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A GENERAL ITERATIVE ALGORITHM FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS IN HILBERT SPACES

J.S. PARK

Gyeongnam National University of Science and Technology, Chinju, Korea

Abstract. In this paepr, we introduce and study a general iterative scheme for finding a common element of the set of common fixed points of an infinite family of nonexpansive mappings, the set of solutions of variational inequality for a relaxed cocoercive mapping and the set of solutions of a equilibrium problems.

Keywords: equilibrium problem; nonexpansive mapping; viscosity approximation method; fixed point

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1. Introduction-Preliminaries

Recently, iterative algorithm have been studied as an effective and powerful tool for investigating equilibrium problems. In this paepr, we investigate hybrid projection algorithms for asymptotically quasi-ϕ-nonexpansive mapping in the framework of Banach spaces. Strong convergence theorems of fixed points are established.

Throughout this paper, Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. We use $F(T)$ to denote the fixed point set of T and P_C to denote the metric projection of H onto C, where C is a nonempty closed convex subset of H. Let $A : C \to H$ be a nonlinear map. The classical variational inequality problem which denoted

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by $VI(C,A)$ is to find $u \in C$ such that

$$\langle Au, v-u\rangle \geq 0, \tag{1.1}$$

for all $v \in C$. It is known that P_C satisfies

$$\langle x-y, P_Cx-P_Cy\rangle \geq \|P_Cx-P_Cy\|^2, \tag{1.2}$$

for $x,y \in H$. Moreover, P_Cx is characterized by the properties: $P_Cx \in C$ and $\langle x-P_Cx, P_Cx-y\rangle \geq 0$ for all $y \in C$.

One can see that the variational inequality (1.1) is equivalent to a fixed point problem.

The element $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ satisfies the relation $u = P_C(u-\lambda Au)$, where $\lambda > 0$ is a constant. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Recall that the following definitions. a mapping B is said to be v-cocoercive, if for each $x,y \in C$, we have

$$\langle Bx-By, x-y\rangle \geq v\|Bx-By\|^2, \quad \text{for a constant } v > 0.$$

B is said to be relaxed (u,v)-cocoercive, if there exist two constants $u,v > 0$ such that

$$\langle Bx-By, x-y\rangle \geq (-u)\|Bx-By\|^2 + v\|x-y\|^2, \quad \forall x,y \in C,$$

for $u = 0$, B is v-strongly monotone. This class of maps is more general than the class of strongly monotone maps.

A mapping $T : C \to C$ is called nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for all $x,y \in C$.

A mapping $f : H \to H$ is said to be a contraction if there exists a coefficient α $(0 < \alpha < 1)$ such that $\|f(x)-f(y)\| \leq \alpha\|x-y\|$, for $\forall x,y \in H$.

An operator A is strong positive if there exists a constant $\bar{\gamma} > 0$ with the property $\langle Ax,x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H$. A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x,y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$

for every $(y,g) \in G(T)$ implies $f \in Tx$. Let B be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v-u, w\rangle \geq 0,\ \forall u \in C\}$ and define

$$Tv = \begin{cases} Bv + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C,B)$; for more details, see [16] and the reference therein.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. The equilibrium problem for $F : C \times C \to \mathbb{R}$ is to find $x \in C$ such that

$$F(x,y) \geq 0, \quad \forall y \in C. \tag{1.3}$$

The set of solution of (1.3) is denoted by $EP(F)$. Give a mapping $T : C \to H$, let $F(x,y) = \langle Tx, y-x\rangle$ for all $x,y \in C$. Then $z \in EP(F)$ if and only if $\langle Tz, y-z\rangle \geq 0$ for all $y \in C$, i.e., z is a solution of the variational inequality. Numerous problems in physics, optimization and economics reduce to find a solution of (1.3). Some methods have been proposed to solve the equilibrium problem; see, for instance, [6,9]. Combettes and Hirstoaga [6] introduced an iterative scheme for finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem. Takahashi et al. [20] introduced a new iterative scheme: $x_0 \in C$ and

$$\begin{cases} F(y_n,u) + \frac{1}{r_n}\langle u-y_n, y_n-x_n\rangle \geq 0, & \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Ty_n, & n \geq 0, \end{cases} \tag{1.4}$$

where f is a contraction on H, F is a bifunction, and T is a nonexpansive maping for approximating a common element of the set of fixed points of a non-self nonexpansive mapping and the set of solutions of the equilibrium problem and obtained a strong convergence theorem in a real Hilbert space. In [15], Plubtieng and Punpaeng further the common element problems.

In [18], Su et al. improved the results of [20] and studied the following iterative algorithm: $x_0 \in H$

$$\begin{cases} F(y_n,u) + \frac{1}{r_n}\langle u-y_n, y_n-x_n\rangle \geq 0, & \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)TP_C(I-\lambda_n A)y_n, & n \geq 0, \end{cases} \tag{1.6}$$

where f is a contraction on H, F is a bifunction and A is inverse-strongly monotone operator of C into H, T is a nonexpansive mapping. They proved the sequence $\{x_n\}$ defined by above iterative algorithm converges strongly to a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of the equilibrium problems and the set of solutions of variational inequality problems.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [7,12,21-23] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x\in C}\frac{1}{2}\langle Ax,x\rangle-\langle x,b\rangle, \tag{1.8}$$

where A is a linear bounded operator, C is the fixed point set of a nonexpansive mapping S and b is a given point in H. In [22], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0\in H$ chosen arbitrarily,

$$x_{n+1}=(I-\alpha_nA)Sx_n+\alpha_nb,\ \ n\geq 0,$$

converges strongly to the unique solution of the minimization problem (1.4) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Recently, Marino and Xu [12] introduced a new iterative scheme by the viscosity approximation method [13]:

$$x_0\in H,\quad x_{n+1}=(I-\alpha_nA)Sx_n+\alpha_n\gamma f(x_n),\ \ n\geq 0.$$

They proved the sequence $\{x_n\}$ generated by above iterative scheme converges strongly to the unique solution of the variational inequality $\langle(A-\gamma f)x^*,x-x^*\rangle\geq 0$, $x\in C$, which is the optimality condition for the minimization problem $\min_{x\in C}\frac{1}{2}\langle Ax,x\rangle-h(x)$, where C is the fixed point set of a nonexpansive mapping S, h is a potential function for γf (i.e., $h'(x)=\gamma f(x)$ for $x\in H$).

Concerning a family of nonexpansive mappings has been considered by many authors, see e.g., [2,17,18,22,25] and the references therein. The well-known convex feasibility problem reduces to finding a point in the intersection of the fixed point sets of a family of nonexpansive mappings; see, e.g., [1,4]. The problem of finding an optimal point that minimizes a given cost function over common set of fixed points of a family of nonexpansive mappings is of wide interdisciplinary interest and practical importance; see, e.g., [2,5,7,11,24]. A simple algorithmic solution to the problem of minimizing a quadratic function over common set of fixed points of a family of nonexpansive mappings is of extreme value in many applications including set theoretic signal estimation; see, e.g., [11,24].

In this paper, we study the mapping W_n defined by

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n T_n U_{n,n+1} + (1-\gamma_n)I,\\
&U_{n,n-1} = \gamma_{n-1} T_{n-1} U_{n,n} + (1-\gamma_{n-1})I,\\
&\vdots\\
&U_{n,k} = \gamma_k T_k U_{n,k+1} + (1-\gamma_k)I, \qquad (1.9)\\
&U_{n,k-1} = \gamma_{k-1} T_{k-1} U_{n,k} + (1-\gamma_{k-1})I,\\
&\vdots\\
&U_{n,2} = \gamma_2 T_2 U_{u,3} + (1-\gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 T_1 U_{n,2} + (1-\gamma_1)I,
\end{aligned}$$

where $\{\gamma_1\},\{\gamma_2\},\ldots$ are real numbers such that $0 \le \gamma \le 1$, $T_1,T_2,\ldots$ be an infinite family of mappings of C into itself.

Concerning W_n we have the following lemmas which are important to prove our main results.

Lemma 1.1 [17]. Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1,T_2,\ldots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\gamma_1,\gamma_2,\ldots$ be real numbers such that $0 < \gamma_n \le b < 1$ for any $n \ge 1$. Then, for every $x \in C$ and $k \in N$, the limit $\lim_{n\to\infty} U_{n,k}x$ exists.

Using Lemma 1.1, one can define the mapping W of C into itself as follows.

$$Wx = \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1}x,$$

for every $x \in C$. Such a W is called the W-mapping generated by $T_1,T_2,\ldots$ and $\gamma_1,\gamma_2,\ldots$. Throughout this paper, we will assume that $0 < \gamma_n \le b < 1$ for all $n \ge 1$.

Lemma 1.2 [17]. Let C be a nonempty closed convex subset of a strictly convex Banach space E. Let $T_1, T_2, \dots$ be nonexpansive mappings of C into itself such that $\cap_{n=1}^{\infty} F(T_n)$ is nonempty, and let $\gamma_1, \gamma_2, \dots$ be real numbers such that $0 < \gamma \leq b < 1$ for any $n \geq 1$. Then, $F(W) = \cap_{n=1}^{\infty} F(T_n)$.

In this paper, we consider a more general iterative process for equilibrium problems and fixed point problems. The results are obtained in this paper improve and extend the recent ones announced by Combettes and Hirstoaga [6], Marino and Xu [12], Iiduka and Takahashi [10], Moudafi [13], Plubtieng and Punpaeng [15], Su et al. [18], Takahashi and Takahashi [20], and Many others.

Recall that A space X is said to satisfy Opial's condition [14] if for each sequence $\{x_n\}_{n=1}^{\infty}$ in X which converges weakly to point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \ \forall y \in X, \ y \neq x.$$

For solving the equilibrium problem for a bifunction $F : C \times C \to \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, i.e., $F(x,y) + F(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x,y);$$

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex and lower semicontinuous.

Lemma 1.3. In a real Hilbert space H, there holds the the following inequality

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle,$$

for all $x, y \in H$.

Lemma 1.4 [21] *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \delta_n,$$

where γ_n *is a sequence in (0,1) and* $\{\delta_n\}$ *is a sequence such that*

(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(ii) $\limsup_{n\to\infty}\delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 1.5 [12] *Assume B is a strong positive linear bounded operator on a Hilbert space H with coefficient* $\bar{\gamma} > 0$ *and* $0 < \rho \le \|B\|^{-1}$. *Then* $\|I - \rho B\| \le 1 - \rho\bar{\gamma}$.

Lemma 1.6 [3] *Let C be a nonempty closed convex subset of H and let B be a bifunction of* $C \times C$ *into* $\mathbb{R}$ *satisfying* (A1)-(A4). *Let* $r > 0$ *and* $x \in H$. *Then, there exists* $z \in C$ *such that*

$$F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,\ \forall y \in C.$$

Lemma 1.7 [6] *Assume that* $F : C \times C \to \mathbf{R}$ *satisfies* $(A1)-(A4)$. *For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows:*

$$T_r(x) = \{z \in C :\ F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,\ \forall y \in C\}.$$

Then, the following hold:

(1) T_r *is single-valued;*

(2) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \le \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ *is closed and convex.*

Lemma 1.8 [19] *Let* $\{x_n\}$ *and* $\{y_n\}$ *be bounded sequences in a Banach space X and let* β_n *be a sequence in [0,1] with* $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. *Suppose* $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ *for all integers* $n \ge 0$ *and*

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

2. Main results

Theorem 2.1 *Let C be a nonempty closed convex subset of a Hilbert space H. Let F be a bifunction from $C\times C$ to $\mathbb{R}$ which satisfies (A1)-(A4), let $\{T_n\}_{n=1}^{\infty}$ be an infinite family of nonexpansive mappings of C into H and let B be a μ-Lipschitzian, relaxed (u,v)-cocoercive map of C into H such that $F = \cap_{i=1}^{\infty} F(T_i)\cap EP(F)\cap VI(C,B) \neq \emptyset$. Let A be a strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Let f be a contraction of H into itself with a coefficient α ($0 < \alpha < 1$) and let $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} F(z_n,\eta) + \frac{1}{r_n}\langle \eta - z_n, z_n - x_n\rangle \geq 0, \quad \forall \eta \in C, \\ y_n = \beta_n \gamma f(z_n) + (I - \beta_n A) W_n P_C (I - s_n B) z_n \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) y_n, \quad n \geq 1, \end{cases}$$

where $\alpha_n \subset [0,1]$, W_n is defined by (1.9) and $\{r_n\}, \{s_n\} \subset [0,\infty)$ satisfy

(i) $\lim_{n\to\infty} \beta_n = 0, \sum_{n=1}^{\infty} \beta_n = \infty$;

(ii) $\lim_{n\to\infty} |r_{n+1} - r_n| = 0, \lim_{n\to\infty} |s_{n+1} - s_n| = 0$;

(iii) $\liminf_{n\to\infty} r_n > 0$, *there exists $c, d \in (0,1)$ such that $c < \alpha_n < d$ for all $n \geq 0$;*

(iv) $\{s_n\} \in [a,b]$ *for some* a,b *with* $0 \leq a \leq b \leq \frac{2(v-u\mu^2)}{\mu^2}$.

Then, $\{x_n\}$ converges strongly to $q \in F$, where $q = P_F(\gamma f + (I-A))(q)$, which solves the following variational inequality $\langle \gamma f(q) - Aq, p - q\rangle \leq 0, \ \forall p \in F$.

Proof. First, we show $I - s_n B$ is nonexpansive. Indeed, from the relaxed (u,v)-cocoercive and μ-Lipschitzian definition on B and condition (iv), we have

$$\begin{aligned} &\|(I-s_nB)x - (I-s_nB)y\|^2 \\ &\leq \|x-y\|^2 - 2s_n[-u\|Bx-By\|^2 + v\|x-y\|^2] + s_n^2\|Bx-By\|^2 \\ &\leq \|x-y\|^2 + 2s_n\mu^2 u\|x-y\|^2 - 2s_n v\|x-y\|^2 + \mu^2 s_n^2\|x-y\|^2 \\ &= (1 + 2s_n\mu^2 u - 2s_n v + \mu^2 s_n^2)\|x-y\|^2 \\ &\leq \|x-y\|^2, \end{aligned}$$

which implies the mapping $I-s_nB$ is nonexpansive. Now, we observe that $\{x_n\}$ is bounded. Indeed, pick $p\in F$. Since $z_n=T_{r_n}x_n$, we have $\|z_n-p\|=\|T_{r_n}x_n-T_{r_n}p\|\le\|x_n-p\|$. Putting $\rho_n=P_C(I-s_nB)z_n$, we have

$$\|\rho_n-p\|\le\|(I-s_nB)z_n-p\|\le\|z_n-p\|\le\|x_n-p\|.$$

Since $\beta_n\to 0$ by the condition (i), we may assume, with no loss of generality, that $\beta_n<\|A\|^{-1}$ for all n. From Lemma 1.5, we know that if $0<\rho\le\|A\|^{-1}$, then $\|I-\rho A\|\le 1-\rho\bar{\gamma}$. Therefore, we obtain

$$\begin{aligned}\|y_n-p\|\le&\beta_n\|\gamma f(z_n)-Ap\|+\|I-\beta_nA\|\|W_n\rho_n-p\|\\ \le&\beta_n[\gamma\|f(z_n)-f(p)\|+\|\gamma f(p)-Ap\|]+(1-\beta_n\bar{\gamma})\|\rho_n-p\|\\ \le&[1-(\bar{\gamma}-\gamma\beta_n)\beta_n]\|x_n-p\|+\beta_n\|\gamma f(p)-Ap\|,\end{aligned}$$

which yields that

$$\begin{aligned}\|x_{n+1}-p\|=&\|\alpha_n(x_n-p)+(1-\alpha_n)(y_n-p)\|\\ \le&\alpha_n\|x_n-p\|+(1-\alpha_n)\|y_n-p\|\\ \le&\alpha_n\|x_n-p\|+(1-\alpha_n)[1-(\bar{\gamma}-\gamma\beta_n)\beta_n]\|x_n-p\|\\ &+(1-\alpha_n)\beta_n\|\gamma f(p)-Ap\|.\end{aligned}$$

This in turn implies that

$$\|x_n-p\|\le\max\{\|x_0-p\|,\ \frac{\|\gamma f(p)-Ap\|}{\bar{\gamma}-\gamma\alpha}\},\quad n\ge 0. \tag{2.1}$$

Therefore, we obtain that $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$. Let M denotes the possible different constant appearing in following argument. Observing that $z_n=T_{r_n}x_n$ and $z_{n+1}=T_{r_{n+1}}x_{n+1}$, we have

$$F(z_n,\eta)+\frac{1}{r_n}\langle\eta-z_n,z_n-x_n\rangle\ge 0,\quad\text{for all }\eta\in C \tag{2.2}$$

and

$$F(z_{n+1},\eta)+\frac{1}{r_{n+1}}\langle\eta-z_{n+1},z_{n+1}-x_{n+1}\rangle\ge 0,\quad\text{for all }\eta\in C. \tag{2.3}$$

Putting $\eta=z_{n+1}$ in (2.2) and $\eta=z_n$ in (2.3), we have

$$F(z_n,z_{n+1})+\frac{1}{r_n}\langle z_{n+1}-z_n,z_n-x_n\rangle\ge 0$$

and

$$F(z_{n+1},z_n)+\frac{1}{r_{n+1}}\langle z_n-z_{n+1},z_{n+1}-x_{n+1}\rangle\geq 0.$$

It follows from (A2) that

$$\langle z_{n+1}-z_n,z_n-z_{n+1}+z_{n+1}-x_n-\frac{r_n}{r_{n+1}}(z_{n+1}-x_{n+1})\rangle\geq 0.$$

Without loss of generality, let us assume that there exists a real number m such that $r_n>m>0$ for all n. It follows that

$$\|z_{n+1}-z_n\|^2\leq\|z_{n+1}-z_n\|(\|x_{n+1}-x_n\|+|1-\frac{r_n}{r_{n+1}}|\|z_{n+1}-x_{n+1}\|).$$

It follows that

$$\begin{aligned}\|z_{n+1}-z_n\|\leq&\|x_{n+1}-x_n\|+|1-\frac{r_n}{r_{n+1}}|\|z_{n+1}-x_{n+1}\|\\ \leq&\|x_{n+1}-x_n\|+\frac{M}{m}|r_{n+1}-r_n|.\end{aligned}\tag{2.4}$$

It follows that

$$\begin{aligned}\|\rho_{n+1}-\rho_n\|\leq&\|(I-s_{n+1}B)z_{n+1}-(I-s_nB)z_n\|\\ =&\|(I-s_{n+1}B)z_{n+1}-(I-s_{n+1}B)z_n+(s_n-s_{n+1})Bz_n\|\\ \leq&\|z_{n+1}-z_n\|+|s_n-s_{n+1}|\|Bz_n\|.\end{aligned}\tag{2.5}$$

Substitute (2.4) into (2.5) yields that

$$\|\rho_{n+1}-\rho_n\|\leq\|x_{n+1}-x_n\|+M(|r_{n+1}-r_n|+|s_n-s_{n+1}|).\tag{2.6}$$

Observe that

$$\begin{aligned}&\|y_n-y_{n+1}\|\\ &=\|(I-\beta_{n+1}A)(W_{n+1}\rho_{n+1}-W_n\rho_n)-(\beta_{n+1}-\beta_n)AW_n\rho_n\\ &\quad+\gamma[\beta_{n+1}(f(z_{n+1})-f(z_n))+f(z_n)(\beta_{n+1}-\beta_n)]\|\\ &\leq(1-\beta_{n+1}\bar{\gamma})(\|\rho_{n+1}-\rho_n\|+\|W_{n+1}\rho_n-W_n\rho_n\|)\\ &\quad+|\beta_{n+1}-\beta_n|\|AW_n\rho_n\|+\gamma[\beta_{n+1}\alpha\|z_{n+1}-z_n\|+|\beta_{n+1}-\beta_n|\|f(z_n)\|].\end{aligned}\tag{2.7}$$

Next we estimate $\|W_{n+1}\rho_n-W_n\rho_n\|$. Since T_i and $U_{n,i}$ are nonexpansive, from (1.9), we have

$$\|W_{n+1}\rho_n-W_n\rho_n\|\leq M\prod_{i=1}^{n}\gamma_i.\tag{2.8}$$

Substitute (2.4), (2.6) and (2.8) into (2.7) yields that

$$\lim_{n\to\infty}\{\|y_n - y_{n+1}\| - \|x_{n+1} - x_n\|\} \le 0.$$

By virtue of Lemma 1.8, we obtain that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{2.9}$$

On the other hand, from (1.10), we have

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{2.10}$$

Observing $y_n = \beta_n\gamma f(z_n) + (I - \beta_n A)W_n\rho_n$, we have $\|y_n - W_n\rho_n\| = \beta_n\|\gamma f(z_n) - AW_n\rho_n\|$ which combines with the condition (i) give

$$\lim_{n\to\infty}\|y_n - W_n\rho_n\| = 0. \tag{2.11}$$

For $p \in F$, we have $\|z_n - p\|^2 \le \|x_n - p\|^2 - \|x_n - z_n\|^2$. It follows that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|^2 + (1-\alpha_n)(1-\beta_n\bar{\gamma})\|\rho_n - p\|^2 \\
&\quad + 2(1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\| \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|^2 + (1-\alpha_n)(1-\beta_n\bar{\gamma})\|z_n - p\|^2 \\
&\quad + 2(1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\| \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|^2 + (1-\alpha_n)\|x_n - p\|^2 \\
&\quad - (1-\alpha_n)(1-\beta_n\bar{\gamma})\|x_n - z_n\|^2 + 2(1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\|,
\end{aligned}$$

from which it follows that

$$\begin{aligned}
&(1-\alpha_n)(1-\beta_n\bar{\gamma})\|x_n - z_n\|^2 \\
&\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + (1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|^2 \\
&\quad + 2(1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\| \\
&\le (\|x_n - p\| - \|x_{n+1} - p\|)\|x_n - x_{n+1}\| + (1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|^2 \\
&\quad + 2(1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\|
\end{aligned}$$

It follows from (i), (iii) and (2.10) that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{2.12}$$

For $p \in F$, we have

$$\begin{aligned}
&\|\rho_n - p\|^2 \\
&\le \|x_n - p\|^2 - 2s_n[-u\|Az_n - Ap\|^2 + v\|z_n - p\|^2] + s_n^2\|Az_n - Ap\|^2 \\
&\le \|x_n - p\|^2 + 2s_n u\|Az_n - Ap\|^2 - 2s_n v\|z_n - p\|^2 + s_n^2\|Az_n - Ap\|^2 \\
&\le \|x_n - p\|^2 + (2s_n u + s_n^2 - \frac{2s_n v}{\mu^2})\|Az_n - Ap\|^2.
\end{aligned} \tag{2.13}$$

Observe that

$$\begin{aligned}
&\|x_{n+1} - p\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|\beta_n(\gamma f(z_n) - Ap) + (I - \beta_n A)(W_n\rho_n - p)\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\beta_n\|\gamma f(z_n) - Ap\| + \|I - \beta_n A\|\|W_n\rho_n - p\|)^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)(\beta_n\|\gamma f(z_n) - Ap\| + (1-\beta_n\bar{\gamma})\|\rho_n - p\|)^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|^2 + (1-\alpha_n)\|\rho_n - p\|^2 \\
&\quad + 2(1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\|.
\end{aligned} \tag{2.14}$$

Substituting (2.13) into (2.14), we have

$$\begin{aligned}
&(\frac{2av}{\mu^2} - 2bu - b^2)\|Az_n - Ap\|^2 \\
&\le (1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2 \\
&\quad + 2(1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\| \\
&\le (1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\| \\
&\quad + 2(1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\|
\end{aligned}$$

Since condition (i) and (2.10), we have that

$$\lim_{n\to\infty}\|Az_n - Ap\| = 0. \tag{2.15}$$

On the other hand, we have

$$\begin{aligned}\|\rho_n - p\|^2 =& \|P_C(I-s_nA)z_n - P_C(I-s_nA)p\|^2\\ \leq& \langle (I-s_nA)z_n - (I-s_nA)p, \rho_n - p\rangle\\ =& \frac{1}{2}\{\|(I-s_nA)z_n - (I-s_nA)p\|^2 + \|\rho_n - p\|^2\\ & - \|(I-s_nA)z_n - (I-s_nA)p - (\rho_n - p)\|^2\}\\ \leq& \frac{1}{2}\{\|z_n - p\|^2 + \|\rho_n - p\|^2 - \|(z_n - \rho_n) - s_n(Az_n - Ap)\|^2\}\\ =& \frac{1}{2}\{\|x_n - p\|^2 + \|\rho_n - p\|^2 - \|z_n - \rho_n\|^2 - s_n^2\|Az_n - Ap\|^2\\ & + 2s_n\langle z_n - \rho_n, Az_n - Ap\rangle\},\end{aligned}$$

which yields that

$$\|\rho_n - p\|^2 \leq \|x_n - p\|^2 - \|z_n - \rho_n\|^2 + 2s_n\|z_n - \rho_n\|\|Az_n - Ap\|. \tag{2.16}$$

Substitute (2.16) into (2.14) yields that

$$\begin{aligned}&(1-\alpha_n)\|z_n - \rho_n\|^2\\ &\leq (1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|^2 + \|x_n - p\|^2 - \|x_{n+1} - p\|^2\\ &\quad + 2s_n(1-\alpha_n)\|z_n - p\|\|Az_n - Ap\| + 2(1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\|\\ &\leq (1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|^2 + (\|x_n - p\| + \|x_{n+1} - p\|)\|x_n - x_{n+1}\|\\ &\quad + 2s_n(1-\alpha_n)\|z_n - p\|\|Az_n - Ap\| + 2(1-\alpha_n)\beta_n\|\gamma f(z_n) - Ap\|\|\rho_n - p\|\end{aligned}$$

Since conditions (i) (ii), (2.10) and (2.15), we have that

$$\lim_{n\to\infty}\|z_n - \rho_n\| = 0. \tag{2.17}$$

Observe that

$$\begin{aligned}\|z_n - W_nz_n\| \leq& \|W_nz_n - W_n\rho_n\| + \|W_n\rho_n - y_n\| + \|y_n - x_n\| + \|x_n - z_n\|\\ \leq& \|z_n - \rho_n\| + \|W_n\rho_n - y_n\| + \|y_n - x_n\| + \|x_n - z_n\|.\end{aligned}$$

Since (2.9), (2.11), (2.12) and (2.17), we have $\lim_{n\to\infty}\|z_n - W_nz_n\| = 0$. On the other hand, we have

$$\|Wz_n - z_n\| \leq \|Wz_n - W_nz_n\| + \|W_nz_n - z_n\|.$$

For any $\varepsilon > 0$, there is N such that $\|Wz - W_n z\| \leq \varepsilon$ for all $z \in \{z_n\}$ and for all $n \geq N$. Therefore, we have $\|Wz_n - W_n z_n\| \to 0$ as $n \to \infty$. It follows that

$$\lim_{n\to\infty} \|z_n - Wz_n\| = 0. \tag{2.18}$$

From Lemma 1.5, we know that if $0 < \rho \leq \|A\|^{-1}$, then $\|I - \rho A\| \leq 1 - \rho\bar{\gamma}$. In this paper, we will assume that $\|I - A\| \leq 1 - \bar{\gamma}$. Observe that $P_F(\gamma f + (I - A))$ is a contraction. Indeed, for $\forall x, y \in H$, we have $\|P_F(\gamma f + (I-A))(x) - P_F(\gamma f + (I-A))(y)\| < \|x - y\|$. Banach's Contraction Mapping Principle guarantees that $P_F(\gamma f + (I - A))$ has a unique fixed point, say $q \in H$. That is, $q = P_F(\gamma f + (I - A))(q)$. To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q\rangle = \lim_{i\to\infty} \langle \gamma f(q) - Aq, x_{n_i} - q\rangle.$$

Correspondingly, there exists a subsequence $\{z_{n_i}\}$ of $\{z_n\}$. Since $\{z_{n_i}\}$ is bounded, there exists a subsequence $\{z_{n_{i_j}}\}$ of $\{z_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $z_{n_i} \rightharpoonup w$.

Next, we show $w \in F$. First, we prove $w \in EP(F)$. Since $z_n = T_{r_n} x_n$, we have $\langle \eta - z_n, \frac{z_n - x_n}{r_n}\rangle \geq F(\eta, z_n)$. It follows that

$$\langle \eta - z_{n_i}, \frac{z_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq F(\eta, z_{n_i}).$$

Since $\frac{z_{n_i} - x_{n_i}}{r_{n_i}} \to 0$, $z_{n_i} \rightharpoonup w$ and (A4), we have $F(\eta, w) \leq 0$ for all $\eta \in C$. For t with $0 < t \leq 1$ and $\eta \in C$, let $\eta_t = t\eta + (1-t)w$. Since $\eta \in C$ and $w \in C$, we have $\eta_t \in C$ and hence $F(\eta_t, w) \leq 0$. So, from (A1) and (A4), we have

$$0 = F(\eta_t, \eta_t) \leq tF(\eta_t, \eta) + (1-t)F(\eta_t, w) \leq tF(\eta_t, \eta).$$

That is, $F(\eta_t, \eta) \geq 0$. It follows from (A3) that $F(w, \eta) \geq 0$ for all $\eta \in C$ and hence $w \in EP(F)$. Since Hilbert spaces are $Opial's$ spaces, from (2.18), we have

$$\begin{aligned}
\liminf_{i\to\infty} \|z_{n_i} - w\| &< \liminf_{i\to\infty} \|z_{n_i} - Ww\| \\
&= \liminf_{i\to\infty} \|z_{n_i} - Wz_{n_i} + Wz_{n_i} - Ww\| \\
&\leq \liminf_{i\to\infty} \|Wz_{n_i} - Ww\| \\
&\leq \liminf_{i\to\infty} \|z_{n_i} - w\|,
\end{aligned}$$

which derives a contradiction. Thus, we have $w \in F(W)$. It follows from $F(W) = \cap_{i=1}^{\infty} F(T_i)$; (see Lemma 1.2) that $w \in \cap_{i=1}^{\infty} F(T_i)$. Next, let us first show that $w \in VI(C,A)$. Put

$$Tw_1 = \begin{cases} Bw_1 + N_C w_1, & w_1 \in C \\ \emptyset, & w_1 \overline{\in} C. \end{cases}$$

Since B is relaxed (u,v)-cocoercve and condition (iv), we have

$$\langle Bx - By, x - y\rangle \geq (-u)\|Bx - By\|^2 + v\|x - y\|^2 \geq (v - u\mu^2)\|x - y\|^2 \geq 0,$$

which yields that B is monotone. Thus T is maximal monotone. Let $(w_1, w_2) \in G(T)$. Since $w_2 - Aw_1 \in N_C w_1$ and $\rho_n \in C$, we have $\langle w_1 - \rho_n, w_2 - Bw_1\rangle \geq 0$. On the other hand, from $\rho_n = P_C(I - s_n B)z_n$, we have

$$\begin{aligned}\langle w_1 - \rho_{n_i}, w_2\rangle &\geq \langle w_1 - \rho_{n_i}, Bw_1\rangle \geq \langle w_1 - \rho_{n_i}, Bw_1\rangle \\ &\quad - \langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - z_{n_i}}{s_{n_i}} + Bz_{n_i}\rangle \\ &= \langle w_1 - \rho_{n_i}, Bw_1 - \frac{\rho_{n_i} - z_{n_i}}{s_{n_i}} - Bz_{n_i}\rangle \\ &= \langle w_1 - \rho_{n_i}, Bw_1 - B\rho_{n_i}\rangle + \langle w_1 - \rho_{n_i}, B\rho_{n_i} - Bz_{n_i}\rangle \\ &\quad - \langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - z_{n_i}}{s_{n_i}}\rangle \\ &\geq \langle w_1 - \rho_{n_i}, B\rho_{n_i} - Bz_{n_i}\rangle - \langle w_1 - \rho_{n_i}, \frac{\rho_{n_i} - z_{n_i}}{s_{n_i}}\rangle,\end{aligned}$$

which implies that $\langle w_1 - w, w_2\rangle \geq 0$. We have $w \in T^{-1}0$ and hence $w \in VI(C,A)$. That is, $w \in F$. Since $q = P_F(\gamma f + (I - A))(q)$, we have

$$\begin{aligned}\limsup_{n\to\infty} \langle \gamma f(q) - Aq, x_n - q\rangle &= \lim_{n\to\infty} \langle \gamma f(q) - Aq, x_{n_i} - q\rangle \\ &= \langle \gamma f(q) - Aq, w - q\rangle \leq 0.\end{aligned} \tag{2.19}$$

It follows from Lemma 1.4 that

$$\begin{aligned}\|y_n-q\|^2 &\le (1-\beta_n\bar{\gamma})^2\|z_n-q\|^2+2\beta_n\gamma\langle f(z_n)-f(q),y_n-q\rangle\\ &\quad +2\beta_n\langle\gamma f(q)-Aq,y_n-q\rangle\\ &\le (1-\beta_n\bar{\gamma})^2\|x_n-q\|^2+2\beta_n\gamma\alpha\|z_n-q\|\|y_n-q\|\\ &\quad +2\beta_n\langle\gamma f(q)-Aq,y_n-q\rangle\\ &\le (1-\beta_n\bar{\gamma})^2\|x_n-q\|^2+\beta_n\gamma\alpha(\|z_n-q\|^2+\|y_n-q\|^2)\\ &\quad +2\beta_n\langle\gamma f(q)-Aq,y_n-q\rangle,\end{aligned}$$

which implies that

$$\begin{aligned}\|y_n-q\|^2 &\le \frac{(1-\beta_n\bar{\gamma})^2+\beta_n\gamma\alpha}{1-\beta_n\gamma\alpha}\|x_n-q\|^2+\frac{2\beta_n}{1-\beta_n\gamma\alpha}\langle\gamma f(q)-Aq,y_n-q\rangle\\ &= \frac{(1-2\beta_n\bar{\gamma}+\beta_n\alpha\gamma)}{1-\beta_n\gamma\alpha}\|x_n-q\|^2+\frac{\beta_n^2\bar{\gamma}^2}{1-\beta_n\gamma\alpha}\|x_n-q\|^2\\ &\quad +\frac{2\beta_n}{1-\beta_n\gamma\alpha}\langle\gamma f(q)-Aq,y_n-q\rangle\\ &\le [1-\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}]\|x_n-q\|^2\\ &\quad +\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}[\frac{1}{\bar{\gamma}-\alpha\gamma}\langle\gamma f(q)-Aq,y_n-q\rangle+\frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M].\end{aligned}\tag{2.20}$$

On the other hand, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n(x_n-p)+(1-\alpha_n)(y_n-p)\|^2\\ &\le \alpha_n\|x_n-p\|^2+(1-\alpha_n)\|y_n-p\|^2\end{aligned}\tag{2.21}$$

Substitute (2.20) into (2.21) yields that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le [1-(1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}]\|x_n-q\|^2\\ &\quad +(1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha\gamma)}{1-\beta_n\gamma\alpha}[\frac{1}{\bar{\gamma}-\alpha\gamma}\langle\gamma f(q)-Aq,y_n-q\rangle+\frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M].\end{aligned}\tag{2.22}$$

Put $l_n=(1-\alpha_n)\frac{2\beta_n(\bar{\gamma}-\alpha_n\gamma)}{1-\beta_n\alpha\gamma}$ and

$$t_n=\frac{1}{\bar{\gamma}-\alpha\gamma}\langle\gamma f(q)-Aq,x_{n+1}-q\rangle+\frac{\beta_n\bar{\gamma}^2}{2(\bar{\gamma}-\alpha\gamma)}M.$$

That is,

$$\|x_{n+1}-q\|^2 \leq (1-l_n)\|x_n-q\|^2 + l_n t_n. \tag{2.23}$$

It follows from condition (i) and (2.19) that

$$\lim_{n\to\infty} l_n = 0, \sum_{n=1}^{\infty} l_n = \infty \text{ and } \limsup_{n\to\infty} t_n \leq 0.$$

Apply Lemma 2.1 to (2.23) to conclude $x_n \to q$. This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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