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ON THE EXISTENCE OF STRONGLY CONTINUOUS SOLUTION $u \in C_{1-\alpha}(I,E)$ OF A WEIGHTED CAUCHY TYPE PROBLEM OF A FRACTIONAL-ORDER DIFFERENTIAL EQUATION

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Abstract. In this paper, we study the existence of strongly continuous solution $u \in C_{1-\alpha}(I,E)$ of a weighted Cauchy type problem of a fractional-order differential equation.

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1. Preliminaries

In this section we introduce the definitions which will be used in the paper. Also; we stated the definitions and the general properties of the fractional order integration and differentiation. The Nonlinear alternative of Leray-Schauder type Theorem will be stated.

Let $\Gamma(.)$ be the gamma function , and let $L_1(I)$ be the space of all Lebesgue integrable functions

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on the interval $I = [0,1]$.

Denote by $C(I,E)$ the space of strongly continuous functions defined on $I = [0,1]$ with norm

$$||u||_C = \sup_{t\in[0,1]} ||u(t)||_E.$$

Also; define the space $C_{1-\alpha}(I,E)$ by

$$C_{1-\alpha}(I,E) = \left\{u : t^{1-\alpha}u(t) \text{ is continuous on } I = [0,1]\right\},$$

with norm

$$||u||_{C_{1-\alpha}} = \sup_{t\in[0,1]} ||t^{1-\alpha}\, u(t)||_E.$$

The fractional calculus is one of the singular integral and integro-differential operators. For the definition and some properties of the fractional order operators we have the following

Definition1.1

The fractional-order integral of the function $f \in L_1[a,b]$ of order $\beta > 0$ is defined by (see [4] - [7])

$$I_a^\beta\, f(t) = \int_a^t \frac{(t - s)^{\beta - 1}}{\Gamma(\beta)}\, f(s)\, ds,$$

or

$$I_a^\beta\, f(t) = \int_0^{t - a} \frac{u^{\beta - 1}}{\Gamma(\beta)}\, f(t - u)\, du.$$

When $a = 0$, we can write $I^\beta\, f(t) = I_0^\beta\, f(t) = f(t) \star \phi_\beta(t)$, where

$$\phi_\beta(t) = \begin{cases} \frac{t^{\beta - 1}}{\Gamma(\beta)} & t > 0 \\ 0 & t \le 0 \end{cases}$$

and ϕ satisfies the property

$$\phi_{\beta_1}(t) \star \phi_{\beta_2}(t) = \phi_{\beta_1 + \beta_2}(t).$$

Also $\phi_\beta(t) \to \delta(t)$ as $\beta \to 0$, where $\delta(t)$ is the Dirac-delta function.

Remark1.1

It is noted that (see [7]), when $f \in L_1(I), I = [a,b]$, then $I_a^\beta f(t)$ exists for a.e. $t \in [a,b]$. But when $f \in C[a,b]$, then $I_a^\beta f(t)$ exists for all $t \in [a,b]$.

For the fractional (arbitrary) order derivative we have the following definition:

Definition1.2 The Riemann-Liouville fractional-order derivative of $f(t)$ of order $\alpha \in (0,1)$ is defined as (see [4] - [6])

$$D_a^{\alpha} f(t) = \frac{d}{dt} \int_a^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s)\, ds$$

or

$$D_a^{\alpha} f(t) = \frac{d}{dt} I_a^{1-\alpha} f(t).$$

And the derivative of fractional-order $\alpha \in (n-1,n)$ is defined as:

$$D_a^{\alpha} f(t) = \frac{d^n}{dt^n} I_a^{n-\alpha} f(t)\ ,n = 1,2,\cdots.$$

Now, we state Nonlinear alternative of Leray-Schauder type Theorem:

Theorem 1.1. ***(Nonlinear alternative of Leray-Schauder type)***[1]
Let E be a Banach space and Ω be a bounded open subset of E, $0 \in \Omega$ and $T : \bar{\Omega} \to E$ be a completely continuous operator. Then, either there exists $u \in \partial\Omega, \lambda > 1$ such that $T(u) = \lambda u$, or there exists a fixed point $u^* \in \bar{\Omega}$.

2. **Introduction**

In this paper, we study the existence of solutions, in the Banach space $C_{1-\alpha}(I,E)$, for the nonlinear weighted Cauchy-type problem of the following type

$$(1) \qquad \begin{cases} D^{\alpha} u(t) = f(t,u(t)),\ t > 0, \\ \\ t^{1-\alpha} u(t)|_{t=0} = b,\ b > 0. \end{cases}$$

The weighted Cauchy-type problem was studied by many author, for example; In [2] the author studied the existence of L_1-solution of the nonlinear weighted Cauchy-type problem:

$$\begin{cases} D^{\alpha}\, u(t) \;=\; f(t,u(\phi(t))),\; t \;>\; 0, \\ \\ t^{1-\alpha}\, u(t)|_{t=0} \;=\; b,\; b \;>\; 0. \end{cases}$$

such that:

(i) $f:(0,1)\times R\rightarrow R$ satisfies the following assumptions:
  (a) for each $t\in(0,1), f(t,.)$ is continuous,
  (b) for each $u\in R, f(.,u)$ is measurable,
  (c) there exist two real functions $t\rightarrow a(t), t\rightarrow b(t)$ such that

$$|f(t,u)| \;\leq\; a(t) \;+\; b(t)\;|u|, \text{ for each } t\in(0,1),\; u\in R,$$

  where $a(.)\in L_1(0,1)$ and $b(.)$ is measurable and bounded.

(ii) $\phi:(0,1)\rightarrow(0,1)$ is nondecreasing and there is a constant $M>0$ such that $\phi'\geq M$ a.e. on $(0,1)$.

Also, in [3] the author studied the existence of L_p-solution of the same problem mentioned in [2], such that

(i) $f:(0,1)\times R\rightarrow R$ be a function with the following properties:
  (a) for each $t\in(0,1), f(t,.)$ is continuous,
  (b) for each $u\in R$, $f(.,u)$ is measurable,
  (c) for each $t\in(0,1), u\in R$, $f(t,u)$ satisfies the growth condition

$$|f(t,u)| \;\leq\; a(t) \;+\; k\;|u|,$$

  where $a(.)\in L_p(0,1)$ and $k\;\geq\; 0$ be a constant.

(ii) $\phi:(0,1)\rightarrow(0,1)$ is nondecreasing and there is a constant $M>0$ such that $\phi'\geq M$ a.e. on $(0,1)$.

Here we study the existence of strongly continuous solution of the weighted Cauchy problem (1) in the space $C_{1-\alpha}(I,E)$ such that the function $f:I\times E\rightarrow E$ satisfies the following assumptions

(1) for each $t \in I, f(t,.)$ is continuous,

(2) for each $u \in E$, $f(.,u)$ is measurable,

(3) for each $t \in I, u \in E$, there exists a constant M such that $||f(t,u)|| \leq M$.

3. Existence of solution

Now, we are in a position to formulate and prove our main result.

Consider the fractional-order integral equation:

$$u(t) = b\, t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds,\ t \in [0,1]. \tag{2}$$

And define the integral operator $T : E \to E$ by:

$$T\, u(t) = b\, t^{\alpha-1} + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds,\ t \in [0,1].$$

We will solve equation (2) by finding a fixed point of the operator T.

Theorem 3.2. *Let the assumptions (1) - (3) be satisfied, then equation (2) has at least one strongly continuous solution $u \in C_{1-\alpha}(I,E)$.*

Proof: Let $m = b + \frac{M}{\Gamma(1+\alpha)}, \Omega = \{u \in C_{1-\alpha}(I,E) : ||t^{1-\alpha}u||_E \leq m\}$. Suppose $u \in \partial\Omega, \lambda > 1$ such that $Tu = \lambda u$, then

$$\begin{aligned} \lambda\, m &= \lambda\, ||t^{1-\alpha}\, u||_E = ||t^{1-\alpha}\, T\, u||_E \leq \sup_{t\in[0,1]} ||t^{1-\alpha}\, T\, u(t)||_E \\ &\leq b + t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,u(s))\, ds \\ &\leq b + \frac{M}{\Gamma(1+\alpha)}. \end{aligned} \tag{3}$$

Therefore,

$$\lambda < 1,$$

this contradicts $\lambda > 1$. Therefore by Theorem 1.1, if $T : \bar{\Omega} \to E$ is a completely continuous operator, then it has a fixed point $u \in \bar{\Omega}$.

In what follows we show that T is a completely continuous operator. For this for any $u \in \bar{\Omega}$, let $t, \tau \in [0,1]$, then we have

$$\begin{aligned}
||t^{1-\alpha}\, Tu(t) \; - \; \tau^{1-\alpha}\, Tu(\tau)||_E \;\; &\leq \;\; ||\, t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f(s,u(s))\, ds \\
&- \;\; \tau^{1-\alpha} \int_0^\tau \frac{(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}\, f(s,u(s))\, ds\, || \\
&\leq \;\; || \int_0^\tau \frac{t^{1-\alpha}\,(t-s)^{\alpha-1} \; - \; \tau^{1-\alpha}\,(\tau-s)^{\alpha-1}}{\Gamma(\alpha)}\, f(s,u(s))\, ds|| \\
&+ \;\; || \int_\tau^t \frac{t^{1-\alpha}\,(t-s)^{\alpha-1}}{\Gamma(\alpha)}\, f(s,u(s))\, ds|| \\
&\leq \;\; \frac{M}{\Gamma(\alpha)} \,(\int_0^\tau |t^{1-\alpha}\,(t-s)^{\alpha-1} \; - \; \tau^{1-\alpha}\,(\tau-s)^{\alpha-1}|\, ds \\
&+ \;\; \int_\tau^t |t^{1-\alpha}\,(t-s)^{\alpha-1}|\, ds) \\
&\leq \;\; \frac{M}{\Gamma(1+\alpha)}\,(2\,(t-\tau)^\alpha \; + \; |t-\tau|)\,. \qquad (4)
\end{aligned}$$

The above inequality shows that

$$||t^{1-\alpha}\, Tu(t) \; - \; \tau^{1-\alpha}\, Tu(\tau)|| \; \to \; 0 \text{ as } t \; \to \; \tau, \qquad (5)$$

then $t^{1-\alpha}Tu$ is uniformly continuous in $[0,1]$, and hence $T : \bar{\Omega} \to E$ is well defined.

Finally, we use Arzela-Ascoli Theorem to show that $T : \bar{\Omega} \to E$ is compact. Immediately we obtain from inequality (3) that $T(U)$ is uniformly bounded, while the equicontinuity of $T(U)$ follows from inequality (5).

The Arzela-Ascoli Theorem thus guarantees that $T : \bar{\Omega} \to E$ is compact operator, which complete the proof. ■

Now, we are looking for sufficient conditions to ensure the existence of strongly continuous solution to the nonlinear weighted Cauchy-type problem (1).

Theorem 3.3. *If $f : I \times E \to E$ satisfies the assumptions of Theorem 3.2, then the nonlinear weighted Cauchy-type problem (1) has at least one solution $u \in C_{1-\alpha}(I,E)$.*

Proof: Firstly, we will prove the equivalence of problem (1) with the corresponding Volterra integral equation:

$$u(t) \;=\; b\, t^{\alpha-1} \;+\; \int_0^t \frac{(t\;-\;s)^{\alpha-1}}{\Gamma(\alpha)}\, f(s,u(s))\, ds,\; t \in [0,1]. \tag{6}$$

Indeed: Let $u(t)$ be a solution of (6), multiply both sides of (6) by $t^{1-\alpha}$, we get

$$t^{1-\alpha} u(t) \;=\; b \;+\; t^{1-\alpha}\, I^{\alpha}\, f(t,u(t)),$$

which gives

$$t^{1-\alpha} u(t)|_{t=0} \;=\; b.$$

Now, operating by $I^{1-\alpha}$ on both sides of (6), then

$$I^{1-\alpha} u(t) \;=\; b_1 \;+\; I\, f(t,u(t)).$$

Differentiating both sides we get

$$D^{\alpha}\, u(t) \;=\; f(t,u(t)).$$

Conversely, let $u(t)$ be a solution of (1), integrate both sides, then

$$I^{1-\alpha} u(t) \;-\; I^{1-\alpha} u(t)|_{t=0} \;=\; I\, f(t,u(t)),$$

operating by I^{α} on both sides of the last equation, then

$$I u(t) \;-\; I^{\alpha}\, C \;=\; I^{1+\alpha}\, f(t,u(t)),$$

differentiate both sides, then

$$u(t) \;-\; C_1\, t^{\alpha-1} \;=\; I^{\alpha}\, f(t,u(t)),$$

from the initial condition, we find that $C_1 = b$, then we obtain (6), i.e. Problem (1) and equation (6) are equivalent to each other. ■

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