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DIAMETER APPROXIMATE FIXED POINT THEOREMS ON MAP T_α

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Abstract. We define diameter approximate fixed point for $T_\alpha : X \to X$ in normed spaces. Two general lemmas are given regarding approximate fixed point and diameter it's on normed spaces. Using these results we prove theorems for various types of well-known generalized contractions on normed spaces. .

Keywords: $\varepsilon-$fixed points; $T_\alpha-$Kannan operator; $T_\alpha-$Chatterjea operator; $T_\alpha-$ Zamfirescu operator.

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1. Introduction

Nowadays, Fixed point property is used to solve many problems in mathematics, particularly in applied mathematics. But in some situations it will be difficult to derive conditions for existence of fixed points for certain types of mappings and the requirement will be only an approximation to the fixed points. In such a case naturally we will use the concept of approximate fixed points.

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In 1966, Edelstein [6] succeed in relaxing the condition of uniform convexity and proved Theorem of Krasnoselskii [7] for strictly convex Banach spaces for map as fallow: $T_\alpha = \alpha I + (1-\alpha)T,\ 0<\alpha<1$.

In 1976 Ishikawa obtained a surprising result, a special case of which may be stated as follows: Let K be an arbitrary bounded closed convex subset of a Banach space X, $T: K \to K$ non-expansive, and $\alpha \in (0,1)$. Set $T_\alpha = \alpha I + (1-\alpha)T,\ 0<\alpha<1$, then for each $x \in K$, $\|T_\alpha^n(x) - T_\alpha^{n+1}(x)\| \to 0$. In 1978, Edelstein and O'Brien [5] proved that $\{T_\alpha^n(x) - T_\alpha^{n+1}(x)\}$ converges to 0 uniformly for $x \in K$, in 1983 Goebel and Kirk [9] proved that this convergence is even uniform for $T \in \zeta$, where ζ denotes the collection of all non-expansive self-mappings of K. In 1971, Petryshyn [13] extended the result to densifying non-expansive mappings also Diaz and Metcalf [4] gave a theorem for strictly convex Banach spaces for sequences of the type:

$$x_{n+1} = \alpha x_n + (1-\alpha)T(x_n),$$

where $T: K \to K$ is a non-expansive mapping and K is a arbitrary bounded closed convex subset of a Banach space X. Also, we obtain some result on T_α and research about approximate fixed point property and diameter approximate fixed point for various types of operators. Let $(X, ||.||)$ be a normed space.

2. Preliminaries

We begin by recalling some needed definitions and results.

Definition 2.1. [1] *Let $T: X \to X$, $\varepsilon > 0$, $x_0 \in X$. Then $x_0 \in X$ is $\varepsilon-$fixed point for T if $\|Tx_0 - x_0\| < \varepsilon$.*

Remark 2.2. [1] *In this paper we will denote the set of all $\varepsilon-$ fixed points of T, for a given ε, by :*

$$AF(T) = \{x \in X \mid x \text{ is an } \varepsilon - \text{fixed point of } T\}.$$

Definition 2.3. [1] Let $T: X \to X$. Then T has the approximate fixed point property (a.f.p.p) if

$$\forall \varepsilon > 0,\ AF(T) \neq \varnothing.$$

Theorem 2.4. [14] *Let* $(X, \|.\|)$ *be a completely norm space,* $T : X \to X$, $x_0 \in X$ *and* $\varepsilon > 0$ *. If* $\|T^n(x_0) - T^{n+k}(x_0)\| \to 0$ *as* $n \to \infty$ *for some* $k > 0$, *then* T^k *has an* $\varepsilon-$ *fixed point.*

Theorem 2.5. [14] *Let* $(X, \|.\|)$ *be a completely norm space and* $T : X \to X$ *be a map also for all* $x, y \in X$,

$$\|Tx - Ty\| \le c\ \|x - y\|\ 0 < c < 1$$

then T has an $\varepsilon-$ *fixed point in completely norm space. Moreover, if* $x, y \in X$ *are* $\varepsilon-$ *fixed points of T, then* $\|x - y\| \le \frac{2\varepsilon}{1-c}$.

Definition 2.6. [14] Let $(X, \|.\|)$ be a completely norm space and $T : X \to X$, and $T_\alpha : X \to X$ be a map as follow:

$$T_\alpha = \alpha I + (1 - \alpha)T,\ 0 < \alpha < 1.$$

Then $x_0 \in X$ is $\varepsilon-$fixed point for T_α if $\|T_\alpha x_0 - x_0\| < \varepsilon$.

Remark 2.7. [14] *In this paper we will denote the set of all* $\varepsilon-$ *fixed points of* T_α*, for a given* ε*, by :*

$$AF(T_\alpha) = \{x \in X \mid x\ is\ an\ \varepsilon - fixed\ point\ of\ T_\alpha\}.$$

Theorem 2.8. [14] *Let* $(X, \|.\|)$ *be a completely norm space and* $T : X \to X$ *be a map also for all* $x, y \in X$,

$$\|Tx - Ty\| \le c\ \|x - y\|,\ 0 < c < 1\ (1)$$

If $AF(T)$, *the set of Approximate fixed point of* T, *is nonempty then the mapping*

$$T_\alpha = \alpha I + (1 - \alpha)T,\ 0 < \alpha < 1$$

satisfy in (1) and $AF(T) = AF(T_\alpha)$. *Moreover* $\|T_\alpha^n(x) - T_\alpha^{n+k}(x)\| \to 0$ *as* $n \to \infty$, *for some* $k > 0$, $\varepsilon > 0$.

Definition 2.9. [14] Let $T : X \to X$ and $\varepsilon > 0$. We define diameter $AF(T)$ by

$$diam(AF(T)) = \sup\{\|x - y\| :\ x, y \in AF(T)\}.$$

Remark 2.10. *The following result (see* [10]*) gives conditions under which the existence of fixed points for a given mapping is equivalent to that of approximate fixed points.*

Theorem 2.11. *Let A be a closed subset of a metric space (X,d) and $T : A \to X$ a compact map. Then T has a fixed point if and only if it has the approximate fixed point property.*

3. Map T_α and approximate fixed point property

The section begins with two lemma which will be used in order to prove quantitative results for various types of operators on a metric space.

Definition 3.1. Let $T : X \to X$, and $T_\alpha : X \to X$ be a map as follow:

$$T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1,$$

and $\varepsilon > 0$. We define diameter $AF(T_\alpha)$ by

$$diam(AF(T_\alpha)) = \sup\{\|x-y\| :\ x,y \in AF(T_\alpha)\}.$$

Lemma 3.2. *Let $T : X \to X$, $T_\alpha : X \to X$ be a map with $T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1$, and $\varepsilon > 0$. We assume that:*

(i) $AF(T_\alpha) \neq \emptyset$;

(ii) *foe each $\theta > 0$ there existe a $\phi(\theta) > 0$ such that*

$$\|x-y\| - \|T_\alpha(x) - T_\alpha(y)\| \quad \leq \quad \theta \Rightarrow \|x-y\| \leq \phi(\theta),\ \forall x,y \in AF(T_\alpha) \neq \emptyset.$$

Then:

$$diam(AF(T_\alpha)) \leq \phi(\varepsilon).$$

Proof: Let $\varepsilon_1, \varepsilon_2 > 0$ and $x,y \in AF(T_\alpha)$. Then:

$$\|x - T_\alpha x\| \leq \varepsilon_1, \|y - T_\alpha y\| \leq \varepsilon_2.$$

We can write:

$$\begin{aligned} \|x-y\| &\leq \|x - T_\alpha x\| + \|T_\alpha x - T_\alpha y\| + \|y - T_\alpha y\| \\ &\leq \|T_\alpha x - T_\alpha y\| + \varepsilon_1 + \varepsilon_2. \end{aligned}$$

Put $\varepsilon = Max\{\varepsilon_1, \varepsilon_2\}$, therfore,

$$\|x-y\| - \|T_\alpha x - T_\alpha y\| \leq \varepsilon.$$

Now by (ii) it follows that $\|x-y\| \leq \phi(\varepsilon)$, so

$$diam(AF(T_\alpha)) \leq \phi(\varepsilon). \square$$

Remark 3.3. *Condition (i) in Lemma 3.2 can be replaced by Theorem 2.4 and Theorem 2.8, the latter ensures (i). So Lemma 3.2 can be given in:*

Lemma 3.4. *Let $T : X \to X$, and $T_\alpha : X \to X$ be a map as follow:*

$$T_\alpha = \alpha I + (1-\alpha)T, \ 0 < \alpha < 1,$$

such that for $\varepsilon > 0$ the following hold:

(i) $\|T^n(x) - T^{n+k}(x)\| \to 0$ *as* $n \to \infty$ *for some* $k > 0$, $\forall x \in X$;

(ii) *foe each* $\theta > 0$ *there existe a* $\phi(\theta) > 0$ *such that*

$$\|x-y\| - \|T_\alpha(x) - T_\alpha(y)\| \quad \leq \quad \theta \Rightarrow \|x-y\| \leq \phi(\theta), \ \forall x,y \in AF(T_\alpha) \neq \emptyset.$$

Then:

$$diam(AF(T_\alpha)) \leq \phi(\varepsilon).$$

Let (X,d) be a metric space. Note that the completeness of the space is not required, as in fixed point theorems.

Definition 3.5. Let $T : X \to X$, and $T_\alpha : X \to X$ be a map as follow:

$$T_\alpha = \alpha I + (1-\alpha)T, \ 0 < \alpha < 1.$$

Map T_α is a $T_\alpha-$**a-contraction** if there exists $a \in (0,1)$ such that

$$\|T_\alpha x - T_\alpha y\| \leq a\|x-y\|, \ \forall x,y \in X.$$

Theorem 3.6. *Let $T : X \to X$, and $T_\alpha : X \to X$ be a map as follow:*

$$T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1.$$

*Suppose that the mapping T_α is a $T_\alpha-$**a-contraction**. Then:*

$$\forall \varepsilon > 0,\ AF(T_\alpha) \neq \emptyset.$$

Proof: Let $\varepsilon > 0$ and $x \in X$.

$$\begin{aligned} \|T_\alpha^n x - T_\alpha^{n+1} x\| &= \|T_\alpha(T_\alpha^{n-1}x) - T_\alpha(T_\alpha^n x)\| \\ &\leq a\|T_\alpha^{n-1}x - T_\alpha^n x\| \leq \cdots \leq a^n\|x - T_\alpha x\|. \end{aligned}$$

But $a \in (0,1)$, Therfore

$$Lim_{n\to\infty}\|T_\alpha^n x - T_\alpha^{n+1}x\| = 0,\ \forall x \in X.$$

Using Theorems 2.4 and 2.8, we find that $AF(T_\alpha) \neq \emptyset, \forall \varepsilon > 0$. □

In 1968, Kannan (see [2] [8]) proved a fixed point theorem for operators which need not be continuous. We it apply on normed spaces for T_α.

Definition 3.7. Let $T : X \to X$, and $T_\alpha : X \to X$ be a map as follow:

$$T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1.$$

Map T_α is a $T_\alpha-$Kannan operator if there exists $\beta \in (0, \frac{1}{2})$ such that

$$\|T_\alpha x - T_\alpha y\| \leq \beta[\|x - T_\alpha x\| + \|y - T_\alpha y\|],\ \forall x, y \in X.$$

Theorem 3.8. *Let $T : X \to X$, and $T_\alpha : X \to X$ be a map as follow:*

$$T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1.$$

Suppose that the mapping T_α is a $T_\alpha-$Kannan operator. Then:

$$\forall \varepsilon > 0,\ AF(T_\alpha) \neq \emptyset.$$

Proof: Let $\varepsilon > 0$ and $x \in X$.

$$\begin{aligned}\|T_\alpha^n x - T_\alpha^{n+1} x\| &= \|T_\alpha(T_\alpha^{n-1}x) - T_\alpha(T_\alpha^n x)\| \\ &\leq \beta[\|T_\alpha^{n-1}x - T_\alpha^n x\| + \|T_\alpha^n x - T_\alpha^{n+1}x\|] \\ &= \beta\|T_\alpha^{n-1}x - T_\alpha^n x\| + \beta\|T_\alpha^n x - T_\alpha^{n+1}x\|.\end{aligned}$$

Therefore $(1-\beta)\|T_\alpha^n x - T_\alpha^{n+1}x\| \leq \beta\|T_\alpha^{n-1}x - T_\alpha^n x\|$. Then

$$\begin{aligned}\|T_\alpha^n x - T_\alpha^{n+1} x\| &\leq \frac{\beta}{1-\beta}\|T_\alpha^{n-1}x - T_\alpha^n x\| \\ &\vdots \\ &\leq (\frac{\beta}{1-\beta})^n\|x - T_\alpha x\|.\end{aligned}$$

But $\beta \in (0, \frac{1}{2})$ hence $\frac{\beta}{1-\beta} \in (0,1)$. Therfore

$$Lim_{n\to\infty}\|T_\alpha^n x - T_\alpha^{n+1}x\| = 0,\ \forall x \in X.$$

Using Theorems 2.4 and 2.8, we find that $AF(T_\alpha) \neq \emptyset, \forall \varepsilon > 0$. $\square$

In 1972, Chatterjea (see [11]) considered another which again does not impose the continuity of the operator. We it apply on normed spaces for T_α.

Definition 3.9. Let $T : X \to X$, and $T_\alpha : X \to X$ be a map with $T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1$. Map T_α is a $T_\alpha-$Chatterjea operator if there exists $\beta \in (0, \frac{1}{2})$ such that

$$\|T_\alpha x - T_\alpha y\| \leq \beta[\|x - T_\alpha y\| + \|y - T_\alpha x\|],\ \forall x, y \in X.$$

Theorem 3.10. *Let $T : X \to X$, and $T_\alpha : X \to X$ be a map such as $T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1$. Suppose that the mapping T_α is a $T_\alpha-$Chatterjea operator. Then:*

$$\forall \varepsilon > 0,\ AF(T_\alpha) \neq \emptyset.$$

Proof:

Let $\varepsilon > 0$ and $x \in X$.

$$\begin{aligned} \|T_\alpha^n x - T_\alpha^{n+1} x\| &= \|T_\alpha(T_\alpha^{n-1} x) - T_\alpha(T_\alpha^n x)\| \\ &\leq \beta[\|T_\alpha^{n-1} x - T_\alpha(T_\alpha^n x)\| + \|T_\alpha^n x - T_\alpha(T_\alpha^{n-1} x)\|] \\ &= \beta[\|T_\alpha^{n-1} x - T_\alpha^{n+1} x\| + \|T_\alpha^n x - T_\alpha^n x\|] = \beta\|T_\alpha^{n-1} x - T_\alpha^{n+1} x\|. \end{aligned}$$

On the other hand

$$\|T_\alpha^{n-1} x - T_\alpha^{n+1} x\| \leq \|T_\alpha^{n-1} x - T_\alpha^n x\| + \|T_\alpha^n x - T_\alpha^{n+1} x\|.$$

Then

$$(1-\beta)\|T_\alpha^n x - T_\alpha^{n+1} x\| \leq \beta\|T_\alpha^{n-1} x - T_\alpha^n x\|,$$

hence

$$\begin{aligned} \|T_\alpha^n x - T_\alpha^{n+1} x\| &\leq \frac{\beta}{1-\beta}\|T_\alpha^{n-1} x - T_\alpha^n x\| \\ &\vdots \\ &\leq (\frac{\beta}{1-\beta})^n \|x - T_\alpha x\|. \end{aligned}$$

But $\beta \in (0, \frac{1}{2})$ hence $\frac{\beta}{1-\beta} \in (0,1)$. Therfore

$$Lim_{n\to\infty}\|T_\alpha^n x - T_\alpha^{n+1} x\| = 0,\ \forall x \in X.$$

Using Theorems 2.4 and 2.8, we find that $AF(T_\alpha) \neq \emptyset, \forall \varepsilon > 0$. □

We, by combining the three independent contraction conditions above obtain another approximate fixed point result on T_α for operators which satisfy the following.

Definition 3.11. Let $T : X \to X$, and $T_\alpha : X \to X$ be a map with $T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1$. Map T_α is a $T_\alpha-$ Zamfirescu operator if there exists $\alpha_1, \beta, \gamma \in R$, $\alpha_1 \in [0,1[, \beta \in [0, \frac{1}{2}[, \gamma \in [0, \frac{1}{2}[$ such that for all $x, y \in X$ at least one of the following is true:

(i) $\|T_\alpha x - T_\alpha y\| \leq \alpha_1 \|x - y\|$;

(ii) $\|T_\alpha x - T_\alpha y\| \leq \beta[\|x - T_\alpha x\| + \|y - T_\alpha y\|]$;

(iii) $\|T_\alpha x - T_\alpha y\| \leq \gamma[\|x - T_\alpha y\| + \|y - T_\alpha x\|]$.

Theorem 3.12. *Let $T : X \to X$, and $T_\alpha : X \to X$ be a map with $T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1$. Suppose that the mapping T_α is a $T_\alpha-$Zamfirescu. Then:*

$$\forall \varepsilon > 0,\ AF(T_\alpha) \neq \emptyset.$$

Proof: Let $x, y \in X$.

Supposing *ii*) holds, we have that:

$$\begin{aligned}
\|T_\alpha x - T_\alpha y\| &\leq \beta[\|x - T_\alpha x\| + \|y - T_\alpha y\|] \\
&\leq \beta\|x - T_\alpha x\| + \beta[\|y - x\| + \|x - T_\alpha x\| + \|T_\alpha x - T_\alpha y\|] \\
&= 2\beta\|x - T_\alpha x\| + \beta\|x - y\| + \beta\|T_\alpha x - T_\alpha y\|.
\end{aligned}$$

Thus

$$\|T_\alpha x - T_\alpha y\| \leq \frac{2\beta}{1-\beta}\|x - T_\alpha x\| + \frac{\beta}{1-\beta}\|x - y\|. \tag{3.1}$$

Supposing) holds, we have that:

$$\begin{aligned}
\|T_\alpha x - T_\alpha y\| &\leq \gamma[\|x - T_\alpha y\| + \|y - T_\alpha x\|] \\
&\leq \gamma[\|x - y) + \|y - T_\alpha y\|] + \gamma[\|y - T_\alpha y\| + \|T_\alpha y - T_\alpha x\|] \\
&= \gamma\|T_\alpha x - T_\alpha y\| + 2\gamma\|y - T_\alpha y\| + \gamma\|x - y\|.
\end{aligned}$$

Thus

$$\|T_\alpha x - T_\alpha y\| \leq \frac{2\gamma}{1-\gamma}\|y - T_\alpha y\| + \frac{\gamma}{1-\gamma}\|x - y\|. \tag{3.2}$$

Similarly:

$$\begin{aligned}
\|T_\alpha x - T_\alpha y\| &\leq \gamma[\|x - T_\alpha y\| + \|y - T_\alpha x\|] \\
&\leq \gamma[\|x - T_\alpha x\| + \|T_\alpha x - T_\alpha y\|] + \gamma[\|y - x\| + \|x - T_\alpha x\|] \\
&= \gamma\|T_\alpha x - T_\alpha y\| + 2\gamma\|x - T_\alpha x\| + \gamma\|x - y\|.
\end{aligned}$$

Then

$$\|T_\alpha x - T_\alpha y\| \leq \frac{2\gamma}{1-\gamma}\|x - T_\alpha x\| + \frac{\gamma}{1-\gamma}\|x-y\|. \tag{3.3}$$

Now looking at i), (2.1), (2.2), (2.3) we can denote:

$$\eta = max\{\alpha_1, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\},$$

and it is easy to see that $\eta \in [0,1[$.

For T satisfying at least one of the conditions i), ii), iii) we have that

$$\|T_\alpha x - T_\alpha y\| \leq 2\eta\,\|x - T_\alpha x\| + \eta\,\|x-y\|. \tag{3.4}$$

and

$$\|T_\alpha x - T_\alpha y\| \leq 2\eta\,\|y - T_\alpha y\| + \eta\,\|x-y\|. \tag{3.5}$$

hold. Using these conditions implied by i) - iii) and taking $x \in X$, we have:

$$\begin{aligned} \|T_\alpha^n x - T_\alpha^{n+1} x\| &= \|T_\alpha(T_\alpha^{n-1}x)\| - T_\alpha(T_\alpha^n x)\| \\ &\leq^{(2.4)} 2\eta\,\|T_\alpha^{n-1}x - T_\alpha(T_\alpha^{n-1}x)\| + \eta\,\|T_\alpha^{n-1}x - T_\alpha^n x\| \\ &= 3\eta\,\|T_\alpha^{n-1}x - T_\alpha^n x\|. \end{aligned}$$

Then

$$\|T_\alpha^n x - T_\alpha^{n+1} x\| \leq \cdots \leq (3\eta)^n \|x - T_\alpha x\|$$

Therfore

$$Lim_{n\to\infty}\|T_\alpha^n x - T_\alpha^{n+1} x\| = 0,\ \forall x \in X.$$

Using Theorems 2.4 and 2.8, we find that $AF(T_\alpha) \neq \emptyset, \forall \varepsilon > 0$. □

Now, we consider the contraction condition given in 2004 by V. Berinde, who also formulated a corresponding fixed point theorem, see [2], for example.

Definition 3.13. Let $T : X \to X$, and $T_\alpha : X \to X$ be a map with $T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1$. Map T_α is a $T_\alpha-$weak contraction operator if there exists $\alpha_1 \in]0,1[$ and $L \geq 0$ such that

$$\|T_\alpha x - T_\alpha y\| \leq \alpha_1\|x-y\| + L\|y - T_\alpha x\|,\ \forall x,y \in X.$$

Theorem 3.14. *Let $T : X \to X$, and $T_\alpha : X \to X$ be a map with $T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1$. Suppose that the mapping T_α is a $T_\alpha-$weak contraction. Then:*

$$\forall \varepsilon > 0,\ AF(T_\alpha) \neq \emptyset.$$

Proof: Let $x \in X$.

$$\begin{aligned} \|T_\alpha^n x - T_\alpha^{n+1} x\| &= \|T_\alpha(T_\alpha^{n-1} x\| - T_\alpha(T_\alpha^n x\| \\ &\leq \alpha_1 \|T_\alpha^{n-1} x - T_\alpha^n x\| + L\|T_\alpha^n x - T_\alpha^n x\| \\ &= \alpha_1 \|T_\alpha^{n-1} x - T_\alpha^n x\| \leq \cdots \leq \alpha_1^n \|x - T_\alpha x\|. \end{aligned}$$

But $\alpha_1 \in]0,1[$. Therfore

$$Lim_{n\to\infty} \|T_\alpha^n x - T_\alpha^{n+1} x\| = 0,\ \forall x \in A \cup B.$$

Using Theorems 2.4 and 2.8, we find that $AF(T_\alpha) \neq \emptyset, \forall \varepsilon > 0$. $\square$

In 1974, Ciric [12] obtained a contraction condition for which the map satisfying it is still a Picard operator. We it apply on normed spaces which show that T_α has approximate fixed point property.

Definition 3.15. Let $T : X \to X$, and $T_\alpha : X \to X$ be a map with $T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1$. Map T_α is a $T_\alpha-$quasi contraction if there exists $h \in]0,1[$ such that

$$\|T_\alpha x - T_\alpha y\| \leq h \cdot max\{\|x-y\|, \|x - T_\alpha x\|, \|y - T_\alpha y\|, \|x - T_\alpha y\|, \|y - T_\alpha x\|\},\ \forall x, y \in X.$$

Corollary 3.16. *Let $T : X \to X$, and $T_\alpha : X \to X$ be a map with $T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1$. Suppose that the mapping T_α is a $T_\alpha-$quasi contraction with $0 < h < \frac{1}{2}$. Then:*

$$\forall \varepsilon > 0,\ AF(T_\alpha) \neq \emptyset.$$

Proof: By Proposition 3 of [3] any $T_\alpha-$quasi contraction with $0 < h < \frac{1}{2}$ is a weak contraction. Therfore by Theorem 3.14, $AF(T_\alpha) \neq \emptyset, \forall \varepsilon > 0$.

4. Map T_α and diameter approximate fixed point

For the same operators we have studied in the previous section, we will formulate and prove using Lemma 3.2, in order to obtain results for diameter approximate fixed point with map T_α.

Theorem 4.1. [14] *Let $T : X \to X$, and $\varepsilon > 0$. If there exists a $c \in [0,1]$ such that for all $x,y \in X$ $\|Tx - Ty\| \le c\|x-y\|$, and $T_\alpha : X \to X$ be a map with $T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1$. Then*

$$diam(AF(T_\alpha) \le \frac{2\varepsilon}{1-c}, \forall \varepsilon > 0.$$

Theorem 4.2. *Let $T : X \to X$, and $T_\alpha : X \to X$ be a map with $T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1$. Suppose that the mapping T_α is a $T_\alpha-$Kannan operator. Then:*

$$diam(AF(T_\alpha) \le 2\varepsilon(1+\beta), \forall \varepsilon > 0.$$

Proof: Let $\varepsilon > 0$. Condition 1) in Lemma 3.2, is satisfied, as one can see in the proof of Theorem 3.8, we only verify that condition 2) in Lemma 3.2, holds.

Let $\theta > 0$ and $x,y \in AF(T_\alpha))$ and assume that $\|x-y\| - \|T_\alpha x - T_\alpha y\| \le \theta$. Then:

$$\|x-y\| \quad \le \quad \beta[\|x - T_\alpha x\| + \|x - T_\alpha y\|] + \theta.$$

As $x,y \in AF(T_\alpha))$, we know that $\|x - T_\alpha x\| \le \varepsilon, \|y - T_\alpha y\| \le \varepsilon$, therfore, $\|x-y\| \le 2\beta\varepsilon + \theta$. So for every $\theta > 0$ there exists $\phi(\theta) = \theta + 2\alpha\varepsilon > 0$ such that

$$\|x-y\| - \|T_\alpha x - T_\alpha y\| \le \theta \Rightarrow \|x-y\| \le \phi(\theta).$$

Now by Lemma 2.2, it follows that

$$diam AF(T_\alpha) \le \phi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$diam(AF(T_\alpha)) \le 2\varepsilon(1+\beta), \forall \varepsilon > 0. \square$$

Theorem 4.3. *Let $T : X \to X$, and $T_\alpha : X \to X$ be a map with $T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1$. Suppose that the mapping T_α is a $T_\alpha-$Chatterjea operator. Then for every $\varepsilon > 0$,*

$$diam(AF(T_\alpha) \leq \frac{2\varepsilon(1+\beta)}{1-2\beta}.$$

Proof: Let $\varepsilon > 0$. We will only verify that condition 2) in Lemma 3.2 holds.

Let $\theta > 0$ and $x, y \in AF(T_\alpha)$ and assume that $\|x-y\| - \|T_\alpha x - T_\alpha y\| \leq \theta$. Then:

$$\begin{aligned} \|x-y\| &\leq \beta[\|x-T_\alpha y\| + \|y-T_\alpha x\|] + \theta \\ &\leq \beta\|x-T_\alpha y\| + \beta\|y-T_\alpha x\| + \theta \\ &\leq \beta[\|x-y\| + \|y-T_\alpha y\|] + \beta[\|y-x\| + \|x-T_\alpha x\|] + \theta. \end{aligned}$$

As $x, y \in AF(T_\alpha)$, we know that $\|x-T_\alpha x\| \leq \varepsilon, \|y-T_\alpha y\| \leq \varepsilon$. Therfore, $\|x-y\| \leq 2\beta\|x-y\| + 2\beta\varepsilon + \theta$. Then $(1-2\beta)\|x-y\| \leq 2\beta\varepsilon + \theta$. therfore,

$$\|x-y\| \leq \frac{2\beta\varepsilon + \theta}{1-2\beta}.$$

So for every $\theta > 0$ there exists $\phi(\theta) = \frac{2\beta\varepsilon+\theta}{1-2\beta} > 0$ such that

$$\|x-y\| - \|T_\alpha x, T_\alpha y\| \leq \theta \Rightarrow \|x-y\| \leq \phi(\theta).$$

Now by Lemma 3.2, it follows that

$$diam(AF(T_\alpha) \leq \phi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$diam(AF(T_\alpha) \leq 2\varepsilon\frac{1+\beta}{1-2\beta}, \forall \varepsilon > 0. \square$$

Theorem 4.4. *Let $T : X \to X$, and $T_\alpha : X \to X$ be a map with $T_\alpha = \alpha I + (1-\alpha)T,\ 0 < \alpha < 1$. Suppose that the mapping T_α is a $T_\alpha-$Zamfirescu operator. Then for every $\varepsilon > 0$,:*

$$diam(AF(T_\alpha) \leq \frac{2\varepsilon(1+\eta)}{1-\eta},$$

where $\eta = max\{\alpha_1, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$, and α_1, β, γ as in Definition 3.11.

Proof: In the proof of Theorem 3.12, we have already shown that if T_α satisfies at least one of the conditions $i), ii) or\ iii)$ from Definition 3.11, then

$$\|T_\alpha x - T_\alpha y\| \leq 2\eta \|x - T_\alpha x\| + \eta \|x - y\|$$

and

$$\|T_\alpha x - T_\alpha y\| \leq 2\eta \|y - T_\alpha y\| + \eta \|x - y\|$$

hold.

Let $\varepsilon > 0$. We will only verify that condition 2) in Lemma 3.2 is satisfied, as 1) holds, see the Proof of Theorem 3.12.

Let $\theta > 0$ and $x, y \in AF(T_\alpha)$ and assume that $\|x - y\| - \|T_\alpha x - T_\alpha y\| \leq \theta$. Then

$$\|x - y\| - \|T_\alpha x - T_\alpha y\| \quad \leq \quad 2\eta \|x - T_\alpha x\| + \eta \|x - y\| + \theta \Rightarrow$$

$$(1-\eta)\|x - y\| \leq 2\eta\varepsilon + \theta$$

$$\|x - y\| \leq \frac{\theta + 2\eta\varepsilon}{1-\eta}.$$

So for every $\theta > 0$ there exists $\phi(\theta) = \frac{\theta+2\eta\varepsilon}{1-\eta} > 0$ such that

$$\|x - y\| - \|T_\alpha x - T_\alpha y\| \leq \theta \Rightarrow \|x - y\| \leq \phi(\theta).$$

Now by Lemma 3.2, it follows that

$$diam(AF(T_\alpha)) \leq \phi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$diam(AF(T_\alpha)) \leq \frac{2\varepsilon(1+\eta)}{1-\eta}, \forall \varepsilon > 0. \ \square$$

Conflict of Interests

The authors declare that there is no conflict of interests.

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