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SOME FIXED POINT RESULTS LINKED TO $\alpha-\beta$ RATIONAL CONTRACTIONS AND MODIFIED MULTIVALUED HARDY-ROGERS OPERATORS

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Abstract. In this paper two new fixed point results are studied. The first result is a theorem that involves $(\alpha-\beta)$ type rational singlevalued contractions, in the sense of Geraghty type operators. The second result consists of multivalued modified Hardy Rogers operators, namely the existence of the fixed point, data dependence, local version involving two metrics and homotopy theorems involving two metrics are studied.

Keywords: fixed point theorems; complete metric space; rational contractions; multivalued; homotopy; Geraghty; Hardy-Rogers.

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1. Preliminaries for rational Geraghty type mappings

The first idea of the present article is that in the third section we want to prove a theorem based on rational $\alpha-\beta-$contractions, so we remind the necessary concepts for this type of operators. For more informations, we let the reader follow [13]. We recall the following crucial concepts.

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Definition 1.1. Let X be a nonempty set and $\alpha : X \times X \to [0,\infty)$ be a mapping. Then $f : X \to X$ is called $\alpha-$admissible, if it satisfies the following condition for each $x,y \in X$, with $\alpha(x,y) \geq 1 \Longrightarrow \alpha(fx,fy) \geq 1$.

Definition 1.2. The mapping $\alpha : X \times X \to [0,\infty)$ is called transitive, if for each $x,y,z \in X$, with $\alpha(x,y) \geq 1$ and $\alpha(y,z) \geq 1$, we have $\alpha(x,z) \geq 1$.

Moreover, let's denote by Y the set of all functions $\beta : [0,\infty) \to [0,1)$, satisfying $\lim_{n\to\infty} \beta(t_n) = 1 \Longrightarrow \lim_{n\to\infty} t_n = 0$.
Also, we recall the definition of $\alpha-\beta-$contractions, given by Sintunavarat in [13].

Definition 1.3. Let (X,d) be a metric space. A mapping $f : X \to X$ is called an $-\alpha-\beta-$contraction, if there exists $\alpha : X \times X \to [0,\infty)$ and $\beta \in Y$, such that
$[\alpha(x,y) - 1 + \delta_*]^{d(fx,fy)} \leq \delta^{\beta(d(x,y))d(x,y)}$, for each $x,y \in X$, with $1 < \delta \leq \delta_*$.

In [13], the author proved a series of theorems such as : existence of a fixed point assuming that the mapping f is continuous, a theorem in which the continuity is dropped and a theorem for the uniqueness of fixed points. Also, this was based on the result of Geraghty [3] from 1973. Moreover, [8] Paunović et. al. extended the result of W. Sintunavarat in the framework of b-metric spaces. Additionally, they have studied fixed points for a given mapping $F : X \to X$, such that
$[\alpha(x,y) - 1 + \delta]^{d(Fx,Fy)} \leq \delta^{\lambda M(x,y)}$, for each $x,y \in X$, with $1 < \delta$, where $\lambda \in \left[0, \frac{1}{s}\right]$ and
$M(x,y) = \left\{d(x,y), d(x,Fx), d(y,Fy), \frac{d(x,Fy)+d(y,Fx)}{2s}\right\}$, where s was the coefficient of the b-metric space (X,d).

Furthermore, Zabihi and Razani [14] considered rational type operators and developed some fixed point results in the framework of complete b-metric spaces. In the context of a metric space (X,d), a self mapping f on X was considered, satisfying
$d(fx,fy) \leq \beta(d(x,y))M(x,y) + L \cdot N(x,y)$, where $L \geq 0$,
$M(x,y) = max\left\{d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\right\}$ and $N(x,y) = min\left\{d(x,fx), d(x,fy), d(y,fx), d(y,fy)\right\}$.

Moreover, since we want to define some other type of $\alpha-\beta-$contractions, we shall recall that the authors in [12] developed new fixed point theorems involving a new type of rational contractive Geraghty mapping in b-metric spaces. This rational Geraghty mapping is introduced as follows, in the case of metric spaces.

Definition 1.4. Let (X,d) be a metric space. A mapping $f: X \to X$ is called a rational Geraghty of type I, if $d(fx,fy) \leq \beta\left(M(x,y)\right)M(x,y)$, for each $x,y \in X$,
where $\beta \in Y$ and $M(x,y) = max\left\{d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(x,y)}, \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\right\}$.

That means that in the third section, we will present a generalized theorem for rational Geraghty mappings of type I.

2. Preliminaries for modified multivalued Hardy-Rogers

In this section, we recall some general notions in the framework of multivalued analysis theory. Also, for the following preliminary notions and lemmas (such as : multivalued weakly Picard operators, data dependence of the fixed point set, Haussdorf metric properties) we refer the reader to [9], [10] and [11].

Let (X,d) be a metric space and $P(X)$ be the family of all nonempty subsets of X.

We denote by $P_{cl}(X)$ the family of all nonempty subsets of X which are closed, by $P_b(X)$ the family of all nonempty subsets of X which are bounded and by $P_{cp}(X)$ the family of all nonempty subsets of X which are compact.

Furthermore, we consider the following functionals

$D: P(X)\times P(X) \to \mathbb{R}_+$, $D(A,B) = \inf\{d(a,b)/a \in A, b \in B\}$

$H: P_b(X)\times P_b(X) \to \mathbb{R}_+$, $H(A,B) = \max\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A)\}$

$\rho: P_b(X)\times P_b(X) \to \mathbb{R}_+$, $\rho(A,B) = \sup\{D(a,B)/a \in A\}$

We recall some useful results concerning the Haussdorf-Pompeiu generalized functional H.

Lemma 2.1. Let $q>1$ and $A,B \in P(X)$.

Then, for each $a \in A$, there exists $b \in B$ such that $d(a,b) \leq qH(A,B)$.

Lemma 2.2. Let (X,d) be a metric space and $A,B \in P(X)$.

Suppose that there exists $\eta > 0$ such that :

(i) for each $a \in A$, there exists $b \in B$ such that $d(a,b) \leq \eta$,

(ii) for each $b \in B$, there exists $a \in A$ such that $d(a,b) \leq \eta$.

Then $H(A,B) \leq \eta$.

Moreover, if Y is a nonempty subset of X and $T : Y \to P(X)$ a multivalued operator, then an element $x \in Y$ is

(a) a fixed point of T if and only if $x \in Tx$;

(b) a strict fixed point of T if and only if $\{x\} = Tx$;

Furthermore, we denote by F_T the set of all fixed points of T and by $(SF)_T$ the set of all strict fixed points of T.

We also remind the definition of the graphic of a multivalued operator, i.e.

$G_T := \{(x,y) \in Y \times X / y \in Tx\}$.

Definition 2.3. Let (X,d) be a metric space and $T : X \to P(X)$ a multivalued operator. We say that T is a multivalued weakly Picard operator (briefly MWP) if for each $x \in X$ and for each $y \in Tx$, there exists a sequence $(x_n) \in X$, satisfying the following

(i) $x_0 = x$, $x_1 = y$;

(ii) $x_{n+1} \in Tx_n$, for each $n \in \mathbb{N}$;

(iii) the sequence (x_n) is convergent to a fixed point of T.

Definition 2.4. Let (X,d) be a metric space and $T : X \to P(X)$ an MWP.

Then T is called a $c-$weakly Picard operator, with $c \in [0,\infty)$, if there exists a selection t^∞ of T^∞, such that

$d(x,t^\infty(x,y)) \leq cd(x,y)$, for each $(x,y) \in G_T$.

Now we focus our attention to the case of Hardy-Rogers type mappings. In [7], the basic notion of singlevalued Hardy-Rogers contraction appeared.

Definition 2.5. Let (X,d) be a metric space and $T : X \to X$ be an operator such that there exists $\alpha,\beta \geq 0$ with $\alpha+\beta < 1$, satisfying

$d(Tx,Ty) \leq \dfrac{\alpha d(y,Ty)\,[1+d(x,Tx)]}{1+d(x,y)} + \beta d(x,y)$, for each $x,y \in X$.

In [6], Oprea A. developed a theorem concerning multivalued rational contractions (of Hardy Rogers type).

A multivalued operator $T : X \to P(X)$ is called a multivalued rational type contraction, if satisfies the following condition
$H(Tx,Ty) \leq \frac{\alpha D(y,Ty)\,[1+D(x,Tx)]}{1+d(x,y)} + \beta d(x,y)$, for each $x,y \in X$.
Oprea has showed that the multivalued rational contractions are MWP-operators and developed theorems for data dependence, fractal theory, Ulam-Hyers stability etc.
In [4], Kumari and Panthi introduced a new type of rational contractions, called modified Hardy-Rogers contractions.
They introduced this as types of cyclic contractions for the case of families of dislocated metric spaces.
We recall the notion of singlevalued contractions in the context of metric spaces, i.e. singlevalued operator that satisfies

$$d\,(Tx,Ty) \leq \alpha d(x,y) + \beta d(x,Ty) + \gamma d(y,Tx) + \delta d(y,Ty) + \eta \frac{d(y,Ty)[1+d(x,Tx)]}{1+d(x,y)} + \lambda \frac{d(y,Ty)+d(y,Tx)}{1+d(y,Ty)d(y,Tx)} + \mu \frac{d(x,Tx)[1+d(y,Tx)]}{1+d(x,y)+d(y,Ty)}.$$

Also, regarding Hardy Rogers mappings, our purpose to define the concept of modified Hardy-Rogers contractions under the multivalued case shall be presented in the last section, along with some fixed point results.

3. Some theorems regarding rational Geraghty $\alpha-\beta-$contractions

In this section we present a generalized theorem for rational Geraghty mappings or type I, using the $\alpha-$admissibility conditions by Sintunavarat.
Moreover, we will use the same terminology from the first section.

Definition 3.1. Let (X,d) be a metric space.
A mapping $f : X \to X$ is called an $-\alpha-\beta-$rational Geraghty mapping of type I if and only if there exists $\alpha : X \times X \to [0,\infty)$ and $\beta \in Y$, such that
$[\alpha(x,y) - 1 + \delta_*]^{d(fx,fy)} \leq \delta^{\beta(M(x,y))M(x,y)}$, for each $x,y \in X$, with $1 < \delta \leq \delta_*$,
where $M(x,y) = max\left\{d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(x,y)}, \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\right\}$.

Our first main result of this section is the existence theorem for $\alpha-\beta-$rational Geraghty mappings of type I, using the assumption that f is continuous. The techniques used in the theorem's proof follow the same lines as in the theorems from [13].

Theorem 3.2. *Let (X,d) be a complete metric space and $f:X\to X$ an $\alpha-\beta$ rational Geraghty mapping of type I. Also, suppose that the following assumptions hold*

(i) f is $\alpha-$admissible,

(ii) α is transitive,

(iii) there exists $x_0\in X$, such that $\alpha(x_0,fx_0)\geq 1$,

(iv) f is continuous.

Then, there exists $x^*\in X$, such that $x^*=fx^*$.

Proof. • Let $x_0\in X$ satisfying $\alpha(x_0,fx_0)\geq 1$.

Let's consider the Picard sequence $x_{n+1}=fx_n$, for each $n\in\mathbb{N}$.

If there exists $n\in\mathbb{N}$ such that $x_n=x_{n-1}$, then x_{n-1} is a fixed point and the conclusion holds.

Suppose that for each $n\in\mathbb{N}$, $x_n\neq x_{n-1}$. So $d(x_{n-1},x_n)>0$, for each $n\in\mathbb{N}$.

From condition (i), we know that f is $\alpha-$admissible. Since $\alpha(x_0,x_1)=\alpha(x_0,fx_0)\geq 1$, then we have that $\alpha(x_1,x_2)=\alpha(x_1,fx_1)=\alpha(fx_0,fx_1)\geq 1$.

Inductively, one can show that $\alpha(x_{n-1},x_n)\geq 1$, for each $n\in\mathbb{N}$.

Now, we estimate

$$\delta^{d(x_n,x_{n+1})}=\delta^{d(fx_{n-1},fx_n)}\leq\delta_*^{d(fx_{n-1},fx_n)}\leq$$
$$[\alpha(x_{n-1},x_n)-1+\delta_*]^{d(fx_{n-1},fx_n)}\leq\delta^{\beta(M(x_{n-1},x_n))M(x_{n-1},x_n)},$$

so $d(x_n,x_{n+1})\leq\beta\left(M(x_{n-1},x_n)\right)\cdot M(x_{n-1},x_n)$.

Moreover, we make the following computations :

$$M(x_{n-1},x_n)=max\Big\{d(x_{n-1},x_n),\frac{d(x_{n-1},fx_{n-1})d(x_n,fx_n)}{1+d(x_{n-1},x_n)},\frac{d(x_{n-1},fx_{n-1})d(x_n,fx_n)}{1+d(fx_{n-1},fx_n)}\Big\}$$
$$=max\Big\{d(x_{n-1},x_n),\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)},\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})}\Big\}$$

Since $\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_{n-1},x_n)}=d(x_n,x_{n+1})\cdot\frac{d(x_{n-1},x_n)}{1+d(x_{n-1},x_n)}\leq d(x_n,x_{n+1})\cdot 1=d(x_n,x_{n+1})$ and $\frac{d(x_{n-1},x_n)d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})}=d(x_{n-1},x_n)\cdot\frac{d(x_n,x_{n+1})}{1+d(x_n,x_{n+1})}\leq d(x_{n-1},x_n)\cdot 1=d(x_{n-1},x_n)$,

we get that $M(x_{n-1},x_n)\leq max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\}$, for each $n\in\mathbb{N}$, $n\geq 1$.

Now, we consider two cases.

(I) If $max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$, then we get

$d(x_n,x_{n+1}) \leq \beta\,(M(x_{n-1},x_n)) \cdot d(x_n,x_{n+1})$.

Since $\beta(M(x_{n-1},x_n)) < 1$, because $\beta \in Y$,we get the contradiction $d(x_n,x_{n+1}) < d(x_n,x_{n+1})$.

(II) Then, only the second case is valid, i.e. $max\{d(x_{n-1},x_n),d(x_n,x_{n+1})\} = d(x_{n-1},x_n)$, that is $d(x_n,x_{n+1}) \leq \beta(M(x_{n-1},x_n))d(x_{n-1},x_n) < d(x_{n-1},x_n)$, for each $n \in \mathbb{N}$.

So, the sequence $(d(x_n,x_{n+1}))$ is strictly decreasing and nonnegative. It implies that there exists $r \geq 0$, such that $d(x_n,x_{n+1}) \to r$ as $n \to \infty$.

Now, we show that $r = 0$.

Let's suppose that $r > 0$.

We know that $d(x_{n+1},x_{n+2}) \leq \beta\,(M(x_n,x_{n+1}))\,d(x_n,x_{n+1})$. Taking the limit as $n \to \infty$, we get that $r \leq \lim\limits_{n\to\infty} \beta\,(M(x_n,x_{n+1})) \cdot r$.

Because $r > 0$,we get that $1 \leq \lim\limits_{n\to\infty} \beta\,(M(x_n,x_{n+1}))$.

But $\beta\,(M(x_n,x_{n+1})) < 1$, so $\lim\limits_{n\to\infty} \beta\,(M(x_n,x_{n+1})) \leq 1$. From all this, we find that $\lim\limits_{n\to\infty} \beta\,(M(x_n,x_{n+1})) = 1$. This implies that $\lim\limits_{n\to\infty} M(x_n,x_{n+1}) = 0$.

Now, because $M(x_n,x_{n+1})$ is the maximum between three elements, if it's limit is 0, so all the elements have the limit 0. This means that $d(x_n,x_{n+1}) \to 0$. This is a contradiction !

• We now show that (x_n) is a Cauchy sequence.

By reductio ad absurdum, let's suppose that (x_n) is not Cauchy. Then there exists $\varepsilon > 0$ and there exists n_k and m_k, such that $n_k > m_k \geq k$, with $d(x_{m_k},x_{n_k})$ and n_k being the smallest index satisfying the following

$$d(x_{m_k},x_{n_k}) \geq \varepsilon \text{ and } d(x_{m_k},x_{n_k-1}) < \varepsilon.$$

By triangular inequality, we have that $\varepsilon \leq d(x_{m_k},x_{n_k}) \leq d(x_{m_k},x_{n_k-1}) + d(x_{n_k-1},x_{n_k}) < \varepsilon + d(x_{n_k-1},x_{n_k})$.

Since $d(x_{n_k-1},x_{n_k}) \to 0$, taking $k \to \infty$, it follows that

$\lim\limits_{k\to\infty} d(x_{m_k},x_{n_k}) = \varepsilon > 0$.

Like in [13], since α is transitive, we observe that $\alpha(x_{m_k},x_{n_k}) \geq 1$, $k \in \mathbb{N}$.

Now, we make the follow estimation

$$\begin{aligned}\delta^{d(x_{m_k},x_{n_k})} &\leq \delta^{d(x_{m_k},x_{m_k+1})+d(x_{m_k+1},x_{n_k+1})+d(x_{n_k+1},x_{n_k})}\\ &\leq \delta^{d(x_{m_k},x_{m_k+1})+d(x_{n_k},x_{n_k+1})}\cdot\delta^{d(fx_{m_k},fx_{n_k})}\\ &\leq \delta^{d(x_{m_k},x_{m_k+1})+d(x_{n_k},x_{n_k+1})}\cdot\delta_*^{d(fx_{m_k},fx_{n_k})}\\ &\leq \delta^{d(x_{m_k},x_{m_k+1})+d(x_{n_k},x_{n_k+1})}\cdot[\alpha(x_{m_k},x_{n_k})-1+\delta_*]^{d(fx_{m_k},fx_{n_k})}\\ &\leq \delta^{d(x_{m_k},x_{m_k+1})+d(x_{n_k},x_{n_k+1})}\cdot\delta^{\beta(M(x_{m_k},x_{n_k}))M(x_{m_k},x_{n_k})}\end{aligned}$$

So, we get that

$d(x_{m_k},x_{n_k})\leq d(x_{m_k},x_{m_k+1})+d(x_{n_k},x_{n_k+1})+\beta(M(x_{m_k},x_{n_k}))M(x_{m_k},x_{n_k})$.

Furthermore

$$\lim_{k\to\infty}M(x_{m_k},x_{n_k})=$$

$$\lim_{k\to\infty} max\left\{d(x_{m_k},x_{n_k}),\frac{d(x_{m_k},x_{m_k+1})d(x_{n_k},x_{n_k+1})}{1+d(x_{m_k},x_{n_k})},\frac{d(x_{m_k},x_{m_k+1})d(x_{n_k},x_{n_k+1})}{1+d(x_{m_k+1},x_{n_k+1})}\right\}.$$

Since $d(x_{m_k},x_{m_k+1})\to 0$ and $d(x_{n_k},x_{n_k+1})\to 0$ as $k\to\infty$, we get that

$\lim_{k\to\infty}M(x_{m_k},x_{n_k})\leq\lim_{k\to\infty}d(x_{m_k},x_{n_k})=\varepsilon>0$.

In the above inequality, taking the limit as $k\to\infty$, we have that

$\varepsilon\leq\lim_{k\to\infty}\beta(M(x_{m_k},x_{n_k}))\cdot\varepsilon$.

Using the fact that $\varepsilon>0$, it follows that $\lim_{k\to\infty}\beta(M(x_{m_k},x_{n_k}))=1$, i.e. $\leq\lim_{k\to\infty}M(x_{m_k},x_{n_k})=0$.

Since $M(x_{m_k},x_{n_k})$ is the maximum of three elements and it has the limit 0, also because $M(x_{m_k},x_{n_k})\geq d(x_{m_k},x_{n_k})$, then all of the elements will have the limit 0, so $d(x_{m_k},x_{n_k})\to 0$, which is false; so (x_n) is Cauchy.

• Since X is complete with respect to the metric d, there exists $x^*\in X$ such that $x^*=\lim_{n\to\infty}x_n$.

Because f is continuous, we infer that

$x^*=\lim_{n\to\infty}x_n=\lim_{n\to\infty}fx_{n-1}=f\left(\lim_{n\to\infty}x_{n-1}\right)=fx^*$, so x^* is a fixed point for the Geraghty-type mapping f.

Now, also based on [13], we give a theorem where we dropped the continuity of the operator f.

Theorem 3.3. *Let (X,d) be a complete metric space and $f:X\to X$ an $\alpha-\beta$ rational Geraghty mappings of type I.*

Let's suppose that the following assumptions hold

(i) f is $\alpha-$admissible,

(ii) α is transitive,

(iii) there exists $x_0 \in X$, such that $\alpha(x_0, fx_0) \geq 1$,

(iv) if (x_n) is a sequence satisfying $\alpha(x_n, x_{n+1}) \geq 1$ and $x_n \to x$ implies that $\alpha(x_n, x) \geq 1$, for each $n \in \mathbb{N}$.

Then, there exists $x^* \in X$, such that $x^* = fx^*$.

Proof. In a similar manner like in the previous proof, we can show that (x_n) is a Cauchy sequence and therefore there exists $x^* \in X$, such that $x_n \to x^*$ when $n \to \infty$.
From (iv), we have that $\alpha(x_n, x^*) \geq 1$, for each $n \in \mathbb{N}$.
We make the following estimation

$$\delta^{d(x^*,fx^*)} \leq \delta^{d(x^*,x_{n+1})+d(x_{n+1},fx^*)} =$$
$$\delta^{d(x^*,x_{n+1})} \cdot \delta^{d(fx_n,fx^*)} \leq \delta^{d(x^*,x_{n+1})} \cdot \delta_*^{d(fx_n,fx^*)}$$
$$\leq \delta^{d(x^*,x_{n+1})} \cdot [\alpha(x_n,x^*) - 1 + \delta_*]^{d(fx_n,fx^*)}$$
$$\leq \delta^{d(x^*,x_{n+1})} \cdot \delta^{\beta(M(x_n,x^*))M(x_n,x^*)}.$$

So, we have that $d(x^*, fx^*) \leq d(x^*, x_{n+1}) + \beta(M(x_n, x^*))M(x_n, x^*) < d(x^*, x_{n+1}) + M(x_n, x^*)$.
Furthermore, we have that
$M(x_n, x^*) = max\left\{d(x_n, x^*), \dfrac{d(x_n, x_{n+1})d(x^*, fx^*)}{1 + d(x_n, x^*)}, \dfrac{d(x_n, x_{n+1})d(x^*, fx^*)}{1 + d(x_{n+1}, fx^*)}\right\}$.
Taking the limit as $n \to \infty$ and using the fact that $d(x_n, x_{n+1}) \to 0$ and that $d(x_n, x^*) \to 0$, we get that $M(x_{n-1}, x_n) \to max\{\lim_{n\to\infty} d(x_n, x^*), 0, 0\} = \lim_{n\to\infty} d(x_n, x^*) = 0$.
Thus $d(x^*, fx^*) \leq \lim_{n\to\infty} d(x_{n+1}, x^*) + \lim_{n\to\infty} d(x_n, x^*) = 0$, so the conclusion holds properly.

Finally, we present the theorem for the uniqueness of the fixed point for the Geraghty type operator.

Theorem 3.4. *Let's suppose that all the assumptions from the last theorem are satisfied. Additionally, let's suppose that one of the following assumptions are valid*

(H0) if a,b are two fixed points , then $\alpha(a, b) \geq 1$

(H1) for each $x,y \in X$, there exists $z \in X$ such that $\alpha(x,z) \geq 1$ and $\alpha(y,z) \geq 1$.

Then, f admits a unique fixed point.

Proof. Let x^*, y^* two fixed points for the mapping f.

We consider two cases

$(H0)$ We have that $\alpha(x^*, y^*) \geq 1$.

From the Geraghty condition, it is easy to see that $d(x^*, y^*) = 0$, so the conclusion is true.

$(H1)$ We have that there exists $z \in X$, with $\alpha(x^*, z) \geq 1$ and $\alpha(y^*, z) \geq 1$.

Since f is α-admissible, by induction, we get that $\alpha(x^*, f^n z) \geq 1$ and $\alpha(y^*, f^n z) \geq 1$.

We make the following estimation

$$\delta^{d(x^*, f^{n+1}z)} = \delta^{d(fx^*, f^{n+1}z)} \leq \delta_*^{d(fx^*, f(f^n z))} \leq$$
$$[\alpha(x^*, f^n z) - 1 + \delta_*]^{d(fx^*, f(f^n z))} \leq$$
$$\delta^{\beta(M(x^*, f^n z))M(x^*, f^n z)}$$

So $d(x^*, f^{n+1}z) \leq \beta(M(x^*, f^n z))M(x^*, f^n z)$, for each $n \in \mathbb{N}$.

Now we show that $d(x^*, f^n z) \to 0$ as $n \to \infty$.

By reductio ad absurdum, we suppose that $0 < l := \lim_{n\to\infty} d(x^*, f^n z) < \infty$.

We know that

$\lim_{n\to\infty} M(x^*, f^n z) = \lim_{n\to\infty} max\left\{d(x^*, f^n z), \frac{d(x^*, fx^*)d(f^n z, f^{n+1}z)}{1 + d(x^*, f^n z)}, \frac{d(x^*, fx^*)d(f^n z, f^{n+1}z)}{1 + d(fx^*, f^{n+1}z)}\right\}$.

Since x^* is a fixed point for f, then $\lim_{n\to\infty} M(x^*, f^n z) \leq \lim_{n\to\infty} d(x^*, f^n z) = l$.

Taking the limit as $n \to \infty$ and using the fact that $l > 0$, we get that

$l \leq \lim_{n\to\infty} \beta(M(x^*, f^n z)) \cdot l$. Now, because $\beta \in Y$, it follows that $\lim_{n\to\infty} \beta(M(x^*, f^n z)) \leq 1$. So, it follows that $\lim_{n\to\infty} \beta(M(x^*, f^n z)) = 1$. This means that $\lim_{n\to\infty} M(x^*, f^n z) = 0$.

By the same reasoning as in the last proof, we get that $\lim_{n\to\infty} d(x^*, f^n z) = 0$. Furthermore, in a similar way, one can show that $f^n z \to y^*$ as $n \to \infty$, so $x^* = y^*$.

Remark 3.5. Taking $\alpha(x,y) = 1$, for each $x,y \in X$, we get the existence and uniqueness for Geraghty mappings of type I as a corollary.

4. Fixed Point Results for Modified Multivalued Hardy-Rogers contractions

In this section we introduce the concept of modified multivalued Hardy-Rogers contractions and then we present some theorems concerning the existence of a fixed point, data dependence, Ulam-Hyers stability. Also, we present a local version involving two metrics and a homotopy theorem.

The first main result of this section is a fixed point theorem for modified multivalued Hardy-Rogers contractions, regarding the existence of fixed points for these types of self-mappings.

Theorem 4.1. *Let (X,d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued modified Hardy Rogers operator, i.e.*

$$H(Tx,Ty) \leq \alpha d(x,y) + \beta D(x,Ty) + \gamma D(y,Tx) + \delta D(y,Ty) +$$
$$\eta \frac{D(y,Ty)[1+D(x,Tx)]}{1+d(x,y)} + \lambda \frac{D(y,Ty)+D(y,Tx)}{1+D(y,Ty)D(y,Tx)} + \mu \frac{D(x,Tx)[1+D(y,Tx)]}{1+d(x,y)+D(y,Ty)},$$

with all the above coefficients positive.

If $\alpha + 2\beta + \eta + \mu + \lambda + \delta < 1$, then there exists $p \in X$, such that $p \in F_T$.

Proof. Let's consider $x_0 \in X$ an arbitrary point and $q > 1$.

Let $x_1 \in Tx_0$.

If $H(Tx_0,Tx_1) = 0$, then $Tx_0 = Tx_1$, that means $x_1 \in Tx_1$, i.e. $x_1 \in F_T$.

Let's suppose that $H(Tx_0,Tx_1) \neq 0$.

For $x_1 \in Tx_0$, we can choose $x_2 \in Tx_1$, such that $d(x_1,x_2) \leq q \cdot H(Tx_0,Tx_1)$.

This means that

$$d(x_1,x_2) \leq q \Bigg[\alpha d(x_0,x_1) + \beta D(x_0,Tx_1) + \gamma D(x_1,Tx_0) + \delta D(x_1,Tx_1) +$$
$$\eta \frac{D(x_1,Tx_1)[1+D(x_0,Tx_0)]}{1+d(x_0,x_1)} + \lambda \frac{D(x_1,Tx_1)+D(x_1,Tx_0)}{1+D(x_1,Tx_1)\cdot D(x_1,Tx_0)} + \mu \frac{D(x_0,Tx_0)[1+D(x_1,Tx_0)]}{1+d(x_0,x_1)+D(x_1,Tx_1)} \Bigg]$$

So, we have that

$$d(x_1,x_2) \leq q \Bigg[\alpha d(x_0,x_1) + \beta d(x_0,x_2) + \gamma d(x_1,x_1) + \delta d(x_1,x_2) + \eta \frac{d(x_1,x_2)[1+d(x_0,x_1)]}{1+d(x_0,x_1)} +$$
$$\lambda \frac{d(x_1,x_2)+d(x_1,x_1)}{1+D(x_1,Tx_1)\cdot D(x_1,Tx_0)} + \mu \frac{d(x_0,x_1)[1+d(x_1,x_1)]}{1+d(x_0,x_1)+D(x_1,Tx_1)} \Bigg]$$

Since

$d(x_1,x_1)=0,\ \dfrac{d(x_1,x_2)}{1+D(x_1,Tx_1)\cdot D(x_1,Tx_0)}\leq d(x_1,x_2),$

that is $1+D(x_1,Tx_1)\cdot D(x_1,Tx_0)\geq 1$ and $\dfrac{d(x_0,x_1)}{1+d(x_0,x_1)+D(x_1,Tx_1)}\leq\dfrac{d(x_0,x_1)}{1+d(x_0,x_1)}\leq d(x_0,x_1),$

because $1+d(x_0,x_1)\geq 1$, we get that

$d(x_1,x_2)\leq q\,[\alpha d(x_0,x_1)+\beta d(x_0,x_1)+\beta d(x_1,x_2)+\delta d(x_1,x_2)+\eta d(x_1,x_2)+\lambda d(x_1,x_2)+\mu d(x_0,x_1)].$

This means that

$d(x_1,x_2)\leq q\cdot\dfrac{\alpha+\beta+\mu}{1-q(\beta+\delta+\eta+\lambda)}\cdot d(x_0,x_1).$

In a similar manner, for $x_2\in Tx_1$, there exists $x_3\in Tx_2$ such that

$d(x_2,x_3)\leq q\cdot H(Tx_1,Tx_2).$

This means that

$$d(x_2,x_3)\leq q\Bigg[\alpha d(x_1,x_2)+\beta D(x_1,Tx_2)+\gamma D(x_2,Tx_1)+\delta D(x_2,Tx_2)+\eta\frac{D(x_2,Tx_2)[1+D(x_1,Tx_1)]}{1+d(x_1,x_2)}+$$

$$\lambda\frac{D(x_2,Tx_2)+D(x_2,Tx_1)}{1+D(x_2,Tx_2)\cdot D(x_2,Tx_1)}+\mu\frac{D(x_1,Tx_1)[1+D(x_2,Tx_1)]}{1+d(x_1,x_2)+D(x_2,Tx_2)}\Bigg]$$

So, we have that

$$d(x_2,x_3)\leq q\Bigg[\alpha d(x_1,x_2)+\beta d(x_1,x_3)+\gamma d(x_2,x_2)+\delta d(x_2,x_3)+\eta\frac{d(x_2,x_3)[1+d(x_1,x_2)]}{1+d(x_1,x_2)}+$$

$$\lambda\frac{d(x_2,x_3)+d(x_2,x_2)}{1+D(x_2,Tx_2)\cdot D(x_2,Tx_1)}+\mu\frac{d(x_1,x_2)[1+d(x_2,x_2)]}{1+d(x_1,x_2)_D(x_2,Tx_2)}$$

Like before, since

$d(x_2,x_2)=0,\ \dfrac{d(x_2,x_3)}{1+D(x_2,Tx_2)\cdot D(x_2,Tx_1)}\leq d(x_2,x_3),$

that is $1+D(x_2,Tx_2)\cdot D(x_2,Tx_1)\geq 1$ and $\dfrac{d(x_1,x_2)}{1+d(x_1,x_2)+D(x_2,Tx_2)}\leq\dfrac{d(x_1,x_2)}{1+d(x_1,x_2)}\leq d(x_1,x_2),$

because $1+d(x_1,x_2)\geq 1$, we get that

$d(x_2,x_3)\leq q\,[\alpha d(x_1,x_2)+\beta d(x_1,x_2)+\beta d(x_2,x_3)+\delta d(x_2,x_3)+\eta d(x_2,x_3)+\lambda d(x_2,x_3)+\mu d(x_1,x_2)].$

This means that

$d(x_2,x_3)\leq q\dfrac{\alpha+\beta+\mu}{1-q(\beta+\delta+\eta+\lambda)}\cdot d(x_1,x_2)\leq\left(q\dfrac{\alpha+\beta+\mu}{1-q(\beta+\delta+\eta+\lambda)}\right)^2\cdot d(x_0,x_1)$

By induction, we infer that $d(x_n,x_{n+1})\leq r^n d(x_0,x_1)$, where $r:=\dfrac{q(\alpha+\beta+\mu)}{1-q(\beta+\eta+\lambda+\delta)}$.

Since $q>1$ is arbitrary taken, we impose the following condition, namely

$r<1$, which means that $q(\alpha+\beta+\mu)<1-q(\beta+\eta+\lambda+\delta)$.

Equivalently, we can take $q<\dfrac{1}{\alpha+2\beta+\mu+\eta+\lambda+\delta}$.

In this way, we can take $q \in \left(1, \dfrac{1}{\alpha+2\beta+\mu+\eta+\lambda+\delta}\right)$.
From the hypotheses, we have that $\alpha+2\beta+\mu+\eta+\lambda+\delta < 1$, which implies that $1 < \dfrac{1}{\alpha+2\beta+\mu+\eta+\lambda+\delta}$, so the definition for q is correct.
For $r \geq 0$ to take place, we need the relation $q < \dfrac{1}{\beta+\eta+\lambda+\mu}$.
Now, since $q < \dfrac{1}{\alpha+2\beta+\eta+\lambda+\mu+\delta} \leq \dfrac{1}{\beta+\eta+\lambda+\mu}$, it follows that $\alpha+\beta+\delta \geq 0$, which is obviously true.

Now, we show that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e.

$$d(x_n,x_{n+p}) \leq d(x_n,x_{n+1}) + \ldots + d(x_{n+p-1},x_{n+p}) \leq$$
$$\left(r^n+\ldots+r^{n+p-1}\right)\cdot d(x_0,x_1) =$$
$$r^n\cdot\left(1+\ldots+r^{p-1}\right)\cdot d(x_0,x_1) =$$
$$r^n\cdot\frac{1-r^p}{1-r}\cdot d(x_0,x_1) \leq r^n\cdot\frac{1}{1-r}d(x_0,x_1).$$

For each $p \geq 1$, letting $n \to \infty$, if follows that (x_n) is a Cauchy sequence.

Because the metric d is a complete, the sequence (x_n) is convergent.

Then exists $p \in X$, such that $x_n \to p$.

We now show that p is a fixed point for the operator T. We estimate

$$D(p,Tp) = \inf_{y\in Tp} d(p,y) \leq d(p,x_{n+1}) + \inf_{y\in Tp} d(x_{n+1},y) \leq$$

$$d(p,x_{n+1}) + D(x_{n+1},Tp) \leq d(p,x_{n+1}) + H(Tx_n,Tp) \leq$$

$$d(p,x_{n+1}) + \alpha d(x_n,p) + \beta D(x_n,Tp) + \gamma D(p,Tx_n) + \delta D(p,Tp) + \eta\frac{D(p,Tp)\left[1+D(x_n,Tx_n)\right]}{1+d(x_n,p)} +$$

$$\lambda\frac{D(p,Tp)+D(p,Tx_n)}{1+D(p,Tp)D(p,Tx_n)} + \mu\frac{D(x_n,Tx_n)\left[1+D(p,Tx_n)\right]}{1+d(p,x_n)+D(p,Tp)}$$

$$\Longrightarrow D(p,Tp) \leq d(p,x_{n+1}) + \alpha d(x_n,p) + \beta d(x_n,p) + \beta D(p,Tp) + \gamma d(p,x_{n+1}) + \delta D(p,Tp) +$$

$$\eta\frac{D(p,Tp)\left[1+d(x_n,x_{n+1})\right]}{1+d(x_n,p)} + \lambda\frac{D(p,Tp)+d(p,x_{n+1})}{1+D(p,Tp)D(p,Tx_n)} + \mu\frac{d(x_n,x_{n+1})\left(1+d(p,x_{n+1})\right)}{1+d(x_n,p)+D(p,Tp)}$$

Now, we have used the following relations

$D(p,Tx_n) \leq d(p,x_{n+1})$, for each $n \in \mathbb{N}$,

$$D(x_n,Tp) = \inf_{y\in Tp} d(x_n,y) \leq d(x_n,p) + \inf_{y\in Tp} d(p,y) = d(x_n,p) + D(p,Tp),$$

$$\frac{D(p,Tp)+D(p,Tx_n)}{1+D(p,Tp)D(p,Tx_n)} \leq \frac{D(p,Tp)+d(p,x_{n+1})}{1+D(p,Tp)D(p,Tx_n)} \leq D(p,Tp) + d(x_{n+1},p),$$

because $D(p,Tp)D(p,Tx_n)+1 \geq 1$.

Letting $n \to \infty$, we have that

$D(p,Tp) \leq \beta D(p,Tp)+\eta D(p,Tp)+\lambda D(p,Tp)+\delta D(p,Tp)+\mu \cdot 0 = (\beta+\eta+\lambda+\delta)D(p,Tp)$.

It follows that $[1-(\beta+\eta+\lambda+\delta)]D(p,Tp) \leq 0$. The inequality $1-(\beta+\eta+\lambda+\delta) > 0$, is satisfied since $\beta+\eta+\lambda+\delta \leq \alpha+2\beta+\mu+\eta+\lambda+\delta < 1$.

We obtain that $D(p,Tp) = 0$. This means that $p \in Tp$, i.e. $p \in F_T$.

We have shown that $d(x_n,x_{n+p}) \leq r^n \frac{1-r^p}{1-r} d(x_0,x_1)$.

Letting $p \to \infty$, we have $d(x_n,p) \leq \frac{r^n}{1-r} d(x_0,x_1)$.

Letting $n = 0$ in the above inequality, we have $d(x_0,p) \leq \frac{1}{1-r} d(x_0,x_1)$, so T is a MWP operator.

Now we present a theorem concerning the fact that T is a MWP operator.

Theorem 4.2. *Let (X,d) be a complete metric space and $T : X \to P_{cl}(X)$ be a multivalued modified Hardy Rogers operator, i.e.*

$$H(Tx,Ty) \leq \alpha d(x,y)+\beta D(x,Ty)+\gamma D(y,Tx)+\delta D(y,Ty)+$$
$$\eta \frac{D(y,Ty)[1+D(x,Tx)]}{1+d(x,y)} + \lambda \frac{D(y,Ty)+D(y,Tx)}{1+D(y,Ty)D(y,Tx)} + \mu \frac{D(x,Tx)[1+D(y,Tx)]}{1+d(x,y)+D(y,Ty)},$$

with all the above coefficients positive.

If $\alpha+2\beta+\eta+\mu+\lambda+\delta < 1$, then the operator T is $\frac{1-(\beta+\eta+\lambda+\delta)}{1-(2\beta+\eta+\lambda+\delta+\alpha+\mu)}$-MWP.

Proof. From the proof of the previous theorem, we have that $d(x_n,x_{n+p}) \leq r^n \frac{1-r^p}{1-r} d(x_0,x_1)$, with r defined as $r = \frac{q(\alpha+\beta+\mu)}{1-q(\beta+\eta+\lambda+\delta)}$.

Since the sequence (x_n) was convergent to a fixed point p of T, letting $p \to \infty$ and then making $n = 1$, we get that $d(x_1,p) \leq \frac{r}{1-r} d(x_0,x_1)$.

Using the triangular inequality, it follows that

$$d(x_0,p) \leq d(x_0,x_1)+d(x_1,p) \leq d(x_0,x_1)+\frac{r}{1-r}d(x_0,x_1) \leq$$
$$\left(1+\frac{r}{1-r}\right) d(x_0,x_1) = \frac{1}{1-r} d(x_0,x_1)$$

From the definition of r, letting $q \searrow 1$, we obtain

$$d(x_0,p) \leq \frac{1}{1-\dfrac{\alpha+\beta+\mu}{1-(\beta+\eta+\lambda+\delta)}} d(x_0,x_1).$$

So $d(x_0,p) \leq \dfrac{1-(\beta+\eta+\lambda+\delta)}{1-(2\beta+\eta+\lambda+\delta+\alpha+\mu)} d(x_0,x_1)$.

But $\dfrac{1-(\beta+\eta+\lambda+\delta)}{1-(2\beta+\eta+\lambda+\delta+\alpha+\mu)} \geq 0$, which is equivaluent to $\beta+\eta+\lambda+\delta \leq 1$ and $\alpha+2\beta+\eta+\mu+\lambda+\delta < 1$, which is valid because $\beta+\eta+\lambda+\delta \leq 2\beta+\eta+\lambda+\delta+\alpha+\mu$, so $\alpha+\mu \geq 0$.

Finally, the conclusion holds properly.

The next two theorems which are presented are related to data dependence and Ulam-Hyers stability. For more information about this notions we remind the articles [6], [10] and [11].

Theorem 4.3. *Let (X,d) be a complete metric space and $T,S : X \to P_{cl}(X)$ be two multivalued modified Hardy Rogers operators, i.e.*

$$H(Tx,Ty) \leq \alpha_T d(x,y) + \beta_T D(x,Ty) + \gamma_T D(y,Tx) + \delta_T D(y,Ty) + \eta_T \frac{D(y,Ty)[1+D(x,Tx)]}{1+d(x,y)} + \lambda_T \frac{D(y,Ty)+D(y,Tx)}{1+D(y,Ty)D(y,Tx)} + \mu_T \frac{D(x,Tx)[1+D(y,Tx)]}{1+d(x,y)+D(y,Ty)} \text{ and}$$

$$H(Sx,Sy) \leq \alpha_S d(x,y) + \beta_S D(x,Sy) + \gamma_S D(y,Sx) + \delta_S D(y,Sy) + \eta_S \frac{D(y,Sy)[1+D(x,Sx)]}{1+d(x,y)} + \lambda_S \frac{D(y,Sy)+D(y,Sx)}{1+D(y,Sy)D(y,Sx)} + \mu_S \frac{D(x,Sx)[1+D(y,Sx)]}{1+d(x,y)+D(y,Sy)},$$

with all the above coefficients positive.

Let's suppose that $\alpha_T+2\beta_T+\eta_T+\lambda_T+\mu_T+\delta_T < 1$ and $\alpha_S+2\beta_S+\eta_S+\lambda_S+\mu_S+\delta_S < 1$.

Also, suppose that there exists $\tau > 0$, such that $H(Sx,Tx) \leq \tau$, for each $x \in X$.

Then

$$H(F_S,F_T) \leq \tau \cdot max\{\frac{1-(\beta_T+\eta_T+\lambda_T+\delta_T)}{1-(2\beta_T+\eta_T+\lambda_T+\delta_T+\mu_T+\alpha_T)}, \frac{1-(\beta_S+\eta_S+\lambda_S+\delta_S)}{1-(2\beta_S+\eta_S+\lambda_S+\delta_S+\mu_S+\alpha_S)}\}.$$

Proof. Let's consider $x_0 \in F_S$. This means that $x_0 \in Sx_0$.

Let $x^* := t^\infty(x,y) \in F_T$, i.e. $x^* \in Tx^*$. We denote by $x := x_0$.

From the proofs of the previous theorems, we remind that we have shown $d(x,t^\infty(x,y)) = d(x,x^*) \leq \dfrac{1}{1-r_T} d(x,y)$,

where $r_T := \dfrac{q(\alpha_T+\beta_T+\mu_T)}{1-q(\beta_T+\eta_T+\lambda_T+\delta_T)}$, with q arbitrary taken as in the previous proofs.
So $d(x,t^\infty(x,y)) \leq \dfrac{1}{1-r_T}d(x,y) = \dfrac{1}{1-r_T}d(x,x_1) \leq \dfrac{q\tau}{1-r_T}$, with $x := x_0$ and $y := x_1 \in Tx_0$. This inequality chain is obtained because for $x = x_0$, there exists $y = x_1 \in Tx_0$, such that $d(x_0,x_1) \leq qH(Sx_0,Tx_0) \leq q\tau$.
Analogous, we have that for $y_0 \in F_T$, there exists $y_1 \in Sy_0$, such that
$d(y_0,t^\infty(y_0,y_1)) \leq \dfrac{q\tau}{1-r_S}$,
where $r_S := \dfrac{\alpha_S+\beta_S+\mu_S}{1-(\beta_S+\eta_S+\lambda_S+\delta_S)}$.
All the above inequalities implies that $H(F_S,F_T) \leq q\tau\cdot\max\{\dfrac{1}{1-r_T},\dfrac{1}{1-r_S}\}$.
Letting $q \searrow 1$, we get that $H(F_S,F_T) \leq \tau\cdot\max\{\dfrac{1}{1-r_T},\dfrac{1}{1-r_S}\}$.
So, the conclusion holds.

Now, the next fixed point theorem involves Ulam-Hyers stability of the fixed point inclusion.

Theorem 4.4. *Let $T : X \to P_{cp}(X)$ be a multivalued modified Hardy Rogers contraction with positive coefficients $(\alpha,\beta,\gamma,\delta,\eta,\lambda,\mu)$, with $\alpha+2\beta+\eta+\lambda+\mu+\delta < 1$.*
Let $\varepsilon > 0$ and $x \in X$, such that $D_d(x,Tx) \leq \varepsilon$.
Then, there exists $x^* \in F_T$ such that $d(x,x^*) \leq \varepsilon\cdot\dfrac{1-(\beta+\eta+\lambda+\delta)}{1-(2\beta+\eta+\alpha+\lambda+\delta+\mu)}$.

Proof. Let $\varepsilon > 0$ and $x \in X$, such that $D_d(x,Tx) \leq \varepsilon$.
Since Tx is compact for the above x, it implies that there exists $y \in Tx$, such that $D(x,Tx) = d(x,y) \leq \varepsilon$.
From the previous proofs, we have that $d(x,t^\infty(x,y)) \leq \dfrac{1}{1-r}d(x,y)$, with $y \in Tx$ considered above.
Then $d(x,x^*) \leq \dfrac{\varepsilon}{1-r}$, that is $d(x,x^*) \leq \varepsilon\cdot\dfrac{1-(\beta+\eta+\lambda+\delta)}{1-(2\beta+\eta+\alpha+\lambda+\delta+\mu)}$.
This means that the conclusion is valid under the theorem's hypotheses.

In the next two theorems and in the last corollary we present local versions involving two metrics and homotopy results with respect to modified multivalued Hardy-Rogers operators. For homotopy-type results we let the reader follow [1] and [2] and [5].

Theorem 4.5. *Let (X,d) be a complete metric space. Let $x_0 \in X$ and $r > 0$.*
Let ρ be another metric on X and let $T : \overline{B}^d_\rho(x_0,r) \to P(X)$ be a multivalued operator.

Let's suppose the following assumptions are satisfied

(1) there exists $c > 0$ *such that* $d(x,y) \leq c\rho(x,y)$*, for each* $x,y \in X$

(2) If $d \neq \rho$*, then* $T : \overline{B}^d_\rho(x_0,r) \to P(X^d)$ *is a closed operator,*

If $d = \rho$*, then* $T : \overline{B}^d_d(x_0,r) \to P_{cl}(X^d)$*,*

(3) for each $x,y \in \overline{B}^d_\rho(x_0,r)$*, we have that* T *is a multivalued modified Hardy Rogers contraction with respect to the metric* ρ*, i.e.*

$$H_\rho(Tx,Ty) \leq \alpha\rho(x,y) + \beta D_\rho(x,Ty) + \gamma D_\rho(y,Tx) + \delta D_\rho(y,Ty) +$$
$$\eta \frac{D_\rho(y,Ty)\left[1 + D_\rho(x,Tx)\right]}{1+\rho(x,y)} + \lambda \frac{D_\rho(y,Ty) + D_\rho(y,Tx)}{1 + D_\rho(y,Ty)D_\rho(y,Tx)} + \mu \frac{D_\rho(x,Tx)\left[1 + D_\rho(y,Tx)\right]}{1+\rho(x,y) + D_\rho(y,Ty)}$$

(4) $D_\rho(x_0,Tx_0) < (1-\theta)r$*, with* $\theta := \dfrac{\alpha+\beta+\mu}{1-(\beta+\delta+\eta+\lambda)} \in [0,1)$*, where all the Hardy-Rogers type coefficients are positive.*

Then, we have that there exists $x^* \in \overline{B}^d_\rho(x_0,r)$*, such that* $x^* \in Tx^*$*.*

Proof. From the hypotheses we have that $D_\rho(x_0,Tx_0) < (1-\theta)r$.

Then, for x_0 there exists $x_1 \in Tx_0$ such that

$\rho(x_0,x_1) < (1-\theta)r$.

This means that $x_1 \in \overline{B}^d_\rho(x_0,r)$. We have that

$$H_\rho(Tx_0,Tx_1) \leq \alpha\rho(x_0,x_1) + \beta D_\rho(x_0,Tx_1) + \gamma D_\rho(x_1,Tx_0) + \delta D_\rho(x_1,Tx_1) +$$
$$\eta \frac{D_\rho(x_1,Tx_1)\left[1 + D_\rho(x_0,Tx_0)\right]}{1+\rho(x_0,x_1)} + \lambda \frac{D_\rho(x_1,Tx_1) + D_\rho(x_1,Tx_0)}{1 + D_\rho(x_1,Tx_1)D_\rho(x_1,Tx_0)} + \mu \frac{D_\rho(x_0,Tx_0)\left[1 + D_\rho(x_1,Tx_0)\right]}{1+\rho(x_0,x_1) + D_\rho(x_1,Tx_1)}$$
$$\leq \alpha\rho(x_0,x_1) + \beta\rho(x_0,x_1) + \beta D_\rho(x_1,Tx_1) + \delta D_\rho(x_1,Tx_1) + \eta D_\rho(x_1,Tx_1) \cdot \frac{1+\rho(x_0,x_1)}{1+\rho(x_0,x_1)} +$$
$$\lambda D_\rho(x_1,Tx_1) \cdot \frac{1}{1 + D_\rho(x_1,Tx_1)D_\rho(x_1,Tx_0)} + \mu\rho(x_0,x_1) \leq$$
$$(\alpha+\beta+\mu)\rho(x_0,x_1) + (\lambda+\delta+\beta+\eta)D_\rho(x_1,Tx_1).$$

Then $H_\rho(Tx_0,Tx_1) \leq (\alpha+\beta+\mu)\rho(x_0,x_1) + (\lambda+\delta+\beta+\eta)D_\rho(x_1,Tx_1)$.

Since $D_\rho(x_1,Tx_1) \leq H_\rho(Tx_0,Tx_1)$, then $D_\rho(x_1,Tx_1) \leq \dfrac{\alpha+\beta+\mu}{1-(\beta+\delta+\lambda+\eta)}\rho(x_0,x_1) = \theta\rho(x_0,x_1) < \theta(1-\theta)r$. So there exists $x_2 \in Tx_1$ such that $\rho(x_1,x_2) < \theta(1-\theta)r$.

So $\rho(x_0,x_2) \leq \rho(x_1,x_2) + \rho(x_0,x_1) < \theta(1-\theta)r + (1-\theta)r = (1-\theta)(1+\theta)r = (1-\theta^2)r \leq r$.

This means that $x_2 \in \overline{B}^d_\rho(x_0,r)$.

In a similar manner, for x_1 and x_2 in $\overline{B}^d_\rho(x_0,r)$ we have that

$$H_\rho(Tx_1,Tx_2) \le \alpha\rho(x_1,x_2)+\beta D_\rho(x_1,Tx_2)+\gamma D_\rho(x_2,Tx_1)+\delta D_\rho(x_2,Tx_2)+$$
$$\eta\frac{D_\rho(x_2,Tx_2)\left[1+D_\rho(x_1,Tx_1)\right]}{1+\rho(x_1,x_2)}+\lambda\frac{D_\rho(x_2,Tx_2)+D_\rho(x_2,Tx_1)}{1+D_\rho(x_2,Tx_2)D_\rho(x_2,Tx_1)}+\mu\frac{D_\rho(x_1,Tx_1)\left[1+D_\rho(x_2,Tx_1)\right]}{1+\rho(x_1,x_2)+D_\rho(x_2,Tx_2)}$$
$$\le \alpha\rho(x_1,x_2)+\beta\rho(x_1,x_2)+\beta D_\rho(x_2,Tx_2)+\delta D_\rho(x_2,Tx_2)+\eta D_\rho(x_2,Tx_2)\cdot\frac{1+\rho(x_1,x_2)}{1+\rho(x_1,x_2)}+$$
$$\lambda D_\rho(x_2,Tx_2)\cdot\frac{1}{1+D_\rho(x_2,Tx_2)D_\rho(x_2,Tx_1)}+\mu\rho(x_1,x_2)\le$$
$$(\alpha+\beta+\mu)\rho(x_1,x_2)+(\lambda+\delta+\beta+\eta)D_\rho(x_2,Tx_2).$$

Then $H_\rho(Tx_1,Tx_2)\le(\alpha+\beta+\mu)\rho(x_1,x_2)+(\lambda+\delta+\beta+\eta)D_\rho(x_2,Tx_2)$.
Since $D_\rho(x_2,Tx_2)\le H_\rho(Tx_1,Tx_2)$, then $D_\rho(x_2,Tx_2)\le\dfrac{\alpha+\beta+\mu}{1-(\beta+\delta+\lambda+\eta)}\rho(x_1,x_2)=\theta\rho(x_1,x_2)<$ $\theta^2(1-\theta)r$. So there exists $x_3\in Tx_2$ such that $\rho(x_2,x_3)<\theta^2(1-\theta)r$.
So, applying triangular inequality, we obtain
$\rho(x_0,x_3)\le\rho(x_0,x_2)+\rho(x_2,x_3)<(1-\theta^2)r+\theta^2(1-\theta)r=(1-\theta^3)r\le r$.
This means that $x_3\in\overline{B}^d_\rho(x_0,r)$.

So, we have created a sequence $(x_n)\subset\overline{B}^d_\rho(x_0,r)$, with the following properties :
(i) $x_{n+1}\in Tx_n$, for each $n\in\mathbb{N}$,
(ii) $\rho(x_{n-1},x_n)\le\theta^{n-1}(1-\theta)r$, for each $n\in\mathbb{N}$,
(iii) $\rho(x_0,x_n)\le(1-\theta^n)r$.
It is easy to see that (x_n) is a Cauchy sequence in (X,ρ).
Using the fact that $d(x,y)\le c\rho(x,y)$, for each $x,y\in X$, it implies that (x_n) is a Cauchy sequence in (X,d).
Because (X,d) is a complete metric space, there exists $x^*\in\overline{B}^d_\rho(x_0,r)$ such that $x_n\xrightarrow{d}x^*$.

Furthermore, we have two cases to analyze.
I If $d\ne\rho$, since $T:\overline{B}^d_\rho(x_0,r)\to P(X^d)$ is a closed operator, then $x^*\in Tx^*$.
II If $d=\rho$, we have that $D_d(x^*,Tx^*)\le d(x^*,x_{n+1})+D_d(x_{n+1},Tx^*)\le d(x^*,x_{n+1})+H_d(Tx_n,Tx^*)=$

$d(x^*,x_{n+1})+H_\rho(Tx_n,Tx^*)$. It follows that

$$D_d(x^*,Tx^*)\le d(x^*,x_{n+1})+\alpha d(x_n,x^*)+\beta D_d(x_n,Tx^*)+\gamma D_d(x^*,Tx_n)+\delta D_d(x^*,Tx^*)+$$
$$\eta\frac{D_d(x^*,Tx^*)\left[1+D_d(x_n,Tx_n)\right]}{1+d(x_n,x^*)}+\lambda\frac{D_d(x^*,Tx^*)+D_d(x^*,Tx_n)}{1+D_d(x^*,Tx^*)\cdot D_d(x^*,Tx_n)}+\mu\frac{D_d(x_n,Tx_n)\left[1+D_d(x^*,Tx_n)\right]}{1+d(x_n,x^*)+D_d(x^*,Tx^*)}\le$$
$$d(x^*,x_{n+1})+\alpha d(x_n,x^*)+\beta D_d(x_n,Tx^*)+\gamma D_d(x^*,Tx_n)+\delta D_d(x^*,Tx^*)+$$
$$\eta D_d(x^*,Tx^*)\left[1+d(x_n,x_{n+1})\right]+\lambda D_d(x^*,Tx^*)+\lambda d(x^*,x_{n+1})+\mu d(x_n,x_{n+1})\left[1+d(x^*,x_n)\right].$$

Letting $n\to\infty$, we get the following inequality

$$D_d(x^*,Tx^*)\le[\beta+\delta+\lambda+\eta]\,D_d(x^*,Tx^*),\ \text{i.e. } D_d(x^*,Tx^*)\,(\beta+\delta+\lambda+\eta)\le 0.$$

Now, since $\theta\in[0,1)$, as in the proofs of the previous theorems, it follows that $\beta+\delta+\eta+\lambda\le\alpha+2\beta+\mu+\eta+\lambda+\delta<1$. Moreover, because $d=\rho$ and T has closed values, we have that $x^*\in Tx^*$.

Now, the last main result of this section involves a theorem regarding the homotopy of a modified multivalued Hardy-Rogers operator.

Theorem 4.6. *Let (X,d) be a complete metric space and d,ρ two metrics on X such that there exists $c>0$, with $d(x,y)\le c\rho(x,y)$, for each $x,y\in X$.*
Let $U\subset(X,\rho)$ an open subset and $V\subset(X,d)$ a closed subset of X, such that $U\subset V$.
Let's consider the multivalued operator $G:V\times[0,1]\to P(X)$, which satisfies the following conditions :
(a) $x\notin G(x,t)$, for each $x\in V\setminus U$ and $t\in[0,1]$
(b) there exists $\alpha,\beta,\gamma,\delta,\lambda,\eta,\mu$ positive coefficients with $\theta\in[0,1)$ as in the previous theorem, such that for each $t\in[0,1]$ and $x,y\in V$, we have that
$H_\rho\left(G(x,t),G(y,t)\right)\le M_{\rho,G(\cdot,t)}(x,y)$, where

$$M_{\rho,G(\cdot,t)}(x,y):=\alpha\rho(x,y)+\beta D_\rho(x,G(y,t))+\gamma D_\rho(y,G(x,t))+\delta D_\rho(y,G(y,t))+$$
$$\eta\frac{D_\rho(y,G(y,t))\left[1+D_\rho(x,G(x,t))\right]}{1+\rho(x,y)}+\lambda\frac{D_\rho(y,G(y,t))+D_\rho(y,G(x,t))}{1+D_\rho(y,G(y,t))D_\rho(y,G(x,t))}+$$
$$\mu\frac{D_\rho(x,G(x,t))\left[1+D_\rho(y,G(x,t))\right]}{1+\rho(x,y)+D_\rho(y,G(y,t))}$$

(c) there exists an increasing, continuous function $\phi : [0,1] \to \mathbb{R}$, such that

$H_\rho(G(x,t), G(x,s)) \leq |\phi(t) - \phi(s)|$, *for each* $s,t \in [0,1]$ *and* $x \in V$

(d) $G : V \times [0,1] \to P(X,d)$ *is a closed operator.*

Then, we have the following equivalence relation

$G(\cdot,0)$ *has a fixed point if and only if* $G(\cdot,1)$ *has a fixed point.*

Proof. Let's suppose that $G(\cdot,0)$ has a fixed point z .

From the assumption (a), we get that $z \in U$.

Let's denote $Q := \{(t,x) \in [0,1] \times U / x \in G(x,t)\}$. Then Q is nonempty, because $(0,z) \in Q$.

On the set Q, we define a partial order relation, i.e.

$(t,x) \leq (s,y)$ if and only if $t \leq s$ and $\rho(x,y) \leq \dfrac{2}{1-\theta}|\phi(t) - \phi(s)|$, for each $t,s \in [0,1]$ and $x,y \in U$.

Let $M \subset Q$, with M being a totally ordered subset of Q.

Moreover, denote by $t^* := \sup\limits_{(t,x) \in M} t$.

Now we define the sequence $(t_n,x_n) \in M$, such that $(t_n,x_n) \leq (t_{n+1},x_{n+1})$, with $t_n \to t^*$.

As in [5], we have that

$\rho(x_n,x_m) \leq \dfrac{2}{1-\theta}|\phi(t_m) - \phi(t_n)|$. This implies that $\rho(x_n,x_m) \to 0$ and therefore (x_n) is a Cauchy sequence with respect to the metric ρ.

Using the fact that (X,d) is a complete metric space and that there exists $c > 0$, such that for each $x,y \in X$, $d(x,y) \leq c\rho(x,y)$, we obtain that $x_n \to x^*$, with $x^* \in (X,d)$.

Since $x_n \in G(x_n,t_n)$ and G is a $d-$closed operator, we have that $x^* \in G(x^*,t^*)$.

From assumption (a), $x^* \in U$ and therefore $(t^*,x^*) \in Q$.

Since M is a totally ordered subset of Q, it follows that $(t,x) \leq (t^*,x^*)$, for each $(t,x) \in M$, so (t^*,x^*) is an upper bound for M.

Using the well known *Zorn's Lemma*, Q admits a maximal element, i.e. $(t_0,x_0) \in Q$.

Now we show that $t_0 = 1$.

Let's suppose the contrary, i.e. that $t_0 < 1$. We choose $r > 0$ and $t \in (t_0,1)$ such that $B_\rho(x_0,r) \subset U$, with $r := \dfrac{2}{1-\theta}|\phi(t) - \phi(t_0)|$.

Then $D_\rho(x_0, G(x_0,t)) \leq D_\rho(x_0, G(x_0,t_0)) + H_\rho(G(x_0,t_0), G(x_0,t)) \leq$

$|\phi(t) - \phi(t_0)| + D_\rho(x_0, G(x_0,t_0))$.

Since $(t_0, x_0) \in Q$, it implies that $x_0 \in G(x_0, t_0)$, therefore $D_\rho(x_0, G(x_0, t_0)) = 0$.
So $D_\rho(x_0, G(x_0, t)) \leq |\phi(t) - \phi(t_0)| \leq \frac{(1-\theta)r}{2} < (1-\theta)r$.
But we know that $\overline{B}^d_\rho(x_0, r) \subset V$, so $G : \overline{B}^d_\rho(x_0, r) \to P_{cl}(X)$ satisfies the assumptions of the previous theorem, therefore for each $t \in [0,1]$, there exists $x \in \overline{B}^d_\rho(x_0, r)$ satisfying the property that G has a fixed point, that is $x \in G(x,t)$, which implies that $(t,x) \in Q$.
But $\rho(x_0, x) \leq \frac{2}{1-\theta}|\phi(t) - \phi(t_0)| = r$. This means that $(t_0, x_0) < (t, x)$, which is a contraction. So $t_0 = 1$.
For the other implication, we show that $G(\cdot, 1)$ has a fixed point by swapping t with $1-t$ in the first part of the proof. So, we get the desired result.

When the metric functionals d and ρ are identical, we have the following corollary.

Corollary 4.7. Let $U \subset V \subset X$, with (X,d) a complete metric space, U open and V closed.
Let $G : V \times [0,1] \to P(X)$ a closed operator , satisfying the following assumptions
(a) $x \notin G(x,t)$, for each $x \in V \setminus U$ and $t \in [0,1]$
(b) $G(\cdot, t)$ be a Hardy Rogers modified multivalued contraction with respect to d, for each $t \in [0,1]$
(c) $H_d(G(x,t), G(x,s)) \leq |\phi(t) - \phi(s)|$, for each $t, s \in [0,1]$ and for each $x \in V$, with $\phi : [0,1] \to \mathbb{R}$ increasing and continuous.
Then
$G(\cdot, 0)$ has a fixed point if and only if $G(\cdot, 1)$ has a fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests.

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