta_n) \|x_{n-1} - p\| + b_n, \qquad (3.5)
\end{aligned}$$

where $\delta_n = \frac{2(d_n + \lambda_n \mu R_1 L)}{1-\tau_2}$ and $b_n = \frac{2\lambda_n \mu}{1-\tau_2} \|A(p)\|$. By using condition (ii) and $\sum_{n=1}^{\infty} d_n < \infty$, it is easy to see that

$$\sum_{n=1}^{\infty} \delta_n < \infty; \quad \sum_{n=1}^{\infty} b_n < \infty.$$

It follows from Lemma 2.1 that $\lim_{n\to\infty} \|x_n - p\|$ exists.

(2) Since $\{\|x_n - p\|\}$ is bounded, there exists $R_2 > 0$ such that

$$\|x_n - p\| \leq R_2, \quad \forall n \geq 1. \qquad (3.6)$$

We can assume that

$$\lim_{n\to\infty} \|x_n - p\| = r, \qquad (3.7)$$

where $r \geq 0$ is some number. Since $\{\|x_n - p\|\}$ is a convergent sequence, $\{x_n\}$ is bounded sequence in K. Let

$$\|x_n - p\| = \left\|\alpha_n (x_{n-1} - p) + (1-\alpha_n)\left(T^{\lambda_n} x_n - p\right)\right\|. \tag{3.8}$$

By condition (ii), $h_{k(n)} \leq R_1$, $\sum_{n=1}^{\infty} d_n < \infty$ and (3.6), (3.7), we have that

$$\begin{aligned}
& \limsup_{n\to\infty} \left\|T^{\lambda_n} x_n - p\right\| \\
= \ & \limsup_{n\to\infty} \left\|T_{i(n)}^{k(n)} x_n - \lambda_n \mu A\left(T_{i(n)}^{k(n)} x_n\right) - p\right\| \\
\leq \ & \limsup_{n\to\infty} \left\|T_{i(n)}^{k(n)} x_n - T_{i(n)}^{k(n)} p\right\| \\
& + \limsup_{n\to\infty} \lambda_n \mu \left\|A\left(T_{i(n)}^{k(n)} x_n\right) - A\left(T_{i(n)}^{k(n)} p\right)\right\| \\
& + \limsup_{n\to\infty} \lambda_n \mu \left\|A\left(T_{i(n)}^{k(n)} p\right)\right\| \\
\leq \ & \limsup_{n\to\infty} h_{k(n)} \|x_n - p\| + \limsup_{n\to\infty} \lambda_n \mu L h_{k(n)} \|x_n - p\| \\
& + \limsup_{n\to\infty} \lambda_n \mu \left\|A\left(T_{i(n)}^{k(n)} p\right)\right\| \\
\leq \ & \limsup_{n\to\infty} (1+d_n) \|x_n - p\| + \limsup_{n\to\infty} \lambda_n \mu L R_1 R_2 \\
& + \limsup_{n\to\infty} \lambda_n \mu \left\|A\left(T_{i(n)}^{k(n)} p\right)\right\| \\
\leq \ & r.
\end{aligned} \tag{3.9}$$

Since E is a uniformly convex Banach space, from (3.7), (3.8), (3.9) and Lemma 2.2 we have that

$$\lim_{n\to\infty} \left\|x_{n-1} - T^{\lambda_n} x_n\right\| = 0. \tag{3.10}$$

From (3.10), we have that

$$\begin{aligned}
\|x_n - x_{n-1}\| &= \left\|(\alpha_n - 1) x_{n-1} + (1-\alpha_n) T^{\lambda_n} x_n\right\| \\
&\leq (1-\alpha_n)\left\|x_{n-1} - T^{\lambda_n} x_n\right\| \\
&\to 0 \quad (\text{as } n \to \infty).
\end{aligned} \tag{3.11}$$

By (3.11), we obtain that

$$\lim_{n\to\infty} \|x_{n+l} - x_n\| = 0, \quad \forall l = 1,2,\cdots,N. \tag{3.12}$$

It follows from (3.10) and (3.11) that

$$\begin{aligned} \left\|x_n - T^{\lambda_n} x_n\right\| &\leq \|x_n - x_{n-1}\| + \left\|x_{n-1} - T^{\lambda_n} x_n\right\| \\ &\to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{3.13}$$

From (3.13) and condition (ii), we obtain

$$\begin{aligned} \|x_n - T_{i(n)}^{k(n)} x_n\| &\leq \left\|x_n - T^{\lambda_n} x_n\right\| + \left\|T^{\lambda_n} x_n - T_{i(n)}^{k(n)} x_n\right\| \\ &\leq \left\|x_n - T^{\lambda_n} x_n\right\| + \lambda_n \mu \left\|A\left(T_{i(n)}^{k(n)} x_n\right)\right\| \\ &\leq \left\|x_n - T^{\lambda_n} x_n\right\| + \lambda_n \mu \left(LR_1R_2 + \left\|A\left(T_{i(n)}^{k(n)} p\right)\right\|\right) \\ &\to 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{3.14}$$

It follows from (3.11) and (3.14) that

$$\begin{aligned} &\|x_n - T_n x_n\| \\ \leq &\left\|x_n - T_{i(n)}^{k(n)} x_n\right\| + \left\|T_{i(n)}^{k(n)} x_n - T_n x_n\right\| \\ \leq &\left\|x_n - T_{i(n)}^{k(n)} x_n\right\| + h_1 \{h_{n-1} \|x_n - x_{n-N}\| \\ &+ \left\|T_{i(n-N)}^{k(n-N)} x_{n-N} - x_{n-N}\right\| + \|x_{n-N} - x_n\|\Big\} \\ = &\left\|x_n - T_{i(n)}^{k(n)} x_n\right\| + h_1 (1 + h_{n-1}) \|x_n - x_{n-N}\| \\ &+ h_1 \left\|T_{i(n-N)}^{k(n-N)} x_{n-N} - x_{n-N}\right\| \\ \to &\ 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{3.15}$$

For any $l = 1, 2, \cdots, N$, from (3.12) and (3.15), we have

$$\begin{aligned} & \|x_n - T_{n+l}x_n\| \\ \leq & \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| + \|T_{n+l}x_{n+l} - T_{n+l}x_n\| \\ \leq & (1+L)\|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l}x_{n+l}\| \\ \to & 0 \quad (\text{as } n \to \infty). \end{aligned} \tag{3.16}$$

Consequently, we obtain

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \forall l = 1, 2, \cdots, N. \tag{3.17}$$

(3) The necessity is obvious. Therefore, we will prove the sufficiency. For arbitrary $p \in \mathfrak{F}$, it follows from (3.5) that

$$\|x_n - p\| \leq (1+\delta_n)\|x_{n-1} - p\| + b_n, \quad \forall n \geq n_0,$$

where $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. Thus, we have

$$d(x_n, \mathfrak{F}) \leq (1+\delta_n)\, d(x_{n-1}, \mathfrak{F}) + b_n, \quad \forall n \geq n_0. \tag{3.18}$$

It follows from Lemma 2.1 and (3.18) that $\lim_{n\to\infty} d(x_n, \mathfrak{F})$ exists. By the assumption, we have $\lim_{n\to\infty} d(x_n, \mathfrak{F}) = 0$.

Next we prove that the sequence $\{x_n\}$ is a Cauchy sequence in K. In fact, since $\sum_{n=1}^{\infty} \delta_n < \infty$, $1 + t \leq e^t$ for all $t > 0$, by (3.5) , we have

$$\|x_n - p\| \leq e^{\delta_n}\|x_{n-1} - p\| + b_n. \tag{3.19}$$

Thus, for any positive integers m, n, from (3.19) it follows that

$$\begin{aligned} \|x_{n+m} - p\| &\leq e^{\delta_{n+m}}\|x_{n+m-1} - p\| + b_{n+m} \\ &\leq e^{\delta_{n+m}}\left[e^{\delta_{n+m-1}}\|x_{n+m-2} - p\| + b_{n+m-1}\right] + b_{n+m} \\ &\leq \ldots \\ &\leq e^{\sum_{i=n}^{n+m} \delta_i}\|x_n - p\| + e^{\sum_{i=n+1}^{n+m} \delta_i} \sum_{i=n+1}^{n+m} b_i \\ &\leq Q\|x_n - p\| + Q\sum_{i=n+1}^{\infty} b_i, \end{aligned}$$

where $Q = e^{\sum_{i=1}^{\infty}\delta_i} < \infty$. Since $\lim_{n\to\infty} d(x_n, \mathfrak{F}) = 0$ and $\sum_{i=1}^{\infty} b_i < \infty$, for any given $\varepsilon > 0$, there exists a positive integer n_0 such that $d(x_n, \mathfrak{F}) < \frac{\varepsilon}{4(Q+1)}$, $\sum_{i=n+1}^{\infty} b_i < \frac{\varepsilon}{2Q}$ for any $n \geq n_0$. Hence, there exists $p_1 \in \mathfrak{F}$ such that $d(x_n, p_1) < \frac{\varepsilon}{2(Q+1)}$ for any $n \geq n_0$. Consequently, for all $m \geq 1$ and for any $n \geq n_0$, we have

$$\begin{aligned}\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p_1\| + \|x_n - p_1\| \\ &\leq Q\|x_n - p_1\| + Q\sum_{i=n+1}^{\infty} b_i + \|x_n - p_1\| \\ &< \varepsilon.\end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in K. Therefore there exists $q \in K$ such that $\{x_n\}$ converges strongly to q. Since $\lim_{n\to\infty}\|x_n - T_l x_n\| = 0$ for each $l \in \{1, 2, \cdots, N\}$, it follows from Lemma 2.3 that $q \in \mathfrak{F}$. This completes the proof.

Remark 3.2. Theorem 3.1 extends the results of [2] and [3] from a finite family of nonexpansive mappings to a finite family of asymptotically nonexpansive mappings.

Theorem 3.3. *Let E be a real uniformly convex Banach space, K be a nonempty closed convex subset of E. Let $\{T_1, T_2, \cdots, T_N\}$:$K \to K$ be N asymptotically nonexpansive mappings of K with sequences $\left\{h_n^{(i)}\right\}$, $1 \leq i \leq N$ and $\mathfrak{F} = \cap_{n=1}^{N} F(T_i) \neq \emptyset$ and one of the mappings in* $\{T_1, T_2, \cdots, T_N\}$ is semicompact. *$A : K \to K$ is an L-Lischitzian mapping. Let $\{\alpha_n\}$ is real sequences in $(0,1)$ and $\{\lambda_n\} \subset [0,1)$ satisfying the following conditions:*

(i) there exists constants τ_1, $\tau_2 \in (0,1)$ such that $\tau_1 \leq (1-\alpha_n) \leq \tau_2$, $\forall n \geq 1$;

(ii) $\sum_{n=1}^{\infty} \lambda_n < \infty$;

(iii) $\sum_{n=1}^{\infty} (h_n - 1) < \infty$, where $h_n = \max\left\{h_n^{(1)},\ h_n^{(2)},\ \cdots, h_n^{(N)}\right\}$.

Then the hybrid implicit iteration $\{x_n\}$ be defined by (1.4) converges strongly to a common fixed point of $\{T_1, T_2, \cdots, T_N\}$.

Proof. Suppose that T_{i_0} is semicompact for some $i_0 \in \{1, 2, \cdots, N\}$. By Theorem 3.1, $\{x_n\}$ is bounded, and $\lim_{n\to\infty}\|x_n - T_{i_0} x_n\| = 0$. Then there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $\lim_{n_j\to\infty} x_{n_j} \to p \in K$. Theorem 3.1 guarantees that $\lim_{n_j\to\infty}\left\|x_{n_j} - T_l x_{n_j}\right\| = 0$ for all $l \in \{1, 2, \cdots, N\}$. Therefore, we have $\|p - T_l p\| = 0$ for all $l \in \{1, 2, \cdots, N\}$. This implies that

$p \in \mathfrak{F}$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists, therefore $\lim_{n\to\infty} \|x_n - p\| = 0$; that is, $\{x_n\}$ converges strongly to a fixed point of $\{T_1, T_2, \cdots, T_N\}$ is K. This completes the proof.

Remark 3.4. Theorem 3.3 in this work improves Theorem 11 of Jiang *et al.* [3]. Also, the condition that $\{T_1, T_2, \cdots, T_N\}$ be semicompact is replaced by the weaker assumption that any one of $\{T_1, T_2, \cdots, T_N\}$ be semicompact.

From Theorem 3.1, we can easily show the following strong convergence theorem, whose proof is omitted.

Theorem 3.5. *Let E be a real uniformly convex Banach space and let K be a nonempty closed convex subset of E. Let $\{T_1, T_2, \cdots, T_N\} : K \to K$ be N asymptotically nonexpansive mappings satisfying condition $(\overline{C})$ and $A : K \to K$ is an L-Lischitzian mapping. $\left\{h_n^{(i)}\right\}$, $1 \leq i \leq N$, $\{\alpha_n\}$ and $\{\lambda_n\}$ are sequences as in Theorem 3.3. If $\mathfrak{F} = \cap_{n=1}^{N} F(T_i) \neq \emptyset$ then the hybrid implicit iteration $\{x_n\}$ be defined by (1.4) converges strongly to a common fixed point of $\{T_1, T_2, \cdots, T_N\}$.*

Theorem 3.6. *Let E be a real uniformly convex Banach space satisfying Opial's condition, K be a nonempty closed convex subset of E. Let $\{T_1, T_2, \cdots, T_N\} : K \to K$ be N asymptotically nonexpansive mappings and $A : K \to K$ be an L-Lischitzian mapping. $\left\{h_n^{(i)}\right\}$, $1 \leq i \leq N$, $\{\alpha_n\}$ and $\{\lambda_n\}$ are sequences as in Theorem 3.3. If $\mathfrak{F} = \cap_{n=1}^{N} F(T_i) \neq \emptyset$ then the hybrid implicit iteration $\{x_n\}$ be defined by (1.4) converges weakly to a common fixed point of $\{T_1, T_2, \cdots, T_N\}$.*

Proof. Since E is uniformly convex, every bounded subset of E is weakly compact. Also, since $\{x_n\}$ is a bounded subset in K, there exists a subsequence $\left\{x_{n_j}\right\} \subset \{x_n\}$ such that $\left\{x_{n_j}\right\}$ converges weakly to $p \in K$ and $\lim_{n_j\to\infty} \left\|x_{n_j} - T_l x_{n_j}\right\| = 0$ for all $l \in \{1, 2, \cdots, N\}$. By Lemma 2.3, we have that $(I - T_l)\, p = 0$, i.e. $p \in F(T_l)$. By arbitrariness of $l \in \{1, 2, \cdots, N\}$, we have $p \in \mathfrak{F} = \cap_{n=1}^{N} F(T_i)$.

Next, we prove that $\{x_n\}$ converges weakly to p. Suppose $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ which converges weakly to some $p_1 \in K$ and $p \neq p_1$. By the similar method as above we have $p_1 \in \mathfrak{F} = \cap_{n=1}^{N} F(T_i)$, then by *Opial's condition, we have*

$$\begin{aligned} \lim_{n\to\infty} \|x_n - p\| &= \lim_{n_j\to\infty} \left\|x_{n_j} - p\right\| \\ &< \lim_{n_j\to\infty} \left\|x_{n_j} - p_1\right\| \\ &= \lim_{n\to\infty} \|x_n - p_1\| \\ &= \lim_{n_k\to\infty} \|x_{n_k} - p_1\| \\ &< \lim_{n_k\to\infty} \|x_{n_k} - p\| \\ &= \lim_{n\to\infty} \|x_n - p\| . \end{aligned}$$

This is a contradiction. Hence, $p = p_1$, which implies that $\{x_n\}$ converges weakly to p. This completes the proof.

Remark 3.7. Theorem 3.6 in this work extends Theorem 9 of Jiang *et al.* [3] to case of a hybrid implicit iteration for a finite of asymptotically nonexpansive mappings.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgements

The authors would like to thank the referees for their helpful comments.

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