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CONSTRUCTION OF MINIMUM NORM SOLUTIONS OF NONLINEAR EQUATIONS WITH p-GENERALIZED STRICTLY PSEUDO-CONTRACTIVE MAPPINGS IN UNIFORMLY CONVEX BANACH SPACES

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Abstract. In a recent paper [15], Saddeek introduced the so-called class of generalized strictly pseudo-contractive mappings and established some strong convergence theorems for the generalized modified Krasnoselskii iterative processes developed by Saddeek [15] for finding the minimum norm solutions of certain nonlinear equations when $p \geq 2$ in the framework of uniformly convex Banach spaces. This paper develops the work presented in [15] by considering separately the case in which $1 < p < 2$.

Keywords: nonlinear equation; minimum norm; p-generalized strictly pseudo-contractive mappings; generalized duality mapping; uniformly convex Banach spaces.

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1. Introduction

The class of generalized strictly pseudo-contractive mappings which has been recently devised by [15] is very general class in the sense that it includes, as special cases, generalized Lipschizian mappings, λ-strictly pseudo-contractive mappings, λ-Lipschitzian mappings, pseudo-contractive mappings and nonexpansive mappings. Such mappings arise in the area of nonlinear functional analysis and its applications, especially those pertinent to fixed point theory and its

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applications to nonlinear pseudomonotone equations and variational inequalities.

Construction of new iterative methods for finding the minimum norm solutions of nonlinear equations involving p-generalized strictly pseudo-contractive mappings and generalized duality mappings in the framework of uniformly convex Banach spaces plays a very significant role in the analysis of many nonlinear fluid dynamics phenomena.

The Krasnoselskii iterative process (see, for example, [9]) is a representative of one of the oldest iterative methods for the solution of nonlinear equations.

In 2008, Saddeek et al. [17, Theorem 2] showed that the Krasnoselskii iterative sequence converges weakly to the solution of nonlinear equation of generalized Lipschitzian mappings in Hilbert spaces. In the same vein, they have presented an application to the stationary problem of filtration.

In an attempt to obtain strong convergence, in 2014, Saddeek [16] proposed the so-called modified Krasnoselskii iterative by boundary point method in the sense of He et al. [6] for finding the minimum norm solutions of certain nonlinear equations with generalized Lipschitzian mappings in Hilbert spaces and proved its strong convergence under some assumptions.

Recently, Saddeek [15] extended the results of Saddeek [16] to the so-called generalized modified Krasnoselskii iterative process with the so-called generalized strictly pseudo-contractive and generalized duality mappings and obtained some strong convergence theorems to the minimum norm solutions of certain nonlinear equations when $p \geq 2$ in the setting of uniformly convex Banach spaces.

The aim of this paper is to develop the study, started in [15] by considering separately the case in which $1 < p < 2$.

2. Preliminaries

Let X be a real Banach space with norm $\|.\|_X$, let X^* be the dual space of X with norm $\|.\|_{X^*}$ and $\langle .,. \rangle$ be the duality pairing between X^* and X. We denote by $\to$ and $\rightharpoonup$ the strong and weak convergence, respectively. Denote by $\mathbb{N}$ the set of all natural numbers.

A Banach space X is said to be uniformly convex if, for any $\varepsilon \in (0,2]$, there exists an increasing positive function $\delta(\varepsilon)$ with $\delta(0) = 0$ such that, for any $x, y \in X$ with $\|x\|_X \leq 1$, $\|y\|_X \leq 1$ and

$\|x-y\|_X \geq \varepsilon,\ \|x+y\|_X \leq 2(1-\delta(\varepsilon))$ holds.

It is known (see, for example, [20]) that every uniformly convex space is reflexive and each real Hilbert space is uniformly convex. Moreover, the sobolev space $\overset{\circ}{W}{}_p^{(1)}$ $(1<p<\infty)$ is uniformly convex.

Let $U=\{x\in X : \|x\|_X=1\}$. Then the norm of X is said to be Gâteaux differentiable (see, for example, [14]) if

$$\lim_{t\to 0^+} \frac{[\|x+ty\|_X - \|x\|_X]}{t}.$$

exists for any $x,y\in U$. The norm of X is said to be uniformly Gâteaux differentiable if, for any $y\in U$, the above limit exists uniformly for all $x\in U$.

Every Hilbert space and $\overset{\circ}{W}{}_p^{(1)}$ $(1<p<\infty)$ space has a uniformly Gâteaux differentiable norm.

The mapping $J_p : X \to 2^{X^*},\ p>1$ defined by

$$J_p x = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|_X^p,\ \|x^*\|_{X^*} = \|x\|_X^{p-1},\ \forall x\in X\},$$

is called the generalized duality mapping.

For $p=2$, the mapping J_2 from X to 2^{X^*} is called the normalized duality mapping.

It is well known (see, for example, [18, Theorem 1.1.17] and [14]) that if X is uniformly convex and has a uniformly Gâteaux differentiable norm, then the generalized duality mapping is single valued (we denote it by j_p), bijective (the inverse of j_p will be denoted by j_p^{-1}), and if C is a nonempty closed and convex subset of X, then there exists a unique $x\in C$ such that $\|x\|_X = \inf_{z\in C}\|z\|_X$, i.e., the metric projection of the origin onto C. Moreover,

$$\langle j_p x, z-x\rangle \geq 0, \forall z\in C. \tag{1}$$

If $X=H$ is a Hilbert space and $p=2$, then the generalized duality mapping becomes the identity mapping of H.

The set of solutions of the variational inequality (1) is denoted by $VI(C, j_p)$, and the element $x\in VI(C, j_p)$, is called the minimum norm solution of (1).

It is well known (see, for example [8] and [11]) that the set $VI(C, j_p)$ is a nonempty closed and convex subset of X.

In [5], Glowinski et al. proved the following properties for the mapping j_p when $1 < p < 2$:

$$\exists \alpha > 0,\ \forall x,y \in X\ \alpha \|x-y\|_X^2 \leq (\|x\|_X + \|y\|_X)^{2-p} \langle j_p x - j_p y, x-y \rangle, \tag{2}$$

$$\exists M > 0,\ \forall x,y \in X\ \|j_p x - j_p y\|_{X^*} \leq M \|x-y\|_X^{p-1}. \tag{3}$$

The following lemma can be founded in [1] and [7].

Lemma 2.1. *Let X be a real uniformly convex Banach space and has a uniformly Gâteaux differentiable norm with X^* as its dual. Then,*

$$\|x^* + y^*\|_{X^*}^2 \leq \|x^*\|_{X^*}^2 + 2\langle y^*, j_p^{-1} x^* - y \rangle,\ \forall x,y \in X, x^*, y^* \in X^*. \tag{4}$$

.

Let us recall the following definition.

Definition 2.1.(see, [4], [11], [13]) For all $x,y \in X$, the mapping $T : X \to X^*$ is said to be as follows:

(i) pseudomonotone, if it is bounded and for every sequence $\{x_n\} \subset X$ such that

$$x_n \rightharpoonup x \in X \textit{ and } \limsup_{n \to \infty} \langle Tx_n, x_n - x \rangle \leq 0$$

we have

$$\liminf_{n \to \infty} \langle Tx_n, x_n - y \rangle \geq \langle Tx, x-y \rangle; \tag{5}$$

(ii) coercive, if

$$\langle Tx, x \rangle \geq \rho(\|x\|_X)\|x\|_X,\ \lim_{\xi \to +\infty} \rho(\xi) = +\infty;$$

(iii) demiclosed at 0, if whenever $\{x_n\}$ is a sequence in X with $x_n \rightharpoonup x$ and $Tx_n \to 0$, then $Tx = 0$;

(iv) potential, if

$$\int_0^1 (\langle T(t(x+y)), x+y \rangle - \langle T(tx), x \rangle)\, dt = \int_0^1 \langle T(x+ty), y \rangle\, dt.$$

Finally, we state the following elementary result on convergence of real sequences.

Lemma 2.2. *(see, for example [12]) Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + b_n + c_n,\ \forall\ n \in \mathbb{N},$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$, $\{b_n\}$ *and* $\{c_n\}$ *are two sequences in* $\mathbb{R}^+$ *such that* (a) $\sum_{n=0}^{\infty}\gamma_n = \infty$; (b) $\limsup_{n\to\infty}\frac{b_n}{\gamma_n} \leq 0$; (c) $\sum_{n=0}^{\infty} c_n < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

3. Generalized iterative for a class of generalized p-strictly pseudo-contractive mappings

Let $p \in (1,2)$ and let $T : X \to X^*$ be a nonlinear mapping. We say that T is p-generalized strictly pseudo-contractive, if for any $x,y \in X$ there exist real valued functions $r_i(x,y) \geq 0, i = 1,2$ satisfying $\sup_{x,y\in X}\{\sum_{i=1}^{2} r_i(x,y)\} = \lambda' < \infty$ such that

$$\|Tx - Ty\|_{X^*}^p \leq r_1(x,y)\|j_p x - j_p y\|_{X^*}^p + r_2(x,y)\|(j_p - T)x - (j_p - T)y\|_{X^*}^p. \tag{6}$$

If $p \geq 2$, then the class of mappings T satisfying (6) with the generalized duality mapping j_p is known as the generalized strictly pseudo-contractive class in the light of Saddeek [15].

We note that for $p = 2$, $r_1(x,y) = 1$, and $r_2(x,y) = \lambda \in [0,1)$ (resp., $r_i(x,y) = 1, i = 1,2$), the class of p-generalized strictly pseudo-contractive mappings coincides with the class of λ-strictly pseudo-contractive (resp., strictly pseudo-contractive) mappings, which was introduced in 1967 by Browder et al. [3] in Hilbert spaces. In addition, if $p = 2$ and $r_2(x,y) = 0$ (resp., $r_1(x,y) = L^2, L > 0, r_2(x,y) = 0$), then we obtain from (6) the class of generalized Lipschitzian (resp., Lipschitzian) mappings, which was studied by Saddeek et al. [17]; if $p = 2$ and $r_1(x,y) = 1$, $r_2(x,y) = 0$ (resp., $r_1(x,y) = \lambda \in (0,1), r_2(x,y) = 0$), then we obtain the class of nonexpansive (resp., λ-contractive) mappings.

For a closed convex subset C of X and $T : C \to X^*$, the sequence $\{x_n\} \subset C$, defined by $x_0 \in C$ and

$$j_p x_{n+1} = (1 - \tau\, h(x_n))\ j_p x_n + \tau\ T_\tau^{j_p} x_n,\ n \geq 0, \tag{7}$$

where $\tau \in (0,1)$, $T_\tau^{j_p} = (1-\tau)\, j_p + \tau\, T$, j_p is the generalized duality mapping, and $h : C \to [0,1]$ is a function defined as follows (see [6]):

$$h(x) = \inf\{\alpha \in [0,1] : \alpha x \in C\},\ \forall\, x \in C, \tag{8}$$

is called the generalized modified Krasnoselskii iterative process, in the sense of Saddeek [15]. If $X = H$, $p = 2$, $h(x_n) = 1$, and $T_\tau^{j_p}$ is replaced by T, then (7) reduces to the so-called Krasnoselskii iterative process (see, [9]).

4. Main results

Now we are in position to state our main result:

Theorem 4.1. *Let C be a closed convex subset of a uniformly convex Banach space X whose norm is uniformly Gâteaux differentiable, let X^* be its dual, and let $j_p : X \to X^*$, $1 < p < 2$ be the generalized duality mapping. Let T be a nonlinear mapping from C to X^* satisfying the condition (3). For $x \in C$ and $\tau \in (0,1)$, let $S_{h(x)} : C \to X^*$ be defined by*

$$S_{h(x)}x = (h(x) + \tau - 1)\, j_p x - \tau T x, \tag{9}$$

where $h(x)$ is given by (8).

Suppose that the constant appearing in (3) satisfies

$$M = \inf_{x,y \in C} [\|x\|_X + \|y\|_X]^{p-2} \|x - y\|_X^{2-p}. \tag{10}$$

Suppose that $S_{h(x)}$ is bounded, coercive, potential, demiclosed at 0, and p-generalized strictly pseudo-contractive with

$$\sup_{x,y \in C} [r_1(x,y) + (2 - h(x))^p r_2(x,y)] = (\lambda')^p < \infty, \tag{11}$$

then, for arbitrary $x_0 \in C$, $0 < \tau = min\{1, \frac{\alpha}{\lambda'}\}$ the generalized modified Krasnoselskii iterative sequence $\{x_n\} \subset C$ defined by

$$j_p x_{n+1} = j_p x_n - \tau\, S_{h(x_n)} x_n,\ n \geq 0, \tag{12}$$

with $\{h(x_n)\}$ *satisfying* $\sum_{n=0}^{\infty} h(x_n)=\infty$, *converges strongly to* $\bar{x}\in VI(S^{-1}_{h(\bar{x})}0, j_p)$, *where* $\bar{x}=proj_{S^{-1}_{h(\bar{x})}0}(0)$, *and* $S^{-1}_{h(\bar{x})}0=\tilde{\mathfrak{F}}(h(\bar{x})j_p, T_\tau^{j_p})=\{\bar{x}\in C: h(\bar{x})j_p\bar{x}=T_\tau^{j_p}\bar{x}\}$.

Proof. Analogously to [15], we define $F: C\to(-\infty,\infty]$ by

$$F(x)=\int_0^1\langle S_{h(x)}(tx),x\rangle\,dt,\ \forall x\in C. \tag{13}$$

We first show that $\{x_n\}$ is bounded. It suffices to show that

$$\{x_n\}\subset S_0,\ \|x_n\|_X\le R_0,\ n\ge 0, \tag{14}$$

where $S_0=\{x\in C: F(x)\le F(x_0)\}$, and $R_0=\sup_{x\in S_0}\ \|x\|_X$.

Using the definition of S_0, it follows directly that $x_0\in S_0$. For $n\in\mathbb{N}$, assume that $x_n\in S_0$.

We now verify that $x_{n+1}\in S_0$. In fact, by the definition of $S_{h(x)}$, (3), (6), and (11), we have

$$\begin{aligned}\|S_{h(x_n)}(x_{n+1}+t(x_n-x_{n+1}))\ &-S_{h(x_n)}(x_n)\|_{X^*}^p\le r_1\|j_p(x_{n+1}+t(x_n-x_{n+1}))-j_px_n\|_{X^*}^p\\ &+r_2\|(j_p-S_{h(x_n)})(x_{n+1}+t(x_n-x_{n+1}))-(j_p-S_{h(x_n)})(x_n)\|_{X^*}^p\\ &\le r_1\|j_p(x_{n+1}+t(x_n-x_{n+1}))-j_p(x_n)\|_{X^*}^p\\ &+r_2[(2-\tau-h(x_n))\|j_p(x_{n+1}+t(x_n-x_{n+1}))-j_p(x_n)\|_{X^*}\\ &+\tau\|T(x_{n+1}+t(x_n-x_{n+1}))-T(x_n)\|_{X^*}]^p\\ &\le(1-t)^{p(p-1)}M^p[r_1+(2-h(x_n))^pr_2]\times\|x_{n+1}-x_n\|_X^{p(p-1)}\\ &\le M^p[r_1+(2-h(x_n))^pr_2]\times\|x_n-x_{n+1}\|_X^{p(p-1)}\\ &\le M^p\lambda'^p\times\|x_n-x_{n+1}\|_X^{p(p-1)},\ \text{for each } t\in[0,1].\end{aligned}$$

This implies that

$$\|S_{h(x_n)}(x_{n+1}+t(x_n-x_{n+1}))-S_{h(x_n)}(x_n)\|_{X^*}\le M\lambda'\|x_n-x_{n+1}\|_X^{p-1}. \tag{15}$$

Therefore, we have

$$|\langle S_{h(x_n)}(x_{n+1}+t(x_n-x_{n+1}))-S_{h(x_n)}(x_n),x_n-x_{n+1}\rangle|\le M\lambda'\|x_n-x_{n+1}\|_X^p. \tag{16}$$

Since $S_{h(x)}$ is potential, it follows from (13) (see [15]) that

$$\begin{aligned} F(x_n)-F(x_{n+1}) &\geq -\int_0^1 |\langle S_{h(x_n)}(x_{n+1}+t(x_n-x_{n+1}))-S_{h(x_n)}x_n, x_n-x_{n+1}\rangle|\, dt \\ &+ \langle S_{h(x_n)}x_n, x_n-x_{n+1}\rangle. \end{aligned} \tag{17}$$

Now from (2), (12), (16), and (17), we have

$$\begin{aligned} F(x_n)-F(x_{n+1}) &\geq -M\lambda'\|x_n-x_{n+1}\|_X^p+\frac{1}{\tau}\langle j_px_n-j_px_{n+1}, x_n-x_{n+1}\rangle \\ &\geq -M\lambda'\|x_n-x_{n+1}\|_X^p+\frac{\alpha}{\tau}[\|x_n\|_X+\|x_{n+1}\|_X]^{p-2}\|x_n-x_{n+1}\|_X^2. \end{aligned} \tag{18}$$

(18) together with (10) give the following inequality

$$F(x_n)-F(x_{n+1}) \geq \mu[\|x_n\|_X+\|x_{n+1}\|_X]^{p-2}\|x_n-x_{n+1}\|_X^2,\ \mu=\frac{\alpha}{\tau}-\lambda'>0. \tag{19}$$

This implies that $F(x_{n+1}) \leq F(x_n) \leq F(x_0)$, and so we obtain $x_{n+1} \in S_0$. Thus, $\{x_n\}$ is bounded and so are $\{S_{h(x_n)}x_n\}$, $\{j_px_n\}$, $\{T_\tau^{j_p}x_n\}$, and $\{Fx_n\}$.

Moreover, we obtain from (19) that the sequence $\{Fx_n)$ is nonincreasing, and thus, we have $\{Fx_n\}$ converges.

This implies from (19) that

$$\lim_{n\to\infty} \|x_n-x_{n+1}\|_X=0, \tag{20}$$

since $p<2$.

Hence, from (3) and (12), we obtain

$$\lim_{n\to\infty} \|j_px_n-j_px_{n+1}\|_{X^*}=0, \tag{21}$$

and

$$\lim_{n\to\infty} \|S_{h(x_n)}x_n\|_{X^*}=0. \tag{22}$$

Since the sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\lim_{k\to\infty}\|x_{n_k}-\bar{x}\|_X$ exists.

From the demiclosedness of $S_{h(x)}$, it follows from (22) that $\bar{x} \in S_{h(\bar{x})}^{-1}0$.

By similar argument as in [15], (20), (21), (22), and (1), we conclude immediately that

$$\limsup_{n\to\infty} \langle S_{h(x_n)}(x_n), x_{n+1}-\bar{x}\rangle \leq 0,\ \forall \bar{x} \in S_{h(\bar{x})}^{-1}0, \tag{23}$$

$$\limsup_{n\to\infty} \langle -j_p\bar{x}, x_{n+1}-\bar{x}\rangle \leq 0, \tag{24}$$

and

$$\bar{x} \in S^{-1}_{h(\bar{x})}0 \cap VI(S^{-1}_{h(\bar{x})}0, j_p), \tag{25}$$

where $\bar{x}$ is the metric projection of the origin onto $S^{-1}_{h(\bar{x})}0$.
Now using Lemma 2.1, Lemma 2.2, $\sum_{n=0}^{\infty} h(x_n) = \infty$, and the arguments of [15], we can obtain

$$\lim_{n\to\infty} \|j_px_n - j_p\bar{x}\|_{X^*} = 0. \tag{26}$$

Now, from (2), we obtain

$$\begin{aligned} \|x_n-\bar{x}\|_X^2 &\leq \frac{1}{\alpha}[\|x_n\|_X + \|\bar{x}\|_X]^{2-p}\langle j_px_n - j_p\bar{x}, x_n-\bar{x}\rangle \\ &\leq \frac{1}{\alpha}[\|x_n\|_X + \|\bar{x}\|_X]^{2-p}\|j_px_n - j_p\bar{x}\|_{X^*}\|x_n-\bar{x}\|_X. \end{aligned} \tag{27}$$

Using again the boundedness of $\{x_n\}$, (20) and (21), we obtain that $\lim_{n\to\infty}\|x_n-\bar{x}\|_X = 0$, i.e., $\lim_{n\to\infty} x_n = \bar{x}$ and the theorem is proved.

Remark 4.1.

1) *Theorem 4.1 is a generalization of both Theorem 4.1 of Saddeek [16] and Theorem 2 of Saddeek et al. [17].*

2) *If $X = H$ and $p = 2$, then restriction (10) of Theorem 4.1 can be removed. In this case, Theorem 4.1 reduces to Corollary 2.1 of Saddeek [15].*

5. Application to nonlinear operator equations

Consider the equation

$$Ax = f, \tag{28}$$

with an arbitrary $f \in X^*$ where $A : C \to X^*$ is a given nonlinear operator.
It is known (see, for example, [19]) that if A is bounded, pseudomonotone and coercive on a separable reflexive Banach space, then there exists a solution $x \in X$ of equation (28).

Such equations occur in many applications, in particular, in the description of stationary problems of filtration of incompressible liquid and the theory of soft shells (see, for example [2], [10]).

Now we shall apply Theorem 4.1 to the case when C is a closed convex subset of the sobolev space $\overset{\circ}{W}{}_p^{(1)}$ $(1 < p < \infty)$, $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz continuous boundary.

Theorem 5.1. *Let $A : C \to X^*$ be a pseudomonotone, coercive, and potential operator such that*

$$\|Ax - Ay\|_{X^*} \leq \|j_p x - j_p y\|_{X^*}, \forall x, y \in C. \tag{29}$$

Then for any $x_0 = x \in C$, the iterative sequence constructed by

$$j_p x_{n+1} = j_p x_n - \tau(Ax_n - f),\ n \geq 0, \tag{30}$$

with τ satisfying the condition $0 < \tau = \min\{1, \alpha\}$, converges strongly to the minimum norm solution of the equation (28). Provided that $\sum_{n=0}^{\infty} h(x_n) = \infty$.

Proof. Following the same arguments as in the proof of Theorem 3.1 of Saddeek [15], we can prove this theorem.

Conflict of Interests

The authors declare that there is no conflict of interests.

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