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APPROXIMATE FIXED POINT THEOREMS FOR A NEW CLASS OF OPERATORS ON METRIC SPACES

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Abstract. In this paper, we will first introduce the approximate fixed point property and a new class of operators and a contraction mapping for a cyclic map T on metric spaces. Also, we prove two general lemmas regarding approximate fixed point of cyclic maps on metric spaces. Using these results we prove several approximate fixed point theorems for a new class of operators and contraction mapping on metric spaces.

Keywords: fixed points; approximate fixed points; Mohseni operator; Mohsenialhosseini operator; diameter approximate fixed point.

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1. Introduction

In 2011, Mohsenalhosseini et al [11], introduced the approximate best proximity pairs and proved the property of approximate best proximity pairs. Also, In 2012 , Mohsenalhosseini et al [12], introduced the approximate fixed point for complete norm spaces and map T_α and proved the property of approximate fixed point. In 2014 , Mohsenalhosseini [13] introduced the Approximate best proximity pairs on metric space for contraction maps. Now we give preliminaries and basic definitions which are used throughout the paper. Also, we study different types

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of well-known operators, and we give a new class of operators and contraction maps regarding approximate fixed point for a cyclic map $T : A\cup B \to A\cup B$ i.e. $T(A) \subseteq B$, $T(B) \subseteq A$ on metric spaces.

2. Preliminaries

This section recalls the following notations and the ones that will be used in what follows.

Definition 2.1. [12] *Let $T : X \to X$, $\varepsilon > 0$, $x_0 \in X$. Then $x_0 \in X$ is an $\varepsilon-$fixed point for T if* $\|Tx_0 - x_0\| < \varepsilon$.

Remark 2.2. [12] *In this paper we will denote the set of all $\varepsilon-$ fixed points of T, for a given ε, by :*

$$F_\varepsilon(T) = \{x \in X \mid x \text{ is an } \varepsilon - fixed\ point\ of\ T\}.$$

Definition 2.3. [12] Let $T : X \to X$. Then T has the approximate fixed point property (a.f.p.p) if

$$\forall \varepsilon > 0,\ F_\varepsilon(T) \neq \varnothing.$$

Theorem 2.4. [12] *Let $(X,\|.\|)$ be a complete norm space, $T : X \to X$, $x_0 \in X$ and $\varepsilon > 0$. If $\|T^n(x_0) - T^{n+k}(x_0)\| \to 0$ as $n \to \infty$ for some $k > 0$, then T^k has an $\varepsilon-$ fixed point.*

Lemma 2.5. [5] *Let (X,d) be a metric space, $T : X \to X$ a operator and $\varepsilon > 0$. We assume that:*

(i) $F_\varepsilon(T) \neq \emptyset$;

(ii) $\forall\theta > 0$, $\exists\phi(\theta) > 0$ *such that*;

$$d(x,y) - d(Tx,Ty) \leq \theta \Rightarrow d(x,y) \leq \phi(\theta),\ \forall x,y \in F_\varepsilon(T) \neq \emptyset.$$

Then:

$$\delta(F_\varepsilon(T)) \leq \phi(2\varepsilon).$$

3. Approximate fixed point for cyclic maps on metric spaces

We begin with three lemmas which will be used in order to prove all the results given in the second and third sections.

Definition 3.1. Let A,B be closed subsets of a metric space X. A cyclic map $T : A\cup B \to A\cup B$ is said to be asymptotically regular at a point $x \in A\cup B$, if

$$lim_{n\to\infty} d(T^n(x_0), T^{n+1}(x_0)) = 0,$$

where T^n denotes the nth iterate of T at x.

Lemma 3.2. *Let A,B be nonempty closed subsets of a complete metric space X and $T : A\cup B \to A\cup B$ be a cyclic map. Let $\varepsilon > 0$ and $x_0 \in A\cup B$. If $T : A\cup B \to A\cup B$ is asymptotically regular at a point $x \in A\cup B$, then T has an $\varepsilon-$ fixed point.*

Proof:. The proof of Lemma is such as the proof of Theorem 2.4 for $x \in A\cup B$.

Lemma 3.3. *Let (X,d) be a completely norm space and $T : A\cup B \to A\cup B$ be a cyclic map. Let $\varepsilon > 0$ and $x_0 \in A\cup B$. If $d(T^n(x_0), T^{n+k}(x_0)) \to 0$ as $n \to \infty$ for some $k > 0$, then T^k has an $\varepsilon-$ fixed point.*

Proof:. The proof of Lemma is such as the proof of Theorem 2.4 for $x \in A\cup B$.

Definition 3.4. Let A,B be nonempty closed subsets of a metric space X, $T : A\cup B \to A\cup B$ a cyclic map and $\varepsilon > 0$. We define diameter of the set $F_\varepsilon(T)$, i.e.,

$$\delta(F_\varepsilon(T)) = \sup\{d(x,y) : \ x,y \in F_\varepsilon(T)\}.$$

Lemma 3.5. *Let A and B be nonempty subsets of a metric space X, $T : A\cup B \to A\cup B$ a cyclic map and $\varepsilon > 0$. We assume that:*

(i) $F_\varepsilon(T) \neq \emptyset$;

(ii) $\forall\theta > 0,\ \exists\phi(\theta) > 0$ *such that*;

$$d(x,y) - d(Tx,Ty) \ \leq \ \theta \Rightarrow d(x,y) \leq \phi(\theta),\ \forall x,y \in F_\varepsilon(T) \neq \emptyset.$$

Then:

$$\delta(F_\varepsilon(T)) \leq \phi(2\varepsilon).$$

Proof:. The proof of Lemma is such as the proof of Lemma 2.5 for $x \in A \cup B$.

4. Approximate fixed point for a new class of operators on metric spaces

In this section a series of qualitative and quantitative results will be obtained regarding the properties of approximate fixed point. Also, we prove approximate fixed point theorems for a new class of operators on metric spaces.

Let (X, d) be a metric space. Note that the completeness of the space is not required, as in fixed point theorems.

In 2001, Rus (see [10]) defined $\alpha-$contraction, and in 2006, Berinde (see [5]) obtained some result on $\alpha-$ contraction for approximate fixed point in metric space. In 1968, Kannan (see [4] [8]) proved a fixed point theorem for operators which need not be continuous. A similar type of contractive condition has been studied by Chatterjea (see[6]). In 1972, by combining the above three independent contraction conditions above, Zamfirescu (see [16]) obtained another fixed point result for operators which satisfy the following. In [4], the author obtained a different contraction condition, also he formulated a corresponding fixed point theorem. Now we, introduce a different group of operators for approximate fixed point on metric spaces.

Definition 4.1. [10] A mapping $T : X \to X$ is a $\alpha-$contraction if there exists $\alpha \in (0,1)$ such that

$$d(Tx,Ty) \leq \alpha d(x,y), \forall x,y \in X.$$

Definition 4.2. Let (X,d) be a metric space. A mapping $T : X \to X$ is a **Mohseni operator** if there exists $\alpha \in (0,\frac{1}{2})$ such that

$$d(Tx,Ty) \leq \alpha[d(x,y)+d(Tx,Ty)] \quad \forall x,y \in X.$$

Theorem 4.3. *Let A and B be nonempty subsets of a metric space X and Suppose that the cyclic map $T : A\cup B \to A\cup B$ is a Mohseni operator. Then for every $\varepsilon > 0$, $F_\varepsilon(T)) \neq \emptyset$.*

Proof: Let $\varepsilon > 0$ and $x \in A \cup B$.

$$\begin{aligned} d(T^n x, T^{n+k}x) &= d(T(T^{n-1}x), T(T^{n+k-1}x) \\ &\leq \alpha[d(T^{n-1}x), T^{n+k-1}x) + d(T^n, T^{n+k})]. \end{aligned}$$

Therefore,

$$(1-\alpha)d(T^n x, T^{n+k}x) \le \alpha d(T^n, T^{n+k}).$$

So,

$$\begin{aligned} d(T^n x, T^{n+k}x) &\le \tfrac{\alpha}{(1-\alpha)} d(T^n, T^{n+k}) \\ &\vdots \\ &\le (\tfrac{\alpha}{(1-\alpha)})^n d(x, T^k). \end{aligned}$$

But $\alpha \in (0, \frac{1}{2})$, therefore $(\frac{\alpha}{1-\alpha}) \in (0,1)$. Hence

$$lim_{n\to\infty} d(T^n, T^{n+k}) = 0, \forall x \in A \cup B.$$

Hence by Lemma 3.3 it follows that $F_\varepsilon(T) \neq \emptyset, \forall \varepsilon > 0$. $\square$

Example 4.4. Let $X = [0,\infty)$ and let d be usual metric on X. Suppose $A = [0,2]$ and $B = [0,1]$. Fix $\beta \in (0,1)$ and define $T : A\cup B \to A\cup B$ as

$$Tx = \begin{cases} 0 & x \in [0, 1-\beta) \\ \frac{x}{4} & x \in [1-\beta, 1) \\ \frac{1-\beta}{4} & x \in [1,2] \end{cases}$$

It is easy to check that $T(A) \subseteq B$ and $T(B) \subseteq A$.

For any $x, y \in A\cup B$ there exists $\alpha \in (0, \frac{1}{2})$ such that $d(Tx,Ty) \le \alpha(d(x,y) + d(Tx,Ty))$. So T satisfies all the conditions of Theorem 4.3 and thus for every $\varepsilon > 0$, $F_\varepsilon(T) \neq \emptyset$.

In 2013, Sumit Chandok et al (see[15]), introduced the example of the inequality of Chatterjea-type cyclic weakly contraction. In the following we show that it has approximate fixed point on Mohseni operator.

Example 4.5. Let X be a subset in R endowed with the usual metric. Suppose $A = [0, 0.8]$ and $B = [0, \frac{1}{2}]$. Define the map $T : A\cup B \to A\cup B$ as $Tx = \frac{x}{4}$ for all $x \in A\cup B$. It is easily to be checked that $T(A) \subseteq B$ and $T(B) \subseteq A$. For any $x, y \in A\cup B$ we have the chain of inequalities

$$d(Tx,Ty) = |\frac{x}{4} - \frac{y}{4}| \le \frac{1}{3}(|x-y| + |\frac{x}{4} - \frac{y}{4}|) = \frac{1}{3}(d(x,y) + d(Tx,Ty))$$

So T satisfies all the conditions of Theorem 4.3 and thus for every $\varepsilon > 0$, $F_\varepsilon(T) \neq \emptyset$.

By combining the three independent contraction conditions: $\alpha-$contraction, Mohseni, and Chatterjea operators we will obtain another approximate fixed point result for operators which satisfy the followings:

Definition 4.6. A mapping $T : A\cup B \to A\cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a **Mohsenialhosseini** operator if there exists $\alpha, \beta, \gamma \in R$, $\alpha \in [0,1[, \beta \in [0, \frac{1}{2}[, \gamma \in [0, \frac{1}{2}[$ such that for all $x, y \in A\cup B$ at least one of the following is true:

i) $d(Tx,Ty) \leq \alpha d(x,y)$;

ii) $d(Tx,Ty) \leq \beta[d(x,y)+d(Tx,Ty)]$;

iii) $d(Tx,Ty) \leq \gamma[d(x,T(y))+d(y,T(x))]$.

Theorem 4.7. *Let A and B be nonempty subsets of a metric space X. Suppose that the cyclic map $T : A\cup B \to A\cup B$ is a Mohsenialhosseini operator. Then for every $\varepsilon > 0$, $F_\varepsilon(T)) \neq \emptyset$.*

Proof: Let $x, y \in A\cup B$. Supposing *ii*) holds, we have that:

$$\begin{aligned} d(Tx,Ty) &\leq \beta[d(x,y)+d(Tx,Ty)] \\ &\leq \beta[d(x,Tx)+d(Tx,y)+d(Tx,Ty)] \\ &\leq \beta[d(x,Tx)+d(Tx,x)+d(x,y)+d(Tx,Ty)] \\ &= 2\beta d(x,Tx)+\beta d(x,y)+\beta d(Tx,Ty)] \end{aligned}$$

Thus

$$d(Tx,Ty) \leq \frac{2\beta}{1-\beta}d(x,Tx)+\frac{\beta}{1-\beta}d(x,y). \tag{4.1}$$

Supposing *iii*) holds, we have that:

$$\begin{aligned} d(Tx,Ty) &\leq \gamma[d(x,Ty)+d(y,Tx)] \\ &\leq \gamma[d(x,y)+d(y,Ty)]+\gamma[d(y,Ty)+d(Ty,Tx)] \\ &= \gamma d(Tx,Ty)+2\gamma d(y,Ty)+\gamma d(x,y). \end{aligned}$$

Thus

$$d(Tx,Ty) \leq \frac{2\gamma}{1-\gamma}d(y,Ty)+\frac{\gamma}{1-\gamma}d(x,y). \tag{4.2}$$

Similarly:

$$\begin{aligned} d(Tx,Ty) &\leq \gamma[d(x,Ty)+d(y,Tx)] \\ &\leq \gamma[d(x,Tx)+d(Tx,Ty)]+\gamma[d(y,x)+d(x,Tx)] \\ &= \gamma d(Tx,Ty)+2\gamma d(x,Tx)+\gamma d(x,y). \end{aligned}$$

Then

$$d(Tx,Ty) \leq \frac{2\gamma}{1-\gamma}d(x,Tx)+\frac{\gamma}{1-\gamma}d(x,y). \tag{4.3}$$

Considering i), (3.1), (3.2), (3.3) we can denote:

$$\eta = max\{\alpha_1, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\},$$

and it is easy to see that $\eta \in [0,1[$.

For T satisfying at least one of the conditions i), ii), iii) we have that

$$d(Tx,Ty) \leq 2\eta d(x,Tx)+\eta d(x,y). \tag{4.4}$$

and

$$d(Tx,Ty) \leq 2\eta d(y,Ty)+\eta d(x,y). \tag{4.5}$$

hold. Using these conditions implied by i) - iii) and taking $x \in X$, we have:

$$\begin{aligned} d(T^n x, T^{n+1}x) &= d(T(T^{n-1}x), T(T^n x)) \\ &\leq^{(3.4)} 2\eta d(T^{n-1}x, T(T^{n-1}x) + \eta d(T^{n-1}x, T^n x) \\ &= 3\eta d(T^{n-1}x, T^n x). \end{aligned}$$

Then

$$d(T^n x, T^{n+1}x) \leq \cdots \leq (3\eta)^n d(x,Tx)$$

So then

$$Lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0, \ \forall x \in A \cup B.$$

Now by Lemma 3.2 it follows that $F_\varepsilon(T) \neq \emptyset, \forall \varepsilon > 0$. $\square$

Example 4.8. Let $X = [0,\infty)$ and let d be usual metric on X. Suppose $A = [0,2]$ and $B = [0,1]$. Fix $\beta \in (0,1)$ and define $T : A\cup B \to A\cup B$ as

$$Tx = \begin{cases} 0 & x \in [0,1-\beta) \\ \frac{x}{4} & x \in [1-\beta,1) \\ \frac{1-\beta}{4} & x \in [1,2] \end{cases}$$

It is easy to check that $T(A) \subseteq B$ and $T(B) \subseteq A$. For any $x,y \in A\cup B$ there exists $\alpha \in (0,\frac{1}{2})$ such that holds at least one of the condition Theorem 4.7. Thus Theorem 4.7 for every $\varepsilon > 0$, $F_\varepsilon(T) \neq \emptyset$.

Example 4.9. Let X be a subset in R endowed with the usual metric. Suppose $A = [0,0.8]$ and $[0,\frac{1}{2}]$. Define the map $T : A\cup B \to A\cup B$ as $Tx = \frac{x}{4}$ for all $x \in A\cup B$. It is easy to check that $T(A) \subseteq B$ and $T(B) \subseteq A$. For any $x,y \in A\cup B$ there exists $\alpha \in (0,\frac{1}{2})$ such that holds at least for one of the condition Theorem 4.7. Thus by Theorem 4.7 for every $\varepsilon > 0$, $F_\varepsilon(T) \neq \emptyset$.

Definition 4.10. A mapping $T : A\cup B \to A\cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ is a **Mohseni-semi** operator if there exists $\alpha \in]0,\frac{1}{2}[$ such that

$$d(Tx,Ty) \leq \alpha[d(x,y)+d(x,T(x))],\ \forall x,y \in A\cup B.$$

Theorem 4.11. *Let A and B be nonempty subsets of a metric space X. Suppose that the cyclic map $T : A\cup B \to A\cup B$ is a Mohseni-semi operator. Then:*

$$\forall \varepsilon > 0,\ F_\varepsilon(T) \neq \emptyset.$$

Proof: Let $x \in A\cup B$.

$$\begin{aligned} d(T^n x, T^{n+1}x) &= d(T(T^{n-1}x), T(T^n x)) \\ &\leq \alpha d(T^{n-1}x, T^n x) + \alpha d(T^{n-1}x, T^n x) \\ &= 2\alpha d(T^{n-1}x, T^n x) \leq \cdots \leq (2\alpha)^n d(x,Tx). \end{aligned}$$

But $\alpha \in]0,\frac{1}{2}[$. Therfore

$$Lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0,\ \forall x \in A\cup B.$$

Now by Lemma 3.2, it follows that $F_\varepsilon(T) \neq \emptyset, \forall \varepsilon > 0$. □

Example 4.12. Let $X = [0,\infty)$ and let d be usual metric on X. Suppose $A = [0,2]$ and $B = [0,1]$. Fix $\beta \in (0,1)$ and define $T : A\cup B \to A\cup B$ as

$$Tx = \begin{cases} 0 & x \in [0,1-\beta) \\ \frac{x}{4} & x \in [1-\beta,1) \\ \frac{1-\beta}{4} & x \in [1,2] \end{cases}$$

It is easy to check that $T(A) \subseteq B$ and $T(B) \subseteq A$. For any $x,y \in A\cup B$ there exists $\alpha \in (0,\frac{1}{2})$ such that $d(Tx,Ty) \leq \alpha(d(x,y)+d(x,Tx))$. So T satisfies all the conditions of Theorem 4.11 and thus for every $\varepsilon > 0$, $F_\varepsilon(T) \neq \emptyset$.

Example 4.13. Let X be a subset in R endowed with the usual metric. Suppose $A = [0,0.8]$ and $[0,\frac{1}{2}]$. Define the map $T : A\cup B \to A\cup B$ as $Tx = \frac{x}{4}$ for all $x \in A\cup B$. It is easy to check that $T(A) \subseteq B$ and $T(B) \subseteq A$. For any $x,y \in A\cup B$ we have the chain of inequalities

$$d(Tx,Ty) = |\frac{x}{4} - \frac{y}{4}| \quad \leq \quad \frac{1}{3}(|x-y| + |x - \frac{x}{4}) = \frac{1}{3}(d(x,y)+d(x,Tx))$$

So T satisfies all the conditions of Theorem 4.11 and thus for every $\varepsilon > 0$, $F_\varepsilon(T) \neq \emptyset$.

5. Diameter approximate fixed point for a new class of operators on metric spaces

In this section a series of qualitative and quantitative results will be obtained regarding the properties of diameter approximate fixed point. Also, we prove diameter approximate fixed point theorems for various types of well known operators on a metric space.

Theorem 5.1. *Let (X,d) be a metric space. Suppose that the cyclic mapping $T : A\cup B \to A\cup B$ is a Mohseni operator. Then for every $\varepsilon > 0$,*

$$\delta(F_\varepsilon(T)) \leq \frac{2\varepsilon(1+\alpha)}{1-2\alpha}.$$

Proof: Let $\varepsilon > 0$. and $x \in A\cup B$. Condition i) in Lemma 3.5 is satisfied, as one can see in the proof of Theorem 4.3. Now we only verify that condition 2) in Lemma 3.5, holds.

Let $\theta > 0$ and $x,y \in F_\varepsilon(T)$. We also assume that $d(x,y) - d(Tx,Ty) \leq \theta$. Then:

$$d(x,y) \quad \leq \quad \alpha[d(x,y)+d(Tx,Ty)]+\theta.$$

Therfore

$$d(x,y) \leq \alpha[d(x,y)+d(Tx,x)+d(x,y)+d(y,Ty)]+\theta.$$

As $x,y \in F_\varepsilon(T)$, we know that

$$d(x,Tx) \leq \varepsilon, d(y,Ty) \leq \varepsilon.$$

Therfore,

$$d(x,y) \leq \frac{2\alpha\varepsilon+\theta}{1-2\alpha}.$$

So for every $\theta > 0$ there exists $\phi(\theta) = \frac{2\alpha\varepsilon+\theta}{1-2\alpha} > 0$ such that

$$d(x,y) - d(Tx,Ty) \leq \theta \Rightarrow d(x,y) \leq \phi(\theta).$$

Now by Lemma 3.5, it follows that

$$\delta(F_\varepsilon(T)) \leq \phi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(T)) \leq \frac{2\varepsilon(1+\alpha)}{1-2\alpha}. \square$$

Theorem 5.2. *Let (X,d) be a metric space. Suppose that the cyclic mapping $T : A\cup B \to A\cup B$ is a Mohseni-Chatterjea operator. Then for every $\varepsilon > 0$,*

$$\delta(F_\varepsilon(T)) \leq 2\varepsilon\frac{1+\eta}{1-\eta},$$

where $\eta = max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$, and α, β, γ as in Definition 4.6

Proof: In the proof of Theorem 4.7, we have already shown that if T satisfies at least one of the conditions $i)$, $ii)$, $iii)$ from Definition 4.6, then

$$d(Tx,Ty) \leq 2\eta d(x,Tx) + \eta d(x,y)$$

and

$$d(Tx,Ty) \leq 2\eta d(y,Ty) + \eta d(x,y)$$

hold.

Let $\varepsilon > 0$. We will only verify that condition 2) in Lemma 3.5 is satisfied, as 1) holds, see the Proof of Theorem 4.7.

Let $\theta > 0$ and $x, y \in F_\varepsilon(T)$ and assume that $d(x,y) - d(Tx,Ty) \leq \theta$. Then

$$\begin{aligned} d(x,y) &\leq d(Tx,Ty) + \theta \Rightarrow \\ d(x,y) &\leq 2\eta d(x,Tx) + \eta d(x,y) + \theta \Rightarrow \end{aligned}$$

$$(1-\eta)d(x,y) \leq 2\eta\varepsilon + \theta$$
$$d(x,y) \leq \frac{2\eta\varepsilon + \theta}{1-\eta}.$$

So for every $\theta > 0$ there exists $\phi(\theta) = \frac{2\eta\varepsilon+\theta}{1-\eta} > 0$ such that

$$d(x,y) - d(Tx,Ty) \leq \theta \Rightarrow d(x,y) \leq \phi(\theta).$$

Now by Lemma 3.5, it follows that

$$\delta(F_\varepsilon(T)) \leq \phi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(T)) \leq 2\varepsilon\frac{1+\eta}{1-\eta}, \forall \varepsilon > 0. \square$$

Theorem 5.3. *Let (X,d) be a metric space. Suppose that the cyclic mapping $T : A \cup B \to A \cup B$ is a Mohseni-semi contraction. Then for every $\varepsilon > 0$,*

$$\delta(F_\varepsilon(T)) \leq \varepsilon\frac{2+\alpha}{1-\alpha}.$$

Proof: Let $\varepsilon > 0$. We will only verify that condition 2) in Lemma 3.5 is satisfied. Let $\theta > 0$ and $x, y \in F_\varepsilon(T)$ and assume that $d(x,y) - d(Tx,Ty) \leq \theta$. Then

$$\begin{aligned} d(x,y) &\leq d(Tx,Ty) + \theta \Rightarrow \\ d(x,y) &\leq \alpha[d(x,y) + d(x,T(x))] + \theta \Rightarrow \end{aligned}$$

$$(1-\alpha)d(x,y) \leq \alpha d(x,T(x)) + \theta$$
$$d(x,y) \leq \frac{\alpha\varepsilon + \theta}{1-\alpha}.$$

So for every $\theta > 0$ there exists $\phi(\theta) = \frac{\alpha\varepsilon+\theta}{1-\alpha} > 0$ such that

$$d(x,y) - d(Tx,Ty) \leq \theta \Rightarrow d(x,y) \leq \phi(\theta).$$

Now by Lemma 3.5, it follows that

$$\delta(F_\varepsilon(T)) \leq \phi(2\varepsilon), \forall\varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(T)) \leq \varepsilon\frac{2+\alpha}{1-\alpha}, \forall\varepsilon > 0. \square$$

Remark 5.4. *Examples 4.4 and 4.5 holds in Theorems section 5.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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