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SOME COMMON FIXED POINT THEOREMS ON S-METRIC SPACE

MUJEEB UR RAHMAN[1,*], MUHAMMAD SARWAR[2] AND MUHIB UR RAHMAN[3]

[1]Department of Mathematics, Government PG Jehanzeb College, Saidu Sharief, Swat, Pakistan

[2]Department of Mathematics University of Malakand, Chakdara Dir(L), Pakistan

[3]Department of Electrical Engineering, Military College of Signals, NUST, Rawalpindi, Pakistan

Abstract. The aim of this manuscript is to study common fixed point theorems for compilable and weakly compatible mappings in the context of S-metric space. A fixed point theorem for three self-mappings is also establish in S-metric space. The derived results generalizes some well known results from the literature in S-metric space.

Keywords: S-metric space; Cauchy sequence; fixed point; compatible mappings and property $(E.A)$.

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1. Introduction and Preliminaries

Jungck [4] initiated the study of fixed points for commuting maps. Further, Jungck [5] made a generalization of commuting maps by introducing the notion of compatible mappings which was further generalized by the same author in [6]. Pant [9] initiated the study of non-compatible maps and introduced pointwise R-weak commutativity of mappings. Aamri and Moutawakil [1] gave a notion of $E.A.$ property which generalized the concept of non-compatible mappings in

*Corresponding author

E-mail address: mujeeb846@yahoo.com

metric space and proved some common fixed point theorems for non-compatible mappings under strict contractive conditions.

Mustafa and Sims [7] introduced the concept of G-metric space as a generalization of metric space. Choudhury et al.[3], established some fixed point theorems for compatible mapping in G-metric space. Mustafa et al.[8] studied sufficient condition for the existence of fixed point in G-metric space. Sedghi et al. [10] initiated the idea of D^*-metric space which is a probable modification of D-metric space.

Recently, Sedghi, Shobe and Aliouche [11] introduced the concept of a S-metric space as a generalization of a G-metric space [7] and D^* -metric space [10]. They studied some of their properties and proved a fixed point theorem for a self-mapping on a complete S-metric space. Moreover, Sedghi et al. [12] investigated some fixed point theorems on S-metric spaces .

In this article, we have discussed some common fixed point theorems in S-metric space. Examples are given in the support of our constructed results.

The following denitions will be needed in the sequel.

Definition 1.1. [11]. Let X be a non-empty set. An S-metric is a function $S : X \times X \times X \to [0,\infty)$ satisfying the following conditions for all $x,y,z,a \in X$:

S_1) $S(x,y,z) = 0$ if and only if $x = y = z$;

S_2) $S(x,y,z) \leq S(x,x,a)+S(y,y,a)+S(z,z,a)$. The pair (X,S) is called S-metric space.

Example 1.1. [11]. Let X=$\mathbb{R}$ the distance function $S : X \times X \times X \to [0,\infty)$ defined by

$$S(x,y,z) = |x-z|+|y-z| \quad \text{for all } \ x,y,z \in X$$

is an S-metric on X.

Every G-metric and D^*-metric spaces are S-metric spaces but the converse is not true which is clear from the above example. For more examples and counter examples see [11, 12].

Definition 1.2. [11]. Let (X,S) be an S-metric space. A sequence $\{x_n\}$ in X converges to $x \in X$ if

$$S(x_n,x_n,x) \to 0 \ \text{ as } \ n \to \infty.$$

We write $x_n \to x$ for brevity.

Definition 1.3. [11]. Let (X,S) be an S-metric space. A sequence $\{x_n\}$ in X is called Cauchy sequence if for $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n,m \geq n_0$ we have

$$S(x_n,x_n,x_m) < \varepsilon$$

Definition 1.4. [11]. An S-metric space (X,S) is said to be complete if every Cauchy sequence in X converges in X.

Lemma 1. [11]. *Limit of the convergent sequence in S-metric space is unique.*

Lemma 2. [11]. *S-metric is jointly continuous on all three variables.*

Lemma 3. [11]. *In an S-metric space, we have*

$$S(x,x,y) = S(y,y,x) \quad \textit{for all} \;\; x,y \in X$$

Definition 1.5. [11]. Let (X,S) be an S-metric space. A mapping $T : X \to X$ is called contraction if

$$S(Tx,Tx,Ty) \leq \alpha \cdot S(x,x,y) \;\text{ for all } x,y \in X \text{with } \alpha \in [0,1).$$

Sedghi et al. [11] proved the following result for contraction mapping in S-metric space.

Theorem 1.1. *Let (X,S) be a complete S-metric space. A mapping $T : X \to X$ be a contraction, then T has a unique fixed point.*

Choudhury et al. [3] proved the following theorems for a pair of compatible, weakly compatible mappings satisfying property (E,A) in G-metric space.

Theorem 1.2. *Let (X,G) be a complete G-metric space and $A,B : X \to X$ be two self-mappings satisfying the following conditions*

(1) $A(X) \subseteq B(X)$;

(2) A or B is continuous;

(3) $S(Ax,Ay,Az) \leq \alpha \cdot S(Ax,By,Bz) + \beta \cdot S(Bx,Ay,Bz) + \gamma \cdot S(Bx,By,Az)$;

for all $x,y,z \in X$ and $\alpha,\beta,\gamma \geq 0$ with $\alpha + 3\beta + 3\gamma < 1$. Then A and B have a unique common fixed point if A and B are compatible mappings.

Theorem 1.3. *Let (X,G) be an G-metric space and $A,B : X \to X$ be a pair of weakly compatible self-mappings satisfying the following conditions:*

(1) A and B satisfy property (E. A);

(2) $B(X)$ is a closed subspace of X;

(3) $S(Ax,Ay,Az) \leq \alpha \cdot S(Ax,By,Bz) + \beta \cdot S(Bx,Ay,Bz) + \gamma \cdot S(Bx,By,Az)$;

for all $x,y,z \in X$ and $\alpha,\beta,\gamma \geq 0$ with $\alpha+3\beta+\gamma < 1$. Then A and B have a unique common fixed point.

Abbas et al. [2] proved the following result in G-metric space.

Theorem 1.4. *Let f,g and h be self-mappings on complete G-metric space X satisfying*

$$G(fx,gy,hz) \leq \alpha \cdot G(x,y,z) + \beta \cdot [G(fx,x,x) + G(y,gy,y) + G(z,z,hz)]$$

$$+\gamma \cdot [G(fx,y,z) + G(x,gy,z) + G(x,y,hz)]$$

for all $x,y,z \in X$, where $\alpha,\beta,\gamma \geq 0$ with $\alpha+3\beta+4\gamma < 1$. Then f,g and h have a unique common fixed point in X. Moreover, any fixed point of f is the fixed point of g and h and conversely.

2. Main Results

Before going to prove the main results we define the notions of compatibility, weak compatibility and property $(E.A)$ for mappings in the setting of S-metric space.

Definition 2.1. Let A and B be two self-mappings on an S-metric space (X,S). The mappings A and B are said to be compatible if

$$\lim_{n\to\infty} S(ABx_n, ABx_n, BAx_n) = 0$$

whenever there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$.

Definition 2.2. Any point $x \in X$ is called coincidence point of the mappings A and B if $Ax = Bx$ and we called $u = Ax = Bx$ is a point of coincidence of A and B.

Definition 2.3. Self-mappings A and B are said to be weakly compatible if they commute at all of their coincidence points.

In [1] Aamri and Moutawakil introduced a generalization of non-compatible mappings as property $(E.A)$ in metric spaces as follows.

Definition 2.4. Let A and B be two self-maps on a metric space (X,d). The pair (A,B) is said to satisfy property (E.A.), if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t \text{ for some } t \in X.$$

In similar manner we use property $(E.A)$ in S-metric space to prove common fixed point theorem for a pair of weakly compatible mappings.

Example 2.1. Let $X = [0,1]$ and S-metric on X^3 is defined by

$$S(x,y,z) = |x-z| + |y-z| \text{ for all } x,y,z \in X.$$

Let us define $Ax = x$ and $Bx = \frac{x}{4}$. Consider the sequence $x_n = \frac{1}{n}$ for all $n \in N$. We see that the mappings A and B are compatible as

$$\lim_{n\to\infty} S(ABx_n, ABx_n, BAx_n) = 0$$

and $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = o \in X$.

Example 2.2. Let $X = \mathbb{R}$, with the usual S-metric. If we define A and B as follows.

$$Ax = [x] \quad \text{and} \quad Bx = \begin{cases} -2 & \text{if } x \leq 0 \\ 0 & \text{if } 0 < x < 2 \\ 2 & \text{if } x \geq 2, \end{cases}$$

where $[x]$ denotes the integral part of x. Then the mappings A and B are weakly compatible as they commute at their coincidence points i.e $x = \pm 2$.

Theorem 2.1. *Let (X,S) be a complete S-metric space and $A,B : X \to X$ be two self-mappings satisfying the following conditions*

(1) $A(X) \subseteq B(X)$;

(2) A or B is continuous;

(3) $S(Ax,Ay,Az) \leq \alpha \cdot S(Bx,By,Bz) + \beta \cdot S(Ax,Ay,Bz) + \gamma \cdot S(Bx,By,Az)$*;*

for all $x,y,z \in X$ *and* $\alpha, \beta, \gamma \geq 0$ *with* $\alpha + \beta + 3\gamma < 1$. *Then A and B have a unique common fixed point if A and B are compatible mappings.*

Proof. Let x_0 be arbitrary in X using (1), we define a sequence $\{y_n\}$ in X by the rule

$$y_n = Ax_n = Bx_{n+1} \text{ for } n = 0,1,2,.....$$

Now to show that $\{y_n\}$ is a Cauchy sequence consider

$$S(y_n,y_n,y_{n+1}) = S(Ax_n,Ax_n,Ax_{n+1}).$$

Using condition (3) we have

$$\leq \alpha \cdot S(Bx_n,Bx_n,Bx_{n+1}) + \beta \cdot S(Ax_n,Ax_n,Bx_{n+1}) + \gamma \cdot S(Bx_n,Bx_n,Ax_{n+1}).$$

$$\leq \alpha \cdot S(y_{n-1},y_{n-1},y_n) + \beta \cdot S(y_n,y_n,y_n) + \gamma \cdot S(y_{n-1},y_{n-1},y_{n+1}).$$

Using (S_1), (S_2) and Lemma 3 we have

$$\leq \alpha \cdot S(y_{n-1},y_{n-1},y_n) + 2\gamma \cdot S(y_{n-1},y_{n-1},y_n) + \gamma S(y_n,y_n,y_{n+1})$$

$$S(y_n,y_n,y_{n+1}) \leq \frac{\alpha + 2\gamma}{1-\gamma} \cdot S(y_{n-1},y_{n-1},y_n)$$

$$S(y_n,y_n,y_{n+1}) \leq k \cdot S(y_{n-1},y_{n-1},y_n)$$

where $k = \frac{\alpha+2\gamma}{1-\gamma} < 1$, because $\alpha + \beta + 3\gamma < 1$. Also

$$S(y_n,y_n,y_{n+1}) \leq k^2 \cdot S(y_{n-2},y_{n-2},y_{n_1}).$$

Continuing the same procedure we have

$$S(y_n,y_n,y_{n+1}) \leq k^n \cdot S(y_0,y_0,y_1). \tag{1}$$

Next to show that $\{y_n\}$ is a Cauchy sequence, for this consider $m,n \in N$ with $m > n$ and using (S_2) we have

$$S(y_n,y_n,y_m) \leq S(y_n,y_n,y_{n+1}) + S(y_n,y_n,y_{n+1}) + S(y_m,y_m,y_{n+1})$$

$$\leq 2S(y_n,y_n,y_{n+1})+S(y_{n+1},y_{n+1},y_m) \text{ (by use of Lemma 3).}$$

Repeating (S_2), Lemma 3 and using Eq. (1), we get the following

$$S(y_n,y_n,y_m) \leq 2(k^n+k^{n+1}+k^{n+2}+......)\cdot S(y_0,y_0,y_1)$$

$$\leq 2\frac{k^n}{1-k}\cdot S(y_0,y_0,y_1).$$

Taking limit $m,n \to \infty$, we have

$$\lim_{n,m\to\infty} S(y_n,y_n,y_m)=0.$$

Hence $\{y_n\}$ is a S-Cauchy sequence in X. Since (X,S) is complete S-metric space so there must exists z in X such that

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_{n+1} = z.$$

Also, since A or B is continuous for definiteness we take that B is continuous therefore

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} BAx_n = \lim_{n\to\infty} BBx_{n+1} = Bz.$$

Further, A and B are compatible, thus

$$\lim_{n\to\infty} S(ABx_n,ABx_n,BAx_n)=0.$$

Implies

(2) $$\lim_{n\to\infty} ABx_n = Bz.$$

Now using (3) we have

$$S(ABx_n,ABx_n,Ax_n) \leq \alpha\cdot S(BBx_n,BBx_n,Bx_n)+$$

$$\beta\cdot S(ABx_n,ABx_n,Bx_n)+\gamma\cdot S(BBx_n,BBx_n,Ax_n).$$

Taking limit $n\to\infty$, and using the above information we have

$$S(Bz,Bz,z) \leq \alpha\cdot S(Bz,Bz,z)+\beta\cdot S(Bz,Bz,z)+\gamma\cdot S(Bz.Bz,z)$$

$$S(Bz,Bz,z) \leq (\alpha+\beta+\gamma)\cdot S(Bz,Bz,z).$$

The above inequality is possible only if $S(Bz,Bz,z)=0$ iff $Bz=z$. Now again using (3), we have

$$S(Ax_n,Ax_n,Az)\leq\alpha\cdot S(Bx_n,Bx_n,Bz)+$$

$$\beta\cdot S(Ax_n,Ax_n,Bz)+\gamma\cdot S(Bx_n,Bx_n,Az).$$

Taking limit $n\to\infty$, using the above information and (S_1) we have

$$S(z,z,Az)\leq\alpha\cdot S(z,z,z)+\beta\cdot S(z,z,z)+\gamma\cdot S(z,z,Az)$$

$$S(z,z,Az)\leq\gamma\cdot S(z,z,Az).$$

The above inequality is possible only if $S(z,z,Az)=0$ iff $Az=z$. Thus z is a common fixed point of A and B.

Uniqueness. Assume that $z\neq z_0$ be two distinct common fixed points of A and B. Then consider,

$$S(z,z,z_0)=S(Az,Az,Az_0)$$

$$\leq\alpha\cdot S(Az,Az,Az_0)+\beta\cdot S(Az,Az,Bz_0)+\gamma\cdot S(Bz,Bz,Az_0)$$

$$\leq\alpha\cdot S(z,z,z_0)+\beta\cdot S(z,z,z_0)+\gamma\cdot S(z,z,z_0)$$

$$S(z,z,z_0)\leq(\alpha+\beta+\gamma)\cdot S(z,z,z_0).$$

The above inequality is possible only if $S(z,z,z_0)=0$ iff $z=z_0$. Thus common fixed point of A and B is unique. □

Corollary 2.1. *Let (X,S) be a complete S-metric space and $A,B:X\to X$ be two compatible self-mappings satisfying the following conditions*

(1) $A(X)\subseteq B(X)$;

(2) A or B is continuous;

(3) $S(Ax,Ay,Az)\leq\alpha\cdot S(Bx,By,Bz)$;

for all $x,y,z\in X$ with $0\leq\alpha<1$. Then A and B have a unique common fixed point.

Corollary 2.2. *Let (X,S) be a complete S-metric space and $A,B:X\to X$ be two compatible self-mappings satisfying the following conditions*

(1) $A(X)\subseteq B(X)$;

(2) A or B is continuous;

(3) $S(Ax,Ay,Az) \leq \alpha \cdot S(Bx,By,Bz) + \beta \cdot S(Ax,Ay,Bz)$;

for all $x,y,z \in X$ *and* $\alpha, \beta \geq 0$ *with* $\alpha + \beta < 1$. *Then A and B have a unique common fixed point.*

Now we give an example to illustrate Theorem 2.1 and its corollaries.

Example 2.3. let $X = [0,1]$ with usual S-metric as defined in Example 1, where the compatible self-mappings on X are defined by

$$Ax = \frac{x}{6} \text{ and } Bx = \frac{x}{2}.$$

Here, we see that $A(X) \subseteq B(X)$ and B is continuous also

$$S(Ax,Ay,Az) = |\frac{x}{6} - \frac{z}{6}| + |\frac{y}{6} - \frac{z}{6}|$$
$$\leq \frac{1}{3}\left(|\frac{x}{2} - \frac{z}{2}| + |\frac{y}{2} - \frac{z}{2}|\right) = \frac{1}{3} \cdot S(Bx,By,Bz).$$

Hold for all $x,y,z \in X$ and $\alpha = \frac{1}{3}$ and $\beta = \gamma = 0$, having 0 is the unique common fixed point of mappings A and B.

Theorem 2.2. *Let* (X,S) *be an S-metric space and* $A,B : X \to X$ *be a pair of weakly compatible self-mappings satisfying the following conditions*

(1) A and B satisfy property (E. A);

(2) $B(X)$ *is a closed subspace of X;*

(3) $S(Ax,Ay,Az) \leq \alpha \cdot S(Bx,By,Bz) + \beta \cdot S(Ax,Ay,Bz) + \gamma \cdot S(Bx,By,Az)$;

for all $x,y,z \in X$ *and* $\alpha, \beta, \gamma \geq 0$ *with* $\alpha + \beta + \gamma < 1$. *Then A and B have a unique common fixed point.*

Proof. Since A and B satisfy property $(E.A)$ therefore there must exist a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t \text{ for some } t \in X.$$

Also, since $B(X)$ is a closed subspace of X, so every convergent sequence in $B(X)$ has a limit point in $B(X)$. Therefore we have $t = Bp$ for some $p \in X$. Now using condition (3) we have

$$S(Ap,Ap,Ax_n) \leq \alpha \cdot S(Bp,Bp,Bx_n) + \beta \cdot S(Ap,Ap,Bx_n) + \gamma \cdot S(Bp,Bp,Ax_n).$$

Taking limit $n \to \infty$, we have

$$S(Ap,Ap,t) \leq \alpha \cdot S(t,t,t) + \beta \cdot S(Ap,Ap,t) + \gamma \cdot S(t,t,t).$$

Using (S_1), we get

$$S(Ap,Ap,t) \leq \beta \cdot S(Ap,Ap,t).$$

The above inequality is possible only if $S(Ap,Ap,t) = 0$ iff $Ap = t$. This prove that $Ap = t = Bp$, i.e that p is coincidence point of A and B. But since A and B are weakly compatible pair therefore

$$At = ABp = BAp = Bt.$$

Again from condition (3) we have

$$S(At,At,Ap) \leq \alpha \cdot S(Bt,Bt,Bp) + \beta \cdot S(At,At,Bp) + \gamma \cdot S(Bt,Bt,Ap).$$

Since $At = Bt$ and $Ap = t = Bp$, using these facts in the above inequality we have

$$S(At,At,t) \leq \alpha \cdot S(At,At,t) + \beta \cdot S(At,At,t) + \gamma \cdot S(At,At,t)$$

$$S(At,At,t) \leq (\alpha + \beta + \gamma) \cdot S(At,At,t).$$

The above inequality is possible only if $S(At,At,t) = 0$ iff $At = t$ iff $Bt = t$. Hence t is common fixed point of A and B.

Uniqueness. Assume that $t \neq t_0$ be two distinct common fixed points of A and B. Then consider

$$S(t,t,t_0) = S(At,At,At_0)$$

$$\leq \alpha \cdot S(Bt,Bt,Bt_0) + \beta \cdot S(At,At,Bt_0) + \gamma \cdot S(Bt,Bt,At_0)$$

$$\leq \alpha \cdot S(t,t,t_0) + \beta \cdot S(t,t,t_0) + \gamma \cdot S(t,t,t_0)$$

$$S(t,t,t_0) \leq (\alpha + \beta + \gamma) \cdot S(t,t,t_0).$$

The above inequality is possible only if $S(t,t,t_0) = 0$ iff $t = t_0$. Thus common fixed point of A and B is unique. □

Corollary 2.3. *Let (X,S) be an S-metric space and $A,B : X \to X$ be a pair of weakly compatible self-mappings satisfying the following conditions*

(1) A and B satisfy property (E. A);

(2) $B(X)$ *is a closed subspace of* X*;*

(3) $S(Ax,Ay,Az) \leq \alpha \cdot S(Bx,By,Bz)$*;*

for all $x,y,z \in X$ *with* $0 \leq \alpha < 1$*. Then* A *and* B *have a unique common fixed point.*

Corollary 2.4. *Let* (X,S) *be an S-metric space and* $A,B : X \to X$ *be a pair of weakly compatible self-mappings satisfying the following conditions*

(1) A *and* B *satisfy property (E. A);*

(2) $B(X)$ *is a closed subspace of* X*;*

(3) $S(Ax,Ay,Az) \leq \alpha \cdot S(Bx,By,Bz) + \beta \cdot S(Ax,Ay,Bz)$*;*

for all $x,y,z \in X$ *and* $\alpha,\beta \geq 0$ *with* $\alpha+\beta < 1$*. Then* A *and* B *have a unique common fixed point.*

Now we prove a common fixed point theorem for three self-mappings in S-metric space.

Theorem 2.3. *Let* f,g *and* h *be self-mappings on complete S-metric space* X *satisfying*

$$S(fx,gy,hz) \leq \alpha \cdot S(x,y,z) + \beta \cdot S(y,gy,hz) + \gamma \cdot S(fx,y,y) \tag{3}$$

for all $x,y,z \in X$*, where* $\alpha,\beta,\gamma \geq 0$ *with* $\alpha+\beta+\gamma < 1$*. Then* f,g *and* h *have a unique common fixed point in* X*. Moreover, any fixed point of* f *is the fixed point of* g *and* h*.*

Proof. Firstly we have to show that any fixed point of f is the fixed point of g and h. let p be afixed point of f i.e $fp = p$ then consider

$$S(p,gp,hp) = S(fp,gp,hp)$$

$$\leq \alpha \cdot S(p,p,p) + \beta \cdot S(p,gp,hp) + \gamma \cdot S(fp,p,p)$$

$$S(p,gp,hp) \leq \beta \cdot S(p,gp,hp).$$

The above inequality is possible only if

$$S(p,gp,hp) = 0 \quad \Leftrightarrow \quad p = hp = gp.$$

Hence every fixed point of f is the fixed point of g and h.

Now we have to show that f,g and h have a unique common fixed point in X, for this purpose we construct a sequence $\{x_n\}$ in X by the rule

$$x_{3n+1} = fx_{3n},\ x_{3n+2} = gx_{3n+1},\ x_{3n+3} = hx_{3n+2} \text{ for all } n = 0,1,2,......$$

Consider

$$S(x_{3n+1}, x_{3n+2}, x_{3n+3}) = S(fx_{3n}, gx_{3n+1}, hx_{3n+2}.$$

Using (3) we have

$$\leq \alpha \cdot S(x_{3n}, x_{3n+1}, x_{3n+2}) + \beta \cdot S(x_{3n+1}, gx_{3n+1}, hx_{3n+2}) + \gamma \cdot S(fx_{3n}, x_{3n+1}, x_{3n+1})$$

$$\leq \alpha \cdot S(x_{3n}, x_{3n+1}, x_{3n+2}) + \beta \cdot S(x_{3n+1}, x_{3n+2}, x_{3n+3}) + \gamma \cdot S(x_{3n+1}, x_{3n+1}, x_{3n+1}).$$

Simplification yields

$$S(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq \frac{\alpha}{1-\beta} \cdot S(x_{3n}, x_{3n+1}, x_{3n+2}).$$

Let $h = \frac{\alpha}{1-\beta} < 1$ because $\alpha + \beta + \gamma < 1$, so the above inequality take the form

$$S(x_{3n+1}, x_{3n+2}, x_{3n+3}) \leq h \cdot S(x_{3n}, x_{3n+1}, x_{3n+2}).$$

Also

$$S(x_{3n}, x_{3n+1}, x_{3n+2}) \leq h \cdot S(x_{3n-1}, x_{3n}, x_{3n+1}).$$

Continuing the same procedure one can have

$$S(x_n, x_{n+1}, x_{n+2}) \leq h^n \cdot S(x_0, x_1, x_2).$$

Following similar procedure like Theorem 2.1 we can show that $\{x_n\}$ is a Cauchy sequence in complete S-metric space, so there must exists $u \in X$ such that

$$\lim_{n\to\infty} x_n = u.$$

Also the subsequences $x_{3n},\ x_{3n+1},\ x_{3n+2}$ and x_{3n+3} are convergent to u.

Next to show that u is the fixed point of f for this aim consider

$$S(fu, x_{3n+2}, x_{3n+3}) = S(fu, gx_{3n+1}, hx3n+2).$$

$$\leq \alpha \cdot S(u, x_{3n+1}, x_{3n+2}) + \beta \cdot S(x_{3n+1}, gx_{3n+1}, hx_{3n+2}) + \gamma \cdot S(fu, x_{3n+1}, x_{3n+1})$$

$$\leq \alpha \cdot S(u, x_{3n+1}, x_{3n+2}) + \beta \cdot S(x_{3n+1}, x_{3n+2}, x_{3n+3}) + \gamma \cdot S(fu, x_{3n+1}, x_{3n+1}).$$

Taking limit $n \to \infty$, we have

$$S(fu, u, u) \leq \gamma \cdot S(fu, u, u).$$

The above inequality is possible only if

$$S(fu,u,u) = 0 \;\Leftrightarrow\; fu = u.$$

Hence u is the fixed point of f and so the fixed point of g and h. Thus u is the common fixed point of f, g and h.

Uniqueness. Let $u \neq v$be two distinct common fixed points of f, g and h. To show that $u = v$ consider

$$S(u,v,v) = S(fu,gv,hv)$$
$$\leq \alpha \cdot S(u,v,v) + \beta \cdot S(v,gv,hv) + \gamma \cdot S(fu,v,v)$$
$$\leq \alpha \cdot S(u,v,v) + \beta \cdot S(v,v,v) + \gamma \cdot S(u,v,v)$$
$$S(u,v,v) \leq (\alpha + \gamma) \cdot S(u,v,v).$$

The above inequality is possible only if

$$S(u,v,v) = 0 \;\Leftrightarrow\; u = v.$$

Hence f, g and h have a unique common fixed point. □

Corollary 2.5. *Let f, g and h be self-mappings on complete S-metric space X satisfying*

$$S(fx,gy,hz) \leq \alpha \cdot S(x,y,z)$$

for all $x,y,z \in X$, with $0 \leq \alpha < 1$. Then f, g and h have a unique common fixed point in X. Moreover, any fixed point of f is the fixed point of g and h.

Example 2.4. Let $X = [0,1]$ with usual S-metric and the self-mappings f, g and h are defined by

$$fx = \frac{x}{8}, \; gx = \frac{x}{4}, \; \text{ and } \; hx = \frac{x}{2}.$$

Then

$$S(fx,gy,hz) = |\frac{x}{8} - \frac{z}{2}| + |\frac{y}{4} - \frac{z}{2}|$$
$$= \frac{1}{2}\left(|\frac{x}{4} - z| + |\frac{y}{2} - z|\right) \leq \frac{1}{2}(|x-z| + |y-z|).$$

Hence

$$S(fx,gy,hz) \leq \alpha \cdot S(x,y,z).$$

Satisfy all the conditions of Corollary 2.5 for $\frac{1}{2} \leq \alpha < 1$ having $x = 0$ is the unique common fixed point of f, g and h.

Conflict of Interests

The authors declare that there is no conflict of interests.

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References

[1] M. Aamri and D.EI. Moutawakil, Some new common fixed point theorems under strict contractive conditions, *Journal of Mathematical Analysis and Applications*, **270**(2002), 181-188.

[2] M. Abbas, T. Nazir and P. Vetro, Common fixed point results for three mappings in G-metric space, *Faculty of Science and Mathematics, University of Nis Serbia*, **25**(2011), 1-17.

[3] B.S. Choudhury, S. Kumar, Asha and K. Das, Some fixed point theorems in G-metric spaces, *Mathematical Sciences Letters*, **1**(2012), 25-31.

[4] G. Jungck, Commuting mappings and xed points, Amer. Math. Monthly 83 (1976) 261263.

[5] G. Jungck, Compatible mappings and common fixed points, *Int. J. Math. Math. Sci.*, **9**(1986), 771-779.

[6] G. Jungck, Common fixed points for non-continuous non-self mappings on non-metric spaces, *Far East J. Math. Sci.*, **4**(1996), 199-212.

[7] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, *Journal of Nonlinear Convex Analysis*, **7**(2006), 289-297.

[8] Z. Mustafa, H. Obiedat and F. Awawdeh, Some fixed point theorem for mapping on complete G-metric spaces, *Fixed Point Theory and Applications*, 2008 (2008), Article ID 189870.

[9] R.P. Pant, Common xed points of four maps, Bull. Calcutta Math. Soc. 90 (1998), 281-286.

[10] S. Sedghi, N. Shobe, H. Zhou, A common fixed point theorem in D^*-metric spaces, *Fixed Point Theory Applications*, 2007 (2007), Article ID 27906.

[11] S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorem in S-metric spaces, *Mat. Vesnik*, **64**(2012), 258266.

[12] S. Sedghi and V.N. Dung, Fixed point theorems on S-metric spaces, *Mat. Veshik*, **66** (2014), 113-124.

[13] S. Sedghi, N. Shobe, H. Zhou, A common fixed point theorem in D^*-metric spaces, *Fixed Point Theory Appl.* 2007 (2007), Article ID 27906.