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GREGUS TYPE FIXED POINT THEOREMS FOR TWO PAIRS SELF MAPPINGS SATISFYING STRICT CONTRACTIVE CONDITION OF INTEGRAL TYPE VIA ALTERING DISTANCE

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Abstract. In the present paper, we prove two common fixed point theorems of Gregus type for two pairs of self mappings satisfying strict contractive condition of integral type by using the weak subsequential continuity property with compatibility of type (E). Our results improve and generalize the results of Beloul et al.[7] and relevant literature.

Keywords: weakly subsequentially continuous; Gregus fixed point theorem; compatible of type (E).

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

Banach principle has been generalized to various ways, as non linear contraction due to Boyd and Wong[9], Meir-Keeler contraction[25], Geraghty contractionmg and some others. Gregus [17] gave new generalization and proved existence of fixed point. Later many authors improved

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and generalized this type as in papers [13, 14, 27, 29]. Recently, Pathak and Shahzad [30] proved some results for Gregus type common fixed point by using the tangential property for two self mappings in metric spaces, Sintunavarat and Kumam [36] extended the results to single and set valued mappings which improved also Chauhan et al. [11], wherein the authors mentioned that the results in paper [36] must need the closure of one of the subspaces by using tangential property.

Jungck[19] introduced the concept of commuting mappings to establish a common fixed point theorem for two self mappings on metric spaces, Sessa [34] defined the weakly commuting mappings which is a generalization to the commuting mappings, later Jungck [19] generalized the two past concepts which called compatible mappings, it is weaken than the last notions after that many authors introduced various type of compatibility, compatibility of types (A), (B), (C) and (P) for two self mappings on metric space respectively in [21, 27, 29] and [28]. In 1996, Jungck [22] introduced the notion of weakly compatible mappings which generalizes the all above type of compatibility and it is weaker than them. Pant [26] is the first who studies and used non-compatible mappings and replaced them by a new concept which called reciprocal continuity to establish a common fixed point, later Aamri and Moutawakil [2] introduced property (E.A) for two self mappings on metric spaces and they used it with generalized contractions. Since then, Al-Thagafi and Shahzad [3] weakened the weak compatibility, they introduced the notion of occasionally weakly compatible mappings on metric spaces which generalized by another concept as subcompatible mappings which given by Bouhadjera and Godet Thobie [8], the same authors in their paper [8] generalized reciprocal continuity to subsequential continuity.

In the present paper, we will prove two common fixed point theorems of Gregus type for four mappings which satisfying strict contractive condition of integral type in metric spaces by using subsequential continuity and compatibility of type (E) due to Singh et al. [35].

2. Preliminaries

Definition 2.1. Two self mappings A and S of a metric space (X,d) are said to be compatible of type (E), if

$$\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = At \text{ and } \lim_{n\to\infty} A^2x_n = \lim_{n\to\infty} ASx_n = St,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

Remark1.1. If $At = St$, then compatible of type (E) implies compatible (compatible of type (A), compatible of type (B), compatible of type (C), compatible of type (P)), however the converse may be not be true. Generally, compatibility of type (E) implies the compatibility of type (B).

Definition 2.2. Two self mappings A and S of a metric space (X,d) are A-compatible of type (E), if

$$\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = At,$$

for some $t \in X$. Also, the pair $\{A,S\}$ is said to be S-compatible of type (E), if $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = At$, for some $t \in X$.

Notice that if two self mappings A and S are compatible of type (E), then they are A-compatible and S-compatible of type (E), but the converse is not true.

Definition 2.3. Two self mappings A and S of a metric space (X,d) are said to be reciprocally continuous, if $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

Definition 2.4. Two self mappings A and S of a metric space (X,d) is called to be subsequentially continuous if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$ and satisfy $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n) = St$.

Clearly that continuous or reciprocally continuous mappings are subsequentially continuous, but the converse may be not be true.

Example 1.1. Let $X = [0,\infty)$ and d is the euclidian metric, we define A, S as follows:

$$Ax = \begin{cases} 2+x, & 0 \leq x \leq 2 \\ \frac{x+2}{2}, & x > 2 \\ \end{cases}, \quad Sx = \begin{cases} 2-x, & 0 \leq x < 2 \\ 2x-2, & x \geq 2 \\ \end{cases}$$

Clearly that A and S are discontinuous at 2.

We consider a sequence $\{x_n\}$ such that for each $n \geq 1$: $x_n = \dfrac{1}{n}$,

clearly that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 2$, also we have:

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A(2-\frac{1}{n}) = 4 = A(2),$$

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S(2+\frac{1}{n}) = 2 = S(2),$$

then the pair (A,S) is subsequentially continuous.

On other hand, let $\{y_n\}$ be a sequence which defined or each $n \geq 1$ by: $y_n = 2+\frac{1}{n}$, we have

$$\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Sy_n = 2,$$

but

$$\lim_{n\to\infty} ASy_n = \lim_{n\to\infty} A(2+\frac{2}{n}) = 2 \neq A(2),$$

$$\lim_{n\to\infty} SAy_n = \lim_{n\to\infty} S(4+\frac{1}{n}) = 6 \neq S(2),$$

then A and S are never reciprocally continuous.

Definition 2.4. Let A and S to be two self mappings of a metric space (X,d), the pair (A,S) is said to be weakly subsequentially continuous if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, for some $z \in X$ and $\lim_{n\to\infty} ASx_n = Az$, $\lim_{n\to\infty} SAx_n) = Sz$.

Notice that the subsequentially continuous or reciprocally continuous mappings are weakly subsequentially continuous, but the converse may be not true.

Example 2.2. Let $X = [0,8]$ and d is the euclidian metric, we define A, S as follows:

$$Ax = \begin{cases} \frac{x+4}{2}, & 0 \leq x \leq 4 \\ x+1, & 4 \leq x \leq 8 \end{cases}, \quad Sx = \begin{cases} 8-x, & 0 \leq x \leq 4 \\ x-2, & 4 \leq x \leq 8 \end{cases}$$

We consider a sequence $\{x_n\}$ such that for each $n \geq 1$: $x_n = 4 - e^{-n}$, clearly that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 4$, also we have:

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A(4+e^{-n}) = 5,$$

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S(4-\frac{1}{2}e^{-n}) = 4 = S(4),$$

then the pair (A,S) is S-subsequentially continuous.

In 2014 the concept of $\mathscr{C}$-class functions was introduced by A.H.Ansari [5]. By using this concept, we can generalize many fixed point theorems in the literature.

Definition 2.5. [5]A mapping $F:[0,\infty)^2 \to \mathbb{R}$ is called *C-class* function if it is continuous and satisfies following axioms:

(1) $F(s,t) \leq s$;

(2) $(s,t) = s$ implies that either $s = 0$ or $t = 0$; for all $s,t \in [0,\infty)$.

Note for some F we have that $F(0,0) = 0$.

Example 2.3. [5]The following functions $F:[0,\infty)^2 \to \mathbb{R}$ are elements of $\mathscr{C}$, for all $s,t \in [0,\infty)$:

(1) $F(s,t) = s-t$, $F(s,t) = s \Rightarrow t = 0$;

(2) $F(s,t) = ms$, $0<m<1$, $F(s,t) = s \Rightarrow s = 0$;

(3) $F(s,t) = \frac{s}{(1+t)^r}$; $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(4) $F(s,t) = \log(t+a^s)/(1+t)$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(5) $F(s,t) = \ln(1+a^s)/2$, $a > e$, $F(s,1) = s \Rightarrow s = 0$;

(6) $F(s,t) = s - \varphi(s), F(s,t) = s \Rightarrow s = 0$,here $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;

(7) $F(s,t) = sh(s,t), F(s,t) = s \Rightarrow s = 0$,here $h:[0,\infty)\times[0,\infty) \to [0,\infty)$is a continuous function such that $h(t,s) < 1$ for all $t,s > 0$;

(8) $F(s,t) = \phi(s), F(s,t) = s \Rightarrow s = 0$,here $\phi : [0,\infty) \to [0,\infty)$ is a upper semicontinuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$,

(9) $F(s,t) = \frac{s}{(1+s)^r}$; $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$.

Definition 2.6. [24]A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is non-decreasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

We let Ψ denote the class of the altering distance functions.

Definition 2.7. [24]An ultra altering distance function is a continuous, nondecreasing mapping $\phi : [0,\infty) \to [0,\infty)$ such that $\phi(t) > 0$ $,t > 0$ and $\phi(0) \geq 0$

Remark 2.3. We denote Φ set ultra altering distance functions.

Definition 2.8. A tripled (ψ,ϕ,F) where $\psi\in\Psi$, $\phi\in\Phi$ and $F\in\mathscr{C}$ is said to be monotone if for any $x,y\in[0,\infty)$

$$x\preceq y\Longrightarrow F(\psi(x),\phi(x))\preceq F(\psi(y),\phi(y)).$$

Example 2.3. Let $F(s,t)=s-t,\phi(x)=\sqrt{x}$

$$\psi(x)=\begin{cases}\sqrt{x} & \text{if } 0\le x\le 1,\\ x^2, & \text{if x} > 1\end{cases},$$

then (ψ,ϕ,F) is monotone.

Example 2.4. let $F(s,t)=s-t,\phi(x)=x^2$

$$\psi(x)=\begin{cases}\sqrt{x} & \text{if } 0\le x\le 1,\\ x^2, & \text{if x} > 1\end{cases},$$

then (ψ,ϕ,F) is not monotone.

Let Ω be set of all continuous functions $\Lambda:\mathbb{R}_+^4\to\mathbb{R}_+$ satisfying: $\lambda(0,0,t,t)=t$.

Example 2.5. $(1)\tau(t_1,t_2,t_3,t_4)=\max\{t_1,t_2,t_3,t_4\}$

$(2)\tau(t_1,t_2,t_3,t_4)=\frac{t_1+t_2+t_3+t_4}{2}$

(3) $\tau(t_1,t_2,t_3,t_4)=\max\{t_1,t_2,\sqrt{t_1t_3},\sqrt{t_3t_4}\}$

3. Main results

Theorem 3.1. *Let $A,B,S,T:X\to X$, be self mappings of a metric space (X,d) such for all x,y in X we have:*

$$\begin{aligned}[1+a(\int_0^{d(Ax,By)}\varphi(t))^p]\Big(\int_0^{d(Sx,Ty)}\varphi(t)dt\Big)^p &< a[\Big(\int_0^{d(Ax,Sx)}\varphi(t)dt\Big)^p\Big(\int_0^{d(By,Ty)}\varphi(t)dt\Big)^p\\ (1)\qquad &+\Big(\int_0^{d(Sx,By)}\varphi(t)dt\Big)^p\Big(\int_0^{d(Ax,Ty)}\varphi(t)dt\Big)^p]+F(M(x,y),\phi(M(x,y))),\end{aligned}$$

where

$$M(x,y)=\frac{1}{\alpha+\beta}(\alpha\Big(\int_0^{d(Ax,By)}\varphi(t)dt\Big)^p)+$$

$$\beta\tau[\Big(\int_0^{d(Ax,Sx)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(By,Ty)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Ax,Ty)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(By,Sx)}\varphi(t)dt\Big)^p])$$

and a,α,β are non-negative numbers such $\alpha+\beta>0, p\in\mathbb{N}^*, \phi\in\Phi, F\in\mathscr{C}$ with (I,ϕ,F) is monotone and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a Lebesgue-integrable function which is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for each $\varepsilon>0, \int_0^{\varepsilon}\varphi(t)dt>0$, if the pair (A,S) is weakly subsequentially continuous and compatible of type (E) as well as (B,T), then A,B,S and T have a unique common fixed point in X.

Proof. Suppose that (A,S) is A-subsequentially continuous, there is a sequence $\{x_n\}$in$X$ such that $\lim\limits_{n\to\infty}Ax_n=\lim\limits_{n\to\infty}Sx_n=z$ and $\lim_{n\to\infty}ASx_n=Az$,also the pair is compatible of type (E) implies that $\lim_{n\to\infty}ASx_n=Sz$, also the pair (A,S) is compatible implies that $\lim\limits_{n\to\infty}ASx_n=Sz$ and $\lim\limits_{n\to\infty}SAx_n=Sz$, which implies $Sz=fz=$ and z is a coincidence point for f and S.

Similarly for the pair (B,T), suppose that (B,T) is B-subsequentially continuous, there exists a sequence $\{y_n\}\in X$ such

$$\lim_{n\to\infty}Ty_n=\lim_{n\to\infty}gy_n=t,$$

and

$$\lim_{n\to\infty}BTy_n=Bt,$$

also the pair (B,T) is compatible of type (E) implies

$$\lim_{n\to\infty}BTy_n=\lim_{n\to\infty}T^2y_n=Tt$$

and

$$\lim_{n\to\infty}TBy_n=\lim_{n\to\infty}B^2y_n=Bt$$

which implies that $Bt=Tt$.

Firstly, we prove $Az=Bt$, if not by using (1) we get

$$\Big(1+a\Big(\int_0^{d(Az,Bt)}\varphi(t)\Big)^p\Big)\Big(\int_0^{d(Sz,Tt)}\varphi(t)dt\Big)^p<$$

$$a\Big(\int_0^{d(Az,Tt)}\varphi(t)dt\Big)^p\Big(\int_0^{d(Bt,Sz)}\varphi(t)dt\Big)^p+F(\frac{1}{\alpha+\beta}[\alpha\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p$$

$$+\beta\tau\Big(0,0,\Big(\int_0^{d(Az,Tt)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Sz,Bt)}\varphi(t)dt\Big)^p\Big)]),$$

$$\phi(\frac{1}{\alpha+\beta}[\alpha\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p+\beta\tau\Big(0,0,\Big(\int_0^{d(Az,Tt)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Sz,Bt)}\varphi(t)dt\Big)^p\Big)]))$$

since $Az = Sz$ and $Bt = Tt$, we get

$$\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p <$$

$$F(\frac{1}{\alpha+\beta}[\alpha(\int_0^{d(Az,Bt)}\varphi(t))^p + \beta\tau\Big(0,0,\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p]$$

$$\phi(\frac{1}{\alpha+\beta}[\alpha(\int_0^{d(Az,Bt)}\varphi(t))^p + \beta\tau\Big(0,0,\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p]))$$

$$\leq F(\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p,\phi(\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p)) < \Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p,$$

So, $\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p = 0$, or, $\phi(\Big(\int_0^{d(Az,Bt)}\varphi(t)dt\Big)^p) = 0$, thus $d(Az,Bt) = 0$, which implies that $Az = Bt$.

Now, we prove $z = Az$, if not by using (1), we get

$$\Big(1+a(\int_0^{d(Ax_nBt)}\varphi(t)\Big)^p\Big(\int_0^{d(Sx_n,Tt)}\varphi(t)dt\Big)^p <$$

$$a\left[\begin{array}{c}\Big(\int_0^{d(Ax_n,Sx_n)}\varphi(t)dt\Big)^p.\Big(\int_0^{d(Bt,Tt)}\varphi(t)dt\Big)^p\\ +\Big(\int_0^{d(Bt,Sx_n)}\varphi(t)dt\Big)^p.\Big(\int_0^{d(Ax_n,Tt)}\varphi(t)dt\Big)^p\end{array}\right]+$$

$$F(\frac{1}{\alpha+\beta}[\alpha\Big(\int_0^{d(Ax_n,Bt)}\varphi(t)dt\Big)^p + \beta\tau\left(\begin{array}{c}\Big(\int_0^{d(Ax_n,Sx_n)}\varphi(t)dt)^p,0,\Big(\int_0^{d(Ax_n,Tt)}\varphi(t)dt\Big)^p,\\ \Big(\int_0^{d(Sx_n,Bt)}\varphi(t)dt\Big)^p\end{array}\right)],$$

$$\phi(\frac{1}{\alpha+\beta}[\alpha\Big(\int_0^{d(Ax_n,Bt)}\varphi(t)dt\Big)^p + \beta\tau\left(\begin{array}{c}\Big(\int_0^{d(Ax_n,Sx_n)}\varphi(t)dt)^p,0,\Big(\int_0^{d(Ax_n,Tt)}\varphi(t)dt\Big)^p,\\ \Big(\int_0^{d(Sx_n,Bt)}\varphi(t)dt\Big)^p\end{array}\right)]))$$

Letting $n \to \infty$, we get:

$$\Big(1+a(\int_0^{d(z,Az)}\varphi(t)\Big)^p\Big(\int_0^{d(z,Az)}\varphi(t)dt\Big)^p <$$

$$a\Big(\int_0^{d(Bt,z)}\varphi(t)dt\Big)^p.\Big(\int_0^{d(z,Tt)}\varphi(t)dt)\Big)^p +$$

$$F(\frac{1}{\alpha+\beta}[\alpha\Big(\int_0^{d(z,Az)}\varphi(t)dt\Big)^p + \beta\tau\Big(0,0,\Big(\int_0^{d(z,Tt)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Az,t)}\varphi(t)dt\Big)^p\Big)],$$

$$\phi(\frac{1}{\alpha+\beta}[\alpha\Big(\int_0^{d(z,Az)}\varphi(t)dt\Big)^p + \beta\tau\Big(0,0,\Big(\int_0^{d(z,Tt)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Az,t)}\varphi(t)dt\Big)^p\Big)]))$$

since $Az = Sz = Bt$, we get

$$\begin{aligned}&\Big(\int_0^{d(z,Az)}\varphi(t)dt\Big)^p\\ &< F(\Big(\int_0^{d(z,Az)}\varphi(t)dt\Big)^p, \phi(\Big(\int_0^{d(z,Az)}\varphi(t)dt\Big)^p))\\ &< \Big(\int_0^{d(z,Az)}\varphi(t)dt\Big)^p,\end{aligned}$$

So, $\Big(\int_0^{d(z,Az)}\varphi(t)dt\Big)^p = 0$, or $,\phi(\Big(\int_0^{d(z,Az)}\varphi(t)dt\Big)^p) = 0$, thus $d(z,Az) = 0$, which implies that $Az = z$, then $z = Az = Sz$.

Nextly we prove $z = t$, if not then by using (1), we get

$$\Big(1 + a(\int_0^{d(Ax_n,By_n)}\varphi(t))^p\Big)\Big(\int_0^{d(Sx_n,Ty_n)}\varphi(t)dt\Big)^p <$$

$$a\left[\begin{array}{c}\Big(\int_0^{d(Ax_n,Sx_n)}\varphi(t)dt\Big)^p.\Big(\int_0^{d(By_n,Ty_n)}\varphi(t)dt\Big)^p\\ +\Big(\int_0^{d(Ax_n,Ty_n)}\varphi(t)dt\Big)^p.\Big(\int_0^{d(By_n,Sx_n)}\varphi(t)dt\Big)^p\end{array}\right]+$$

$$F(\frac{1}{\alpha+\beta}[\alpha\Big(\int_0^{d(Ax_n,By_n)}\varphi(t)dt\Big)^p + \beta\tau\left(\begin{array}{c}\Big(\int_0^{d(Ax_n,Sx_n)}\varphi(t)dt\Big)^p, \Big(\int_0^{d(By_n,Ty_n)}\varphi(t)dt\Big)^p,\\ \Big(\int_0^{d(Ax_n,Ty_n)}\varphi(t)dt\Big)^p, \Big(\int_0^{d(Sx_n,By_n)}\varphi(t)dt\Big)^p\end{array}\right)],$$

$$\phi(\frac{1}{\alpha+\beta}[\alpha\Big(\int_0^{d(Ax_n,By_n)}\varphi(t)dt\Big)^p + \beta\tau\left(\begin{array}{c}\Big(\int_0^{d(Ax_n,Sx_n)}\varphi(t)dt\Big)^p, \Big(\int_0^{d(By_n,Ty_n)}\varphi(t)dt\Big)^p,\\ \Big(\int_0^{d(Ax_n,Ty_n)}\varphi(t)dt\Big)^p, \Big(\int_0^{d(Sx_n,By_n)}\varphi(t)dt\Big)^p\end{array}\right)])).$$

Letting $n \to \infty$, we get

$$\Big(1 + a\Big(\int_0^{d(z,t)}\varphi(t)\Big)^p\Big)\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p <$$

$$a\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p +$$

$$F(\frac{1}{\alpha+\beta}[\alpha\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p + \beta\tau\Big(0,0,\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p, \Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p\Big)],$$

$$\phi(\frac{1}{\alpha+\beta}[\alpha\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p + \beta\tau\Big(0,0,\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p, \Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p\Big)])),$$

and so we have

$$\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p <$$

$$F(\frac{1}{\alpha+\beta}[\alpha\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p + \beta\tau\Big(\Big(0,0,\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p, \Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p\Big)],$$

$$\phi(\frac{1}{\alpha+\beta}[\alpha\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p + \beta\tau\Big(\Big(0,0,\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p, \Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p\Big)])),$$

which implies that

$$\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p \le F(\frac{1}{\alpha+\beta}[\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p],\phi(\frac{1}{\alpha+\beta}[\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p])) \le \Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p.$$

So, $\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p = 0$, or, $\phi(\Big(\int_0^{d(z,t)}\varphi(t)dt\Big)^p) = 0$,thus $d(z,t)=0$, which implies that $z=t$, then z is a common fixed point for A,B,S and T.

For the uniqueness, suppose there is another fixed point w, by using (1) we get

$$\Big(1+a(\int_0^{d(Az,Bw)}\varphi(t))^p\Big)(\int_0^{d(Sz,Tw)}\varphi(t)dt)^p <$$
$$a\Big(\int_0^{d(Az,Tw)}\varphi(t)dt)^p\Big).\Big(\int_0^{d(Bw,Sz)}\varphi(t)dt\Big)^p\Big)+$$
$$F(\frac{1}{\alpha+\beta}[\alpha\Big(\int_0^{d(Az,Bw)}\varphi(t)dt\Big)^p+\beta\tau\Big(0,0,\Big(\int_0^{d(Az,Tw)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Bw,Sz)}\varphi(t)dt\Big)^p\Big)],$$
$$\phi(\frac{1}{\alpha+\beta}[\alpha\Big(\int_0^{d(Az,Bw)}\varphi(t)dt\Big)^p+\beta\tau\Big(0,0,\Big(\int_0^{d(Az,Tw)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Bw,Sz)}\varphi(t)dt\Big)^p\Big)])),$$

since z and w are common fixed points, and so

$$\Big(\int_0^{d(z,w)}\varphi(t)dt\Big)^p \le F(\Big(\int_0^{d(z,w)}\varphi(t)dt\Big)^p,\phi(\Big(\int_0^{d(z,w)}\varphi(t)dt\Big)^p)) \le \Big(\int_0^{d(z,w)}\varphi(t)dt\Big)^p,$$

So, $\Big(\int_0^{d(z,w)}\varphi(t)dt\Big)^p = 0$, or , $\phi(\Big(\int_0^{d(z,w)}\varphi(t)dt\Big)^p) = 0$, thus $d(z,w)=0$, then $z=w$.

With choice $F(s,t)=(\alpha+\beta)s$, $0<\alpha+\beta<1$ in Theorem 3.1, we get to the following corollary:

Corollary 3.1. *Let $A,B,S,T : X\to X$, be self mappings of a metric space (X,d) such for all x,y in X we have:*

$$[1+a(\int_0^{d(Ax,By)}\varphi(t))^p]\Big(\int_0^{d(Sx,Ty)}\varphi(t)dt\Big)^p < a[\Big(\int_0^{d(Ax,Sx)}\varphi(t)dt\Big)^p\Big(\int_0^{d(By,Ty)}\varphi(t)dt\Big)^p$$
$$+\Big(\int_0^{d(Sx,By)}\varphi(t)dt\Big)^p\Big(\int_0^{d(Ax,Ty)}\varphi(t)dt\Big)^p]+\alpha\Big(\int_0^{d(Ax,By)}\varphi(t)dt\Big)^p\Big)$$
$$+\beta\tau[\Big(\int_0^{d(Ax,Sx)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(By,Ty)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Ax,Ty)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(By,Sx)}\varphi(t)dt\Big)^p]$$

where a,α,β are non-negative numbers such $\alpha+\beta<1, p\in\mathbb{N}^*$ and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a Lebesgue-integrable function which is summable on each compact subset of $\mathbb{R}^+$,non-negative, and such that for each $\varepsilon>0, \int_0^{\varepsilon}\varphi(t)dt>0$, if the pair (A,S) is weakly subsequentially continuous and *compatible of type* (E) *as well as* (B,T) *, then* A,B,S *and* T *have a unique common fixed point in* X.

Corollary improves Theorem 2 of Chauhan et al. [10] and some main results of Djoudi and Aliouche [14] and Theorem 2.5 in [31]. If $\alpha = 0$, we obtain the following corollary.

Corollary 3.2. *Let* $A,B,S,T : X \to X$*, be self mappings of a metric space* (X,d) *such for all* x,y *in* X *we have:*

$$\int_0^{d(Sx,Ty)} \varphi(t)dt < F(M(x,y),\phi(M(x,y)))$$

where

$$M(x,y) = \frac{1}{\alpha+\beta}(\alpha\Big(\int_0^{d(Ax,By)} \varphi(t)dt\Big)^p\Big) + \beta\tau\Big(\int_0^{d(Ax,Sx)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(By,Ty)} \varphi(t)dt\Big)^p,$$

$$\Big(\int_0^{d(Ax,Ty)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(By,Sx)} \varphi(t)dt\Big)^p]),$$

$F \in \mathscr{C}, \phi \in \Phi$ *and* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a Lebesgue-integrable function which is summable on each compact subset of* $\mathbb{R}^+$*, non-negative, and such that for each* $\varepsilon > 0, \int_0^{\varepsilon} \varphi(t)dt > 0$*, if the pair* (A,S) *is weakly subsequentially continuous and compatible of type* (E) *as well as* (B,T)*. Hence* A,B,S *and* T *have a unique common fixed point in* X.

Let Λ be a set of all continuous function $\Lambda : \mathbb{R}^5_+ \to \mathbb{R}_+$, such $\lambda(0,0,x,x,x) = kx$, where $0 < k < 1$.

Let $\Lambda_{(F,I,\phi)}$ be a set of all continuous function $\Lambda : \mathbb{R}^5_+ \to \mathbb{R}_+$, such $\lambda(0,0,x,x,x) = F(x,\phi(x))$, where $,\phi \in \Phi, F \in \mathscr{C}$ with (I,ϕ,F) is monotone..

Theorem 3.2. *Let* $A,B,S,T : X \to X$*, be self mappings of a metric space* (X,d) *such for all* x,y *in* X *we have:*

$$[1 + a(\int_0^{d(Ax,By)} \varphi(t))^p]\Big(\int_0^{d(Sx,Ty)} \varphi(t)dt\Big)^p <$$

$$\lambda\left(\begin{array}{c} \Big(\int_0^{d(Ax,Sx)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Ax,Sx)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(By,Ty)} \varphi(t)dt\Big)^p, \\ \Big(\int_0^{d(Ax,Ty)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(By,Sx)} \varphi(t)dt\Big)^p, \end{array}\right) \tag{2}$$

where $\lambda \in \Lambda_{(F,I,\phi)}$*. and* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a Lebesgue-integrable function which is summable on each compact subset of* $\mathbb{R}^+$*, non-negative, and such that for each* $\varepsilon > 0, \int_0^{\varepsilon} \varphi(t)dt > 0$*, assume* *that the two pairs (A,S) and $(B,T)$are weakly subsequentially continuous and compatible of type (E), then A,B,S and T have a unique common fixed point in X.*

Proof. As in proof of Theorem 3.1, z is a coincidence point for A and S and t is a coincidence point for B and T, where

$$\lim_{n\to\infty} By_n = t \text{ and } \lim_{n\to\infty} Ax_n = z,$$

we claim $Az = Bt$, if not by using(2), we get

$$\Big(\int_0^{d(Az,Bt)} \varphi(t)\Big)^p = \Big(\int_0^{d(Sz,Tt)} \varphi(t)dt\Big)^p <$$

$$\lambda\left(\begin{array}{c} 0,0,(\int_0^{d(Az,Tt)} \varphi(t)dt\Big)^p, \\ \Big(\int_0^{d(Sz,Bt)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p \end{array}\right)$$

since $d(Sz,Bt) \leq d(fz,gt)$ and $d(Az,Tt) \leq d(Az,Bt)$, we get

$$\Big(\int_0^{d(Az,Bt)} \varphi(t)\Big)^p < \lambda\left(\begin{array}{c} 0,0,(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p, \\ \Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p \end{array}\right)$$

$$< F(\Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p, \phi(\Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p)) \leq$$

$$\Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p,$$

So, $\Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p = 0$, or, $\phi(\Big(\int_0^{d(Az,Bt)} \varphi(t)dt\Big)^p) = 0$, thus $d(Az,Bt) = 0$, which implies that $Az = Bt$.

Now, we prove $z = Az$, if not by using (2) we get

$$\Big(\int_0^{d(Sx_n,Tt)} \varphi(t)dt\Big)^p \leq \Big(1 + a\Big(\int_0^{d(Ax_n,Bt)} \varphi(t)dt\Big)^p\Big)\Big(\int_0^{d(Sx_n,Tt)} \varphi(t)dt\Big)^p <$$

$$\lambda\left(\begin{array}{c} \Big(\int_0^{d(Ax_n,Sx_n)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Bt,Tt)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Ax_n,Sx_n)} \varphi(t)dt\Big)^p, \\ \Big(\int_0^{d(Bt,Sx_n)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(Ax_n,Bt)} \varphi(t)dt\Big)^p \end{array}\right),$$

letting $n \to \infty$, we get

$$\Big(1 + a\Big(\int_0^{d(z,Bt)} \varphi(t)dt\Big)^p\Big)\Big(\int_0^{d(z,Tt)} \varphi(t)dt\Big)^p <$$

$$\lambda\left(\begin{array}{c} 0,0,\Big(\int_0^{d(z,Tt)} \varphi(t)dt\Big)^p, \\ \Big(\int_0^{d(Bt,z)} \varphi(t)dt\Big)^p, \Big(\int_0^{d(z,Bt)} \varphi(t)dt\Big)^p \end{array}\right),$$

consequently we get

$$\Big(\int_0^{d(z,Az)}\varphi(t)\Big)^p<\lambda\left(\begin{array}{c}0,0,\Big(\int_0^{d(z,Az)}\varphi(t)\Big)^p,\\ \Big(\int_0^{d(z,Az)}\varphi(t)\Big)^p,\Big(\int_0^{d(z,Az)}\varphi(t)\Big)^p\end{array}\right)$$

$$=F(\Big(\int_0^{d(z,Az)}\varphi(t)\Big)^p,\phi(\Big(\int_0^{d(z,Az)}\varphi(t)\Big)^p))\leq\Big(\int_0^{d(z,Az)}\varphi(t)\Big)^p.$$

So, $\Big(\int_0^{d(z,Az)}\varphi(t)\Big)^p=0$, or $,\phi(\Big(\int_0^{d(z,Az)}\varphi(t)\Big)^p)=0$, thus $d(Az,z)=0$n then $z=Az=Sz$. Nextly we claim $z=t$, if not by using (2), we get

$$\Big(1+a\Big(\int_0^{d(Ax_n,By_n)}\varphi(t)dt\Big)^p\Big)\Big(\int_0^{d(Sx_n,Ty_n)}\varphi(t)dt\Big)^p<$$

$$\lambda\left(\begin{array}{c}\Big(\int_0^{d(Ax_n,Sx_n)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(By_n,Ty_n)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Ax_n,Ty_n)}\varphi(t)dt\Big)^p,\\ \Big(\int_0^{d(By_n,Sx_n)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Ax_n,By_n)}\varphi(t)dt)^p\end{array}\right),$$

letting $n\to\infty$, we get

$$\left(1+a\left(\int_0^{d(z,t)}\varphi(t)dt\right)^p\right)\left(\int_0^{d(z,t)}\varphi(t)dt\right)^p<\lambda\left(\begin{array}{c}0,0,\Big(\int_0^{d(z,t)}\varphi(t)\Big)^p,\\ \Big(\int_0^{d(z,t)}\varphi(t)\Big)^p\end{array}\right)$$

$$=F(\left(\int_0^{d(z,t)}\varphi(t)\right)^p,\phi(\left(\int_0^{d(z,t)}\varphi(t)\right)^p))\leq\left(\int_0^{d(z,t)}\varphi(t)\right)^p,$$

So, $\Big(\int_0^{d(z,t)}\varphi(t)\Big)^p=0$, or, $\phi(\Big(\int_0^{d(z,t)}\varphi(t)\Big)^p)=0$, thus $d(z,t)=0$, then z is a common fixed point for A,B,S and T.

For the uniqueness, it is similar as in proof of Theorem 3.1.

With choice $F(s,t)=ks$ and $0<k<1$ in Theorem **??** we get to the following corollary:

Corollary 3.3. *Let $A,B,S,T:X\to X$, be self mappings of a metric space (X,d) such for all x,y in X we have:*

$$[1+a(\int_0^{d(Ax,By)}\varphi(t))^p]\Big(\int_0^{d(Sx,Ty)}\varphi(t)dt\Big)^p<$$

$$\lambda\left(\begin{array}{c}\Big(\int_0^{d(Ax,Sx)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(Ax,Sx)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(By,Ty)}\varphi(t)dt\Big)^p,\\ \Big(\int_0^{d(Ax,Ty)}\varphi(t)dt\Big)^p,\Big(\int_0^{d(By,Sx)}\varphi(t)dt\Big)^p,\end{array}\right)$$

where $\lambda\in\Lambda$ and $\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is a Lebesgue-integrable function which is summable on each compact subset of $\mathbb{R}^+$, non-negative, and such that for each $\varepsilon>0,\int_0^{\varepsilon}\varphi(t)dt>0$, assume that *the two pairs* (A,S) *and* (B,T)*are weakly subsequentially continuous and compatible of type* (E)*, then* A,B,S *and* T *have a unique common fixed point in* X*.*

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] M. Abbas - B. Rhoades, *Common fixed point theorems for hybrid pairs of occasionally weakly compatible mappings satisfying generalized contractive condition of integral type*, Fixed Point Theory Appl. 2007 (2007) Art. ID 54101, 9 pages.

[2] A. Aliouche, *Common fixed point theorems of Gregus type for weakly compatible mappings satisfying generalized contractive conditions*, J. Math. Anal. Appl. 341 (2008), 707–719.

[3] M. A. Al-Thagafi - N. Shahzad, *Generalized I-nonexpansive selfmaps and invariant approximations*, Acta Math. Sinica 24(5) (2008), 867–876.

[4] I. Altun - D. TÃrkoğlu - B.E. Rhoades, *Fixed points of weakly compatible maps satisfying a general contractive condition of integral type*, Fixed Point Theory Appl. 2007 (2007), Article ID 17301, 9 pages.

[5] A.H. Ansari, *Note on "φ-ψ-contractive type mappings and related fixed point*, The 2nd Regional Conference on Mathematics and Applications, Payame Noor University, 2014, 377-380.

[6] S. Beloul, *Some fixed point theorems for weakly subsequentially continuous and compatible of type* (E) *mappings with an application*, Int. J. Nonlianear.Anal.Appl, (7) (1) (2015), 53-62.

[7] S. Beloul and S. Chauhan, *Gregus type fixed points for weakly subsequentially continuous mappings satisfying strict contractive condition of integral type*, vol 71(2) (2015), 3-15.

[8] H. Bouhadjera and C.G. Thobie, *Common fixed point theorems for pairs of subcompatible maps*, arXiv:0906.3159v1 [math.FA] (2009).

[9] D. W. Boyd and S.W.Wong, *On nonlinear contractions*, Proc. Amer. Math. Soc., 20 (1969), 458-464.

[10] S. Chauhan, H. Aydi, W. Shatawani and C. Vetro, *Some integral type fixed-point theorems and an application to systems of functional equations*, Vietnam J. Math. 42(1) (2014), 17–37.

[11] S. Chauhan, M. Imdad, E. Karapınar and B. Fisher, *An integral type fixed point theorem for multi-valued mappings employing strongly tangential property*, J. Egyptian Math. Soc. 22 (2014), 258264.

[12] S. D. Diwan and R. Gupta, *A common fixed point theorem for Gregus Type mappings*, Thai. J. Math., 12 (2014), 103112.

[13] A. Djoudi and A. Aliouche, *Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type*, J. Math. Anal. Appl. 329 (2007), 31–45.

[14] A. Djoudi and L. Nisse, *Gregus type fixed points for weakly compatible mappings*, Bull. Belg. Math. Soc. 10 (2003),, 369–378.

[15] D. Dorić, Z. Kadelburg and S. Radenović, *A note on occasionally weakly compatible mappings and common fixed point*, Fixed Point Theory 13 (2012), 475-480.

[16] M. Geraghty, *On contractive mappings*, Proc. Amer. Math. Soc., 40 (1973), 604-608.

[17] M. M. Gregus Jr., *A fixed point theorem in Banach spaces*, Boll. Unione Mat. Ital. Sez. A 17(5) (1980), 193-198.

[18] M. Imdad, J. Ali and M. Tanveer, *Remarks on some recent metrical common fixed point theorems*, Appl. Math. Lett. 24 (2011), 1165–1169.

[19] G. Jungck, *Commuting mappings and fixed points,* Amer. Math. Monthly, 83(4) (1976), 261–263.

[20] G. Jungck, *Compatible mappings and common fixed points,* Internat. J. Math. Math. Sci. 9 (1986), 771–779.

[21] G. Jungck, P. P. Murthy and Y. J. Cho, *Compatible mappings of type (A) and common fixed points*, Math. Japon. 38 (1993), 381–390.

[22] G. Jungck, *Common fixed points for noncontinuous nonself maps on nonmetric spaces*, Far East J. Math. Sci. 4(2) (1996), 199–215.

[23] G. Jungck and B.E. Rhoades, *Fixed point theorems for occasionally weakly compatible mappings*, Fixed Point Theory 9 (2008), 383–384.

[24] M. S. Khan, M. Swaleh, and S. Sessa, *Fixed point theorems by altering distances between the points*, Bulletin of the Australian Mathematical Society, 30 (1) (1984) 1–9.

[25] k A. Meir and E. Keeler, *A theorem on contraction mappings*, J. Math. Anal. Appl., 28 (1969), 326-329.

[26] R. P. Pant, *A common fixed point theorem under a new condition*, Indian J. Pure Appl. Math. 30(2) (1999), 147–152.

[27] H. K. Pathak and M. S. Khan, *Compatible mappings of type (B) and common fixed point theorems of Gregus type*, Czechoslovak Math. J. 45(120) (1995), 685–698.

[28] H. K. Pathak, Y. J. Cho , S. M. Kang and B. S. Lee, *Fixed point theorems for compatible mappings of type (P) and application to dynamic programming*, Le Matematiche (Fasc. I) 50 (1995), 15–33.

[29] H. K. Pathak, Y. J. Cho, S. M. Khan and B. Madharia, *Compatible mappings of type (C) and common fixed point theorems of Gregus type*, Demonstratio Math. 31(3) (1998), 499–518.

[30] H. K. Pathak, R. R. Lôpez and R. K. Verma, *A common fixed point theorem using implicit relation and property (E.A) in metric spaces*, Filomat 21 (2007), 211–234.

[31] H. K. Pathak and N. Shahzad, *Gregus type fixed point results for tangential mappings satisfying contractive conditions of integral type*, Bull. Belg. Math. Soc. Simon Stevin 16(2) (2009), 277–288.

[32] R. A. Rashwan and M. A. Ahmed, *Common fixed points of Gregus type multi-valued mappings*, Archivum Math. 38(1) (2002), 37–47.

[33] V. Popa, *A general fixed point theorem for four weakly compatible mappings satisfying an implicit relation*, Filomat 19 (2005), 45–51.

[34] S. Sessa, *On a weak commutativity condition of mappings in fixed point considerations*, Publ. Inst. Math. Beograd 32(46) (1982), 149–153.

[35] S. L. Singh and S. N. Mishra, *Coincidence and fixed points of reciprocally continuous and compatible hybrid maps*, Internat. J. Math. Math. Sci. 10 (2002), 627–635.

[36] W. Sintunavarat and P. Kumam, *Gregus-type common fixed point theorems for tangential multivalued mappings of integral type in metric spaces*, International J. Math. Math. Sci. 2011 (2011), Article ID 923458.