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FIXED POINT RESULTS FOR THE MULTIVALUED MAPPING IN HAUSDORFF FUZZY METRIC SPACE

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Abstract. In this paper we established some fixed point results for multivalued mapping in a complete fuzzy metric space through rational inequality. Our results unify, extend and generalize several results in the existing literature. Some applications are also given to support our results.

Keywords: fixed point; complete fuzzy metric space; multivalued mapping; Hausdorff fuzzy metric space.

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1. Introduction and Preliminaries

The notion of fuzzy sets was first present by Zadeh [10]. Later on many authors worked on its different areas. Kramosil and et al. [6] with the help of defintion of fuzzy sets introduced a new concept of fuzzy metric space and prove some fixed point results. Grabiec [3] proved the fixed point theorem of banach and eldestien to fuzzy metric space in the sence of Kramosil and et al. In 1994, George et al. [2] modified the definiton of Kramosil et al. and proved many fixed point results. Later on Lopez et al. [8] used this concept in the sence of compact sets and present the definition of Hausdorff fuzzy metric and prove many well known results. In this paper we

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also used the notion of Hausdorff fuzzy metric with the help of multivalued mapping and prove some results presented by Vishal et al. [5].

Definition 1.1 [2]**:** A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is said to be a continuous t-norm if it is satisfies the following conditions:

i) $*$ is associative and commutative;

ii) $*$ is continuous;

iii) $a*1=a$ for all $a\in[0,1]$;

iv) $a*b\leq c*d$ whenever $a\leq c$ and $b\leq d$ for each $a,b,c,d\in[0,1]$.

Definition 1.2 [6]**:** Let X be any non empty set, $*$ be a continuous t-norm, and F is a fuzzy set on $X^2\times[0,\infty)$. Consider the following conditions holds for all $x,y,z\in X$ and $t,s>0$:

F1) $F(x,y,0)=0$;

F2) $F(x,y,t)=1$ iff $x=y$;

F3) $F(x,y,t)=F(y,x,t)$;

F4) $F(x,y,t+s)\geq F(x,z,t)*F(z,y,s)$;

F5) $F(x,y,.):(0,\infty)\to[0,1]$ is left-continuous;

Then, F is called a fuzzy metric on X and $F(x,y,t)$ denotes the degree of nearness between x and y with to respect t.

Example 1.3 [2]**:** Let (X,d) be a metric space. Define $a*b=ab$ (or $a*b=\min\{a,b\}$) for all $a,b\in[0,1]$. Then, one can define a fuzzy metric by

$$F(x,y,t)=\frac{t}{t+d(x,y)},\quad \text{for all } x,y\in X \text{ and } t>0.$$

called $(X,F,*)$ is a fuzzy metric space induced by $d(x,y)$.

Definition 1.4 [5]**:** Let $(X,F,*)$ be a fuzzy metric space. Then, we have

i) A sequence $\{x_n\}$ in X is said to be convergent to a point $x\in X$ denoted $x_n\to x$, if $\lim_{n\to\infty}F(x_n,x,t)=1$ for each $t>0$.

ii) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence, if $\lim_{n\to\infty}F(x_n,x_{n+p},t)=1$ for each $t>0$, $p>0$.

iii) A fuzzy metric space $(X,F,*)$ in which every Cauchy sequence is convergent is called a complete fuzzy metric space.

Definition 1.5 [1]**:** Let B be any non empty subset of a fuzzy metric space $(X,F,*)$ for $a\in X$

and $t > 0$ then,

$$F(a,B,t) = \sup\{F(a,b,t) : b \in B\}.$$

Definition 1.6 [8]**:** Let $(X,F,*)$ be a fuzzy metric space. Define a function H_F on $\hat{C}_0(X) \times \hat{C}_0(X) \times (0,\infty)$ by

$$H_F(A,B,t) = \min\left\{\inf_{a\in A} F(a,B,t), \inf_{b\in B} F(A,b,t)\right\},$$

for all $A,B \in \hat{C}_0(X)$ and $t > 0$, where $\hat{C}_0(X)$ is the collection of all nonempty compact subsets of X.

Lemma 1.7 [8]**:** Let $(X,F,*)$ be a complete fuzzy metric space. Then, for each $a \in X$, $B \in \hat{C}_0(X)$ and for $t > 0$ there exists $b_o \in B$ such that

$$F(a,b_o,t) = F(a,B,t).$$

Lemma 1.8 [9]**:** Let $(X,F,*)$ be a complete fuzzy metric space, such that $(\hat{C}_0(X),H_F,*)$ is a hausdorff fuzzy metric space on $\hat{C}_0(X)$. Then, for all $A,B \in \hat{C}_0(X)$, for each $a \in A$ and for $t > 0$ there exists $b_a \in B$, satisfies $F(a,B,t) = F(a,b_a,t)$, then

$$H_F(A,B,t) \leq F(a,b_a,t).$$

Proof: If

$$H_F(A,B,t) = \inf_{a\in A} F(a,B,t),$$

then,

$$H_F(A,B,t) \leq F(a,B,t). \quad \text{for each } a \in A$$

Hence, for each $a \in A$ there exist $b_a \in B$ satisfies

$$F(a,B,t) = F(a,b_a,t),$$

then

$$H_F(A,B,t) \leq F(a,b_a,t).$$

Now, if

$$\begin{aligned} H_F(A,B,t) &= \inf_{b\in B} F(A,b,t) \\ &\leq \inf_{a\in A} F(a,B,t) \leq F(a,B,t) = F(a,b_a,t) \\ H_F(A,B,t) &\leq F(a,b_a,t). \end{aligned}$$

for some $b_a \in B$. Hence, in both cases, we proved the result.

Lemma 1.9 [5]**:** If there exist $k \in (0,1)$, such that

$$M(x,y,kt) \geq M(x,y,t)$$

for all $x,y,\in X$ and $t \in (0,\infty)$, then $x = y$.

2. Main Results

Now, we present our main results.

Theorem 2.1: Let $(X,M,*)$ be a fuzzy metric space and $S : X \to \hat{C}_0(X)$ be a multivalued mapping satisfying the following conditions:

$$a) \lim_{t\to\infty} F(x,y,t) = 1, \tag{2.1}$$

$$b)\ H_F(Sx,Sy,kt) \geq \mu(x,y,t). \tag{2.2}$$

where

$$\mu(x,y,t) = \min\left\{\frac{F(y,Sy,t)[1+F(x,Sx,t)]}{[1+F(x,y,t)]}, F(x,y,t)\right\},$$

for all $x,y \in X$ and $k \in (0,1)$. Then, S has a fixed point.

Proof: Let x_o be an arbitrary point in X. We construct a sequence $\{x_n\}$ of points in X as follows: Let $x_1 \in X$ such that $x_1 \in Fx_o$. By using lemma 1.8, we can choose $x_2 \in Fx_1$ such that

$$F(x_1,x_2,t) \geq H_F(Sx_o,Sx_1,t). \quad \text{for all } t > 0$$

By induction we have $x_{n+1} \in Sx_n$, for all $n \in \mathbb{N}$, satisfying

$$F(x_n,x_{n+1},t) \geq H_F(Sx_{n-1},Sx_n,t). \quad \text{for all } t > 0$$

Now,

$$\begin{aligned} F(x_2,x_3,t) &\geq H_F(Sx_1,Sx_2,t) \\ &\geq \mu(x_1,x_2,\frac{t}{k}). \quad \text{by (2.2)} \end{aligned} \tag{2.3}$$

where,

$$\begin{aligned} \mu(x_1,x_2,\frac{t}{k}) &= \min\left\{\frac{F(x_2,Sx_2,\frac{t}{k})[1+F(x_1,Sx_1,\frac{t}{k})]}{[1+F(x_1,x_2,\frac{t}{k})]}, F(x_1,x_2,\frac{t}{k})\right\}, \\ &= \min\left\{\frac{F(x_2,x_3,\frac{t}{k})[1+F(x_1,x_2,\frac{t}{k}]}{[1+F(x_1,x_2,\frac{t}{k})]}, F(x_1,x_2,\frac{t}{k})\right\}, \\ &= \min\left\{F(x_2,x_3,\frac{t}{k}), F(x_1,x_2,\frac{t}{k})\right\}. \end{aligned}$$

If,

$$F(x_1,x_2,\frac{t}{k}) \geq F(x_2,x_3,\frac{t}{k}).$$

Then, by (2.3), we have

$$F(x_2,x_3,t) \geq F(x_2,x_3,\frac{t}{k}).$$

So, by the Lemma 1.9 nothing left to prove. Now, if we have

$$F(x_2,x_3,\frac{t}{k}) \geq F(x_1,x_2,\frac{t}{k}).$$

Then, again by Lemma 1.8, we have

$$\begin{aligned} F(x_2,x_3,t) &\geq F(x_1,x_2,\frac{t}{k}) \geq H_F(Sx_o,Sx_1,\frac{t}{k}) \\ &\geq \mu(x_o,x_1,\frac{t}{k^2}). \end{aligned} \tag{2.4}$$

where,

$$\begin{aligned} \mu(x_o,x_1,\frac{t}{k^2}) &= \min\left\{\frac{F(x_1,Sx_1,\frac{t}{k^2})[1+F(x_o,Sx_o,\frac{t}{k^2})]}{[1+F(x_o,x_1,\frac{t}{k^2})]}, F(x_o,x_1,\frac{t}{k^2})\right\}, \\ &= \min\left\{\frac{F(x_1,x_2,\frac{t}{k^2})[1+F(x_o,x_1,\frac{t}{k^2})]}{[1+F(x_o,x_1,\frac{t}{k^2})]}, F(x_o,x_1,\frac{t}{k^2})\right\}, \\ &= \min\left\{F(x_1,x_2,\frac{t}{k^2}), F(x_o,x_1,\frac{t}{k^2})\right\}. \end{aligned}$$

If,

$$F(x_o,x_1,\frac{t}{k^2}) \geq F(x_1,x_2,\frac{t}{k^2}).$$

Then, again by lemma 1.9, nothing left to prove. If

$$F(x_1,x_2,\frac{t}{k^2}) \geq F(x_o,x_1,\frac{t}{k^2}),$$

then, by (2.4) we have

$$F(x_2,x_3,t) \geq F(x_o,x_1,\frac{t}{k^2}).$$

Consequently,

$$F(x_n,x_{n+1},t) \geq F(x_o,x_1,\frac{t}{k^n}) \tag{2.5}$$

Now, for $m > n$, that is $m = n + p$ we have

$$F(x_n,x_{n+p},t) \geq F(x_n,x_{n+1},\frac{t}{p}) * \cdots (p) \cdots * F(x_{n+p-1},x_{n+p},\frac{t}{p}).$$

By using (2.5), we get

$$F(x_n,x_{n+p},t) \geq F(x_o,x_1,\frac{t}{pk^n}) * \cdots (p) \cdots * F(x_o,x_1,\frac{t}{pk^n}).$$

Now, taking $\underset{n\to\infty}{Lim}$ and using (2.1) we have

$$\underset{n\to\infty}{Lim} F(x_n,x_{n+p},t) = 1.$$

Hence, $\{x_n\}$ is a Cauchy sequence in X. So, by the completeness there exists $z \in X$, such that $x_n \to z$. Now,we claim that z is a fixed point for S. Consider,

$$\begin{aligned} F(z,Sz,t) &\geq F(z,x_{n+1},(1-k)t) * F(x_{n+1},Sz,kt), \\ &\geq F(z,x_{n+1},(1-k)t) * H_F(Sx_n,Sz,kt), \\ &\geq F(z,x_{n+1},(1-k)t) * \mu(x_n,z,t). \end{aligned} \tag{2.6}$$

where,

$$\begin{aligned} \mu(x_n,z,t) &= \min\left\{\frac{F(z,Sz,t)[1+F(x_n,Sx_n,t)]}{[1+F(x_n,z,t)]}, F(x_n,z,t)\right\}, \\ &= \min\left\{\frac{F(z,Sz,t)[1+F(x_n,x_{n+1},t)]}{[1+F(x_n,z,t)]}, F(x_n,z,t)\right\}. \end{aligned}$$

Taking $\underset{n\to\infty}{Lim}$ in above inequality, we get

$$\mu(z,z,t) = \min\{F(z,Sz,t),1\}.$$

If

$$F(z,Sz,t) \geq 1,$$

then, we get z is the fixed point for S. If

$$F(z,Sz,t) \leq 1,$$

then, by (2.6)

$$F(z,Sz,t) \geq F(z,x_{n+1},(1-k)t) * F(z,Sz,t).$$

Now, taking $\underset{n\to\infty}{Lim}$ we get

$$z \in Sz.$$

Let us define $\Phi = \{\varphi/\varphi : [0,1] \to [0,1]\}$ is a continuous function such that $\varphi(1) = 1$, $\varphi(0) = 0$, $\varphi(a) > a$ for each $0 < a < 1$.

Theorem 2.2: Let $(X,M,*)$ be a fuzzy metric space and $S : X \to \hat{C}_0(X)$ be a multivalued mapping satisfying the following conditions :

$$a)\lim_{t\to\infty} F(x,y,t) = 1,$$

$$b)\ H_F(Sx,Sy,kt) \geq \varphi\{\mu(x,y,t)\}.$$

where

$$\mu(x,y,t) = \min\left\{\frac{F(y,Sy,t)[1+F(x,Sx,t)]}{[1+F(x,y,t)]}, F(x,y,t)\right\},$$

for all $x,y \in X$, $k \in (0,1)$ and $\varphi \in \Phi$. Then, S has a fixed point.

Proof: Since $\varphi \in \Phi$. This implies that $\varphi(a) > a$ for each $0 < a < 1$. Thus from above condition

$$H_F(Sx,Sy,kt) \geq \varphi\{\mu(x,y,t)\} \geq \mu(x,y,t)$$

Now, applying Theorem 2.1, we obtain the desired result.

3. Application

Let us define the following function

$$\theta : [0,\infty) \to [0,\infty) \text{ as } \theta(t) = \int_0^t \lambda(t)dt \quad \forall\, t > 0$$

be a nondecreasing and continuous function. Morever, for each $\delta > 0$, $\lambda(\delta) > 0$. Also $\lambda(t) = 0$ iff $t = 0$.

Theorem 3.1: Let $(X, M, *)$ be a fuzzy metric space and $S : X \to \hat{C}_0(X)$ be a multivalued mapping satisfying the following conditions :

$$a) \lim_{t\to\infty} F(x,y,t) = 1,$$

$$b) \int_0^{H_F(Sx,Sy,kt)} \lambda(t)dt \geq \int_0^{\mu(x,y,t)} \lambda(t)dt.$$

where,

$$\mu(x,y,t) = \min\left\{ \frac{F(y,Sy,t)[1+F(x,Sx,t)]}{[1+F(x,y,t)]}, F(x,y,t) \right\},$$

for all $x, y \in X$, $\lambda \in \theta$ and $k \in (0,1)$. Then S has a fixed point.

Proof: Let us take $\lambda(t)dt = 1$, and applying Theorem 2.1 we get the desired result.

Theorem 3.2: Let $(X, M, *)$ be a fuzzy metric space and $S : X \to \hat{C}_0(X)$ be a multivalued mapping satisfying the following conditions :

$$a) \lim_{t\to\infty} F(x,y,t) = 1,$$

$$b) \int_0^{H_F(Sx,Sy,kt)} \lambda(t)dt \geq \varphi\left\{ \int_0^{\mu(x,y,t)} \lambda(t)dt \right\}.$$

where,

$$\mu(x,y,t) = \min\left\{ \frac{F(y,Sy,t)[1+F(x,Sx,t)]}{[1+F(x,y,t)]}, F(x,y,t) \right\},$$

for all $x, y \in X$, $\lambda \in \theta$, $\varphi \in \Phi$ and $k \in (0,1)$. Then S has a fixed point.

Proof: Since $\varphi \in \Phi$. This implies that $\varphi(a) > a$ for each $0 < a < 1$, taking $\lambda(t)dt = 1$, and applying Theorem 2.2 we get the desired result.

Conflict of Interests

The authors declare that they have no competing interests.

Authors Contribution

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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