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A SOLUTION OF RANDOM NONLINEAR INTEGRAL EQUATION VIA RANDOM HYBRID ITERATIVE SCHEME

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Abstract. In this paper, we introduce random Jungck-Picard-Krasnoselskii hybrid iterative process which is a hybrid of random Jungck, Picard and krasnoselskii iterative processes. We prove under $\varphi-$contractive condition, our random hybrid iterative scheme converges faster than all of random Jungck-Picard, Jungck-Mann, Jungck-Krasnoselskii and Jungck-Ishikawa iterative processes. Finally, we use it to find a solution of random nonlinear integral equation. Our results generalize and improve several known results in stochastic and deterministic cases.

Keywords: random fixed point; random hybrid iterative processes; separable Banach spaces; random nonlinear integral equation.

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1. Introduction

Let X be a Banach space, $T : X \to X$ be a self mapping of X. Suppose that $F_T = \{x \in X : T(x) = x\}$ is the set of all fixed points of T, E be a nonempty subset of X and $\mathbb{N}$ be the set of all positive integers.

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In 1976, Jungck [1] introduced the following iterative scheme: Let Y be a subspace of X and $S,T : Y \to X$ such that $T(Y) \subseteq S(Y)$, for every $x_\circ \in Y$, the sequence $\{Sx_n\}_{n=0}^{\infty}$ defined by

$$Sx_{n+1} = Tx_n,\ n = 0,1,... \tag{1.1}$$

This scheme is called Jungck iterative scheme.

A lot of author generalized this scheme to Jungck-Mann, Jungck-Ishikawa, Jungck-Noor and others, these iterations are used to approximate the common fixed point for mappings under suitable contractive conditions see [2, 3, 4, 5, 6].

The Picard iterative process [7] is defined by the sequence $\{u_n\}$ as follows:

$$\begin{cases} u_1 = u \in E, \\ u_{n+1} = Tu_n,\ n \in \mathbb{N}. \end{cases} \tag{1.2}$$

The Mann iterative process [8] is defined by the sequence $\{v_n\}$ as the following:

$$\begin{cases} v_1 = v \in E, \\ v_{n+1} = (1-\alpha_n)v_n + \alpha_n Tv_n,\ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\alpha_n \in (0,1)$.

The sequence $\{w_n\}$ defined by

$$\begin{cases} w_1 = w \in E, \\ w_{n+1} = (1-\lambda)w_n + \lambda Tw_n,\ n \in \mathbb{N}, \end{cases} \tag{1.4}$$

is known as Krasnoselskii iterative process [9], where $\lambda \in (0,1)$.

The sequence $\{z_n\}$ defined by

$$\begin{cases} z_1 = z \in E, \\ z_{n+1} = (1-\alpha_n)z_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)z_n + \beta_n Tz_n,\ n \in \mathbb{N}, \end{cases} \tag{1.5}$$

is known as Ishikawa iterative process [10], where $\alpha_n, \beta_n \in (0,1)$.

Recently, Khan [11] introduced the Picard-Mann hybrid iterative process. This iterative process for one mapping case is given by the sequence $\{m_n\}$ as follows:

$$\begin{cases} m_1 = m \in E, \\ m_{n+1} = Tz_n, \\ z_n = (1-\alpha_n)m_n + \alpha_n T m_n, \; n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\alpha_n \in (0,1)$. This new iterative process can be seen as a hybrid of Picard and Mann iterative processes. He proved that the Picard-Mann hybrid iterative process converges faster than all of Picard, Mann and Ishikawa iterative processes in the sense of Berinde [13].

Motivated by this facts, Okeke and Abbas [12] introduced Picard-krasnoselskii hybrid iterative process for the sequence $\{x_n\}$ as follows:

$$\begin{cases} x_1 = x \in E, \\ x_{n+1} = Ty_n, \\ y_n = (1-\lambda)x_n + \lambda T x_n, \; n \in \mathbb{N}, \end{cases} \tag{1.7}$$

where $\lambda \in (0,1)$. They proved that this iterative process converges faster than all of Picard, Mann, Krasnoselskii and Ishikawa iterative schemes in the sense of Berinde [13].

The following definitions are due to Berinde [13].

Definition 1.1. [13] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b, respectively. The sequence $\{a_n\}$ is said to converge faster than $\{b_n\}$ if

$$\lim_{n\to\infty} \frac{|a_n - a|}{|b_n - b|} = 0. \tag{1.8}$$

Definition 1.2. [13] Let $\{u_n\}$ and $\{v_n\}$ be two fixed point iteration processes that converge to a certain fixed point p of a given operator T. Suppose that the following error estimates are available:

$$\begin{aligned} \|u_n - p\| &\leq a_n, \\ \|v_n - p\| &\leq b_n, \; \forall n \in \mathbb{N}, \end{aligned}$$

where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{u_n\}$ converges faster than $\{v_n\}$ to p.

Lemma 1.1. [13] *Let* $\{k_n\}$ *be a sequence of positive real numbers which satisfies:*

$$k_{n+1} \leq (1-\rho_n)k_n.$$

If $\{\rho_n\} \subset (0,1)$ *and* $\sum_{n=1}^{\infty} \rho_n = \infty$, *then* $\lim_{n\to\infty} k_n = 0$.

The study of random fixed point problems was initiated by the Prague school of probability research. The first results were studied in 1955-1956 by Špaček and Hanš in the context of Fredholm integral equations with random kernel. In a separable metric space, random fixed point theorems for random contraction mappings were proved by Hanš [15] and Špaček [16]. Bharucha-Reid [17] attracted the attention of several mathematicians and gave wings to this theory. Itoh [14] extended the results of Špaček and Hanš to multi-valued contractive mappings and obtained random fixed point theorems with an application to random differential equations in Banach spaces. Now, it has became a full fledged research area and a vast amount of mathematical activities have been carried out in this direction (see [18, 19]).

Recently, some authors [20, 21] applied a random fixed point theorem to prove the existence of a solution in a separable Banach space of a random nonlinear integral equations. Chang et al. [24], Beg and Abbas [25] proved some convergence theorems of random Ishikawa and Mann iterative processes for strongly pseudo-contractive operators and contraction operators, respectively, in separable reflexive Banach spaces. Rashwan et al. [26] studied the convergence and almost sure (S,T)-stability of Jungck-Noor, Jungck-SP, Jungck-Ishikawa and Jungck-Mann type random iterative processes under suitable contractive condition for random operators in a separable Banach spaces.

2. Some basic concepts

Let (X,β_X) be a separable Banach space, where β_X is a $\sigma-$algebra of Borel subsets of X and let (Ω,β,μ) denote a complete probability measure space with measure μ and β be a $\sigma-$algebra of subsets of Ω.

Definition 2.1. [22] A measurable mapping $x:\Omega\to X$ is called:

(i) $X-$valued random variable, if the inverse image under the mapping x of every Borel set B of X belongs to β, that is, $x^{-1}(B)\in\beta$ for all $B\in\beta$.

(ii) finitely valued random variable, if it is constant on each of a finite number of disjoint sets $A_i \in \beta$ and is equal to 0 on $\Omega - \left(\bigcup_{i=1}^{n} A_i \right)$.

(iii) simple random variable, if it is finitely valued and $\mu\{\omega : \|x(\omega)\| > 0\} < \infty$.

(iv) strong random variable, if there exists a sequence $\{x_n(\omega)\}$ of simple random variables which converges to $x(\omega)$ almost surely, i.e., there exists a set $\mu(A_0) \in \beta$ with $\mu(A_0) = 0$ such that $\lim_{n\to\infty} x_n(\omega) = x(\omega)$, $\omega \in \Omega - A_0$.

(v) weak random variable, if the function $x^*(x(\omega))$ is a real valued random variable, for each $x^* \in X^*$, the space X^* denoting the dual space of X.

Definition 2.2. [22] Let Y be a Banach space.

(i) a measurable mapping $f : \Omega \times X \to Y$ is said to be a random mapping if $f(\omega,x) = Y(\omega)$ is a $Y-$valued random variable for every $x \in X$.

(ii) a measurable mapping $f : \Omega \times X \to Y$ is said to be a continuous random mapping if the set of all $\omega \in \Omega$ for which $f(\omega,x)$ is a continuous function of x has measure one.

(iii) an equation of the type $f(\omega,x(\omega)) = x(\omega)$ is called a random fixed point equation, where $f : \Omega \times X \to X$ is a random mapping.

(iv) any measurable mapping $x : \Omega \to X$ which satisfies the random fixed point equation $f(\omega,x(\omega)) = x(\omega)$ almost surely is said to be a wide sense solution of the fixed point equation.

(v) any $X-$valued random variable $x(\omega)$ which satisfies $\mu\{\omega : f(\omega,x(\omega)) = x(\omega)\} = 1$ is said to be a random solution of the fixed point equation or a random fixed point of f.

(vi) a measurable mapping $x : \Omega \to X$ is called:

(a) a random fixed point of a random operator $f : \Omega \times X \to X$ if $f(\omega,x(\omega)) = x(\omega)$ for every $\omega \in \Omega$.

(b) a random coincidence of random operators $T, f : \Omega \times X \to X$ if $T(\omega,x(\omega)) = f(\omega,x(\omega))$ for every $\omega \in \Omega$.

(c) a common random fixed point of random mappings $T, f : \Omega \times X \to X$ if $T(\omega,x(\omega)) = f(\omega,x(\omega)) = x(\omega)$ for every $\omega \in \Omega$.

Example 2.1. [22] *Let $X = \mathbb{R}$ and C be a non-measurable subset of X. Consider $f : \Omega \times X \to Y$ is a random mapping defined as $f(\omega,x(\omega)) = x^2(\omega) + x(\omega) - 1$ for all $\omega \in \Omega$. It's* *clearly that, the real-valued function $x(\omega) = 1$ is a random fixed point of f. However, the real-valued function $y(\omega) = \begin{cases} -1, & \omega \notin C \\ 1, & \omega \in C \end{cases}$ is a wide sense solution of the fixed point equation $f(\omega, x(\omega)) = x(\omega)$ without being a random fixed point of f. Therefore, a random solution is a wide sense solution of the fixed point equation but the converse is not necessarily true.*

The following definition is a stochastic form of (Definition 2.1, [23]).

Definition 2.3. Let (Ω, β, μ) is a complete probability measure space, E and Y be nonempty subsets of a separable Banach space X and $S, T : \Omega \times E \to Y$ are random operators such that $T(E) \subseteq S(E)$. Then, the random operators S and T are said to be generalized φ-contractive type if there exists a monotone increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ for all $x, y \in E$, $\delta(\omega) \in (0,1)$ and $\omega \in \Omega$, we get

$$\|T(\omega,x) - T(\omega,y)\| \leq \varphi(\|S(\omega,x) - T(\omega,x)\|) + \delta(\omega) \|S(\omega,x) - S(\omega,y)\|. \tag{2.1}$$

Now, we present the stochastic verse of the above iterative processes as the following: Let $u_1, v_1, m_1, z_1, x_1 : \Omega \to E$ be an arbitrary measurable mappings and $\alpha_n, \beta_n, \lambda \in (0,1)$.

The random Jungck-Picard iterative process is defined by the sequence $\{S(\omega, u_n(\omega))\}$ as follows:

$$\begin{cases} u_1(\omega) = u(\omega) \in E, \\ S(\omega, u_{n+1}(\omega)) = T(\omega, u_n(\omega)), \ \ n \in N. \end{cases} \tag{2.2}$$

The random Jungck-Mann iterative process is defined by the sequence $\{S(\omega, v_n(\omega))\}$ as the following:

$$\begin{cases} v_1(\omega) = v(\omega) \in E, \\ S(\omega, v_{n+1}(\omega)) = (1-\alpha_n) S(\omega, v_n(\omega)) + \alpha_n T(\omega, v_n(\omega)), \ \ n \in N. \end{cases} \tag{2.3}$$

The random Jungck-Krasnoselskii iterative process is given by the sequence $\{S(\omega, m_n(\omega))\}$ as follows:

$$\begin{cases} m_1(\omega) = m(\omega) \in E, \\ S(\omega, m_{n+1}(\omega)) = (1-\lambda) S(\omega, m_n(\omega)) + \lambda T(\omega, m_n(\omega)), \ \ n \in N. \end{cases} \tag{2.4}$$

The sequence $\{S(\omega, z_n(\omega)\}$ defined by

$$
(2.5) \qquad \begin{cases} z_1(\omega) = z(\omega) \in E, \\ S(\omega, z_{n+1}(\omega)) = (1-\alpha_n)S(\omega, z_n(\omega)) + \alpha_n T(\omega, r_n(\omega)), \\ S(\omega, r_n(\omega)) = (1-\beta_n)S(\omega, z_n(\omega)) + \beta_n T(\omega, z_n(\omega)), \text{ for all } n \in \mathbb{N}, \end{cases}
$$

is known as random Jungck-Ishikawa iterative process.

The random Jungck-Picard-Krasnoselskii hybrid iterative process is defined by

$$
(2.6) \qquad \begin{cases} x_1(\omega) = x(\omega) \in E, \\ S(\omega, x_{n+1}(\omega)) = T(\omega, y_n(\omega)), \\ S(\omega, y_n(\omega)) = (1-\lambda)S(\omega, x_n(\omega)) + \lambda T(\omega, x_n(\omega)), \ n \in \mathbb{N}. \end{cases}
$$

In this paper, we study the faster convergence of random Jungck-Picard-Krasnoselskii hybrid iterative process and compare it with the above random iterative processes and we prove that, if it converges, then it converges to a solution of random nonlinear integral equation.

3. Rate of convergence

In this section, we prove that the random Jungck-Picard-Krasnoselskii hybrid iterative process (2.6) converges at a rate faster than all of random Jungck-Picard (2.2), Jungck-Mann (2.3), Jungck-Krasnoselskii (2.4) and Jungck-Ishikawa (2.5) iterative processes.

Theorem 3.1. *Let (Ω, β, μ) is a complete probability measure space, let E be a nonempty subset of a separable Banach space X and let $S, T : \Omega \times E \to X$ be random nonself operators satisfying (2.1). Assume that $T(E) \subseteq S(E)$, $S(E)$ is a subset of X and $p(\omega) = T(\omega, x(\omega)) = S(\omega, x(\omega))$. Suppose that each of the random iterative processes (2.2), (2.3), (2.4), (2.5) and (2.6) converges to the same random coincidence point $p(\omega) : \Omega \to X$ of T and S, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$ such that $0 < \alpha \leq \lambda, \alpha_n, \beta_n < 1$, for all $n \in \mathbb{N}$ and for some α. Then the random Jungck-Picard-Krasnoselskii hybrid iterative process (2.6) converges faster than all the other four processes.*

Proof. Suppose that $p(\omega):\Omega\to C$ be a measurable coincidence point of the random operators S and T. Using (2.1) and (2.2), we have

$$\begin{aligned}
\|S(\omega,u_{n+1}(\omega))-p(\omega)\| &= \|T(\omega,u_n(\omega))-p(\omega)\| = \|T(\omega,x(\omega)-T(\omega,u_n(\omega))\| \\
&\leq \varphi(\|S(\omega,x(\omega))-T(\omega,x(\omega))\|)+\delta(\omega)\|S(\omega,x(\omega))-S(\omega,u_n(\omega))\| \\
&= \varphi(0)+\delta(\omega)\|S(\omega,u_n(\omega))-p(\omega)\| \\
&= \delta(\omega)\|S(\omega,u_n(\omega))-p(\omega)\| \leq \ldots \leq \delta^n(\omega)\|S(\omega,u_1(\omega))-p(\omega)\|.
\end{aligned}$$

Since $\varphi(0)=0$. Let

$$a_n=\delta^n(\omega)\|S(\omega,u_1(\omega))-p(\omega)\|. \tag{3.1}$$

By (2.1) and the random Jungck-Mann iterative process (2.3), we get

$$\begin{aligned}
\|S(\omega,v_{n+1}(\omega))-p(\omega)\| &= \|(1-\alpha_n)S(\omega,v_n(\omega))+\alpha_n T(\omega,v_n(\omega))-p(\omega)\| \\
&\leq (1-\alpha_n)\|S(\omega,v_n(\omega))-p(\omega)\|+\alpha_n\|T(\omega,v_n(\omega))-p(\omega)\| \\
&\leq (1-\alpha_n)\|S(\omega,v_n(\omega))-p(\omega)\|+\alpha_n\delta(\omega)\|S(\omega,v_n(\omega))-p(\omega)\| \\
&= [1-\alpha_n(1-\delta(\omega))]\|S(\omega,v_n(\omega))-p(\omega)\| \\
&\leq [1-\alpha(1-\delta(\omega))]\|S(\omega,v_n(\omega))-p(\omega)\| \\
&\leq \ldots\ldots \leq [1-\alpha(1-\delta(\omega))]^n\|S(\omega,v_1(\omega))-p(\omega)\|.
\end{aligned}$$

Put

$$b_n=[1-\alpha(1-\delta(\omega))]^n\|S(\omega,v_1(\omega))-p(\omega)\|. \tag{3.2}$$

Applying (2.1) and the random Jungck-Krasnoselskii iterative process (2.4), we can write

$$\begin{aligned}
\|S(\omega,m_{n+1}(\omega))-p(\omega)\| &= \|(1-\lambda)S(\omega,m_n(\omega))+\lambda T(\omega,m_n(\omega))-p(\omega)\| \\
&\le (1-\lambda)\|S(\omega,m_n(\omega))-p(\omega)\|+\lambda\|T(\omega,m_n(\omega))-p(\omega)\| \\
&\le (1-\lambda)\|S(\omega,m_n(\omega))-p(\omega)\|+\lambda\delta(\omega)\|S(\omega,m_n(\omega))-p(\omega)\| \\
&= [1-\lambda(1-\delta(\omega))]\|S(\omega,m_n(\omega))-p(\omega)\| \\
&\le [1-\alpha(1-\delta(\omega))]\|S(\omega,m_n(\omega))-p(\omega)\| \\
&\le ......\le [1-\alpha(1-\delta(\omega))]^n\|S(\omega,m_1(\omega))-p(\omega)\|.
\end{aligned}$$

Set

$$c_n=[1-\alpha(1-\delta(\omega))]^n\|S(\omega,m_1(\omega))-p(\omega)\|. \tag{3.3}$$

Again, using (2.1) and the random Jungck-Ishikawa iterative process (2.5), it follows that

$$\begin{aligned}
\|S(\omega,r_n(\omega))-p(\omega)\| &= \|(1-\beta_n)(S(\omega,z_n(\omega))-p(\omega))+\beta_n(T(\omega,z_n(\omega))-p(\omega))\| \\
&\le (1-\beta_n)\|S(\omega,z_n(\omega))-p(\omega)\|+\beta_n\|T(\omega,z_n(\omega))-p(\omega)\|
\end{aligned}$$

$$\le [1-\beta_n(1-\delta(\omega))]\|S(\omega,z_n(\omega))-p(\omega)\|, \tag{3.4}$$

also,

$$\begin{aligned}
\|S(\omega,z_{n+1}(\omega))-p(\omega)\| &\le (1-\alpha_n)\|S(\omega,z_n(\omega))-p(\omega)\|+\alpha_n\|T(\omega,r_n(\omega))-p(\omega)\| \\
&\le (1-\alpha_n)\|S(\omega,z_n(\omega))-p(\omega)\|+\alpha_n\delta(\omega)\|S(\omega,r_n(\omega))-p(\omega)\|,
\end{aligned}$$

Applying (3.4), we obtain that

$$\begin{aligned}
\|S(\omega,z_{n+1}(\omega))-p(\omega)\| &\leq (1-\alpha_n)\|S(\omega,z_n(\omega))-p(\omega)\| \\
&\quad +\alpha_n\delta(\omega)[1-\beta_n(1-\delta(\omega))]\|S(\omega,z_n(\omega))-p(\omega)\| \\
&= [1-\alpha_n(1-\delta(\omega)[1-\beta_n(1-\delta(\omega))]]\|S(\omega,z_n(\omega))-p(\omega)\| \\
&\leq [1-\alpha_n(1-\delta(\omega))]\|S(\omega,z_n(\omega))-p(\omega)\| \\
&\leq [1-\alpha(1-\delta(\omega))]\|S(\omega,z_n(\omega))-p(\omega)\| \\
&\leq .... \leq [1-\alpha(1-\delta(\omega))]^n\|S(\omega,z_1(\omega))-p(\omega)\|.
\end{aligned}$$

Consider

$$d_n=[1-\alpha(1-\delta(\omega))]^n\|S(\omega,z_1(\omega))-p(\omega)\|. \tag{3.5}$$

By (2.1) and the random Jungck-Picard-Krasnoselskii hybrid iterative process (2.6), we get

$$\begin{aligned}
\|S(\omega,x_{n+1}(\omega))-p(\omega)\| &= \|T(\omega,y_n(\omega))-p(\omega)\| \\
&\leq \delta(\omega)\|S(\omega,y_n(\omega))-p(\omega)\| \\
&\leq \delta(\omega)\{(1-\lambda)\|S(\omega,x_n(\omega))-p(\omega)\|+\lambda\|T(\omega,x_n(\omega))-p(\omega)\|\} \\
&= \delta(\omega)(1-\lambda)\|S(\omega,x_n(\omega))-p(\omega)\|+\delta^2(\omega)\lambda\|S(\omega,x_n(\omega))-p(\omega)\| \\
&= \delta(\omega)[1-\lambda(1-\delta(\omega))]\|S(\omega,x_n(\omega))-p(\omega)\| \\
&\leq \delta(\omega)[1-\alpha(1-\delta(\omega))]\|S(\omega,x_n(\omega))-p(\omega)\| \\
&\leq .... \leq (\delta(\omega)[1-\alpha(1-\delta(\omega))])^n\|S(\omega,x_1(\omega))-p(\omega)\|.
\end{aligned}$$

Set

$$h_n=(\delta(\omega)[1-\alpha(1-\delta(\omega))])^n\|S(\omega,x_1(\omega))-p(\omega)\|. \tag{3.6}$$

Now, we calculate the rate of convergence of our random iterative process (2.6) as the following:

(i) It is clear that

$$\begin{aligned}\frac{h_n}{a_n} &= \frac{(\delta(\omega)[1-\alpha(1-\delta(\omega))])^n \|S(\omega,x_1(\omega))-p(\omega)\|}{\delta^n(\omega)\|S(\omega,u_1(\omega))-p(\omega)\|} \\ &= [1-\alpha(1-\delta(\omega))])^n \frac{\|S(\omega,x_1(\omega))-p(\omega)\|}{\|S(\omega,u_1(\omega))-p(\omega)\|} \to 0 \text{ as } n\to\infty.\end{aligned}$$

Thus, $\{S(\omega,x_n(\omega))\}$ converges faster than $\{S(\omega,u_n(\omega))\}$ to $p(\omega)$. That is, the random Jungck-Picard-Krasnoselskii hybrid iterative process converges faster than the random Jungck-Picard iterative process to $p(\omega)$.

(ii) Similarly,

$$\begin{aligned}\frac{h_n}{b_n} &= \frac{(\delta(\omega)[1-\alpha(1-\delta(\omega))])^n \|S(\omega,x_1(\omega))-p(\omega)\|}{[1-\alpha(1-\delta(\omega))]^n \|S(\omega,v_1(\omega))-p(\omega)\|} \\ &= \delta^n(\omega)\frac{\|S(\omega,x_1(\omega))-p(\omega)\|}{\|S(\omega,v_1(\omega))-p(\omega)\|} \to 0 \text{ as } n\to\infty.\end{aligned}$$

Hence, $\{S(\omega,x_n(\omega))\}$ converges faster than $\{S(\omega,v_n(\omega))\}$ to $p(\omega)$.

(iii) Clearly,

$$\frac{h_n}{c_n} = \delta^n(\omega)\frac{\|S(\omega,x_1(\omega))-p(\omega)\|}{\|S(\omega,m_1(\omega))-p(\omega)\|} \to 0 \text{ as } n\to\infty.$$

Therefore, $\{S(\omega,x_n(\omega))\}$ converges faster than $\{S(\omega,m_n(\omega))\}$ to $p(\omega)$.

(iv) Finally, $\frac{h_n}{d_n} \to 0$ as $n\to\infty$. Then $\{S(\omega,x_n(\omega))\}$ converges faster than $\{S(\omega,z_n(\omega))\}$ to $p(\omega)$. This completes the proof.

4. Application to a random nonlinear integral equation

In this section, we shall prove that our random hybrid iterative process (2.6) converges strongly to a solution of the following random nonlinear integral equation:

$$x(t;\omega) = h(t;\omega) + \int_0^t f(\omega;t,s,x(\omega;s))ds, \tag{4.1}$$

where,

(i) $\omega \in \Omega$ is a supporting set of the probability measure space (Ω,β,μ),

(ii) $x(t;\omega)$ is unknown vector-valued random variables for each $t\in[0,a]$, $a>0$,

(iii) $h(t;\omega)$ is the stochastic free term defined for $t\in[0,a]$,

(iv) $f : \Omega \times [0,a] \times [0,a] \times \mathbb{R} \to \mathbb{R}$ is a random Carathéodory function and (Ω, β) is a measurable space.

Definition 4.1. A function $f : \Omega \times [0,a] \times \mathbb{R} \to \mathbb{R}$ is called random Carathéodory if the following conditions are satisfied:

(i) the mapping $(t;\omega) \to f(t,x;\omega)$ is jointly measurable for all $x \in \mathbb{R}$,

(ii) the mapping $x \to f(t,x;\omega)$ is continuous for all $t \in [0,a]$ and $\omega \in \Omega$.

The integral equation (4.1) in stochastic case is a similar to Volterra integral equation of the second kind in deterministic case.

Let $C([0,a], \mathbb{R})$ be the space of all continuous functions defined on a closed interval $[0,a]$ endowed with the norm

$$\|x(\omega) - y(\omega)\|_\infty = \max_{t \in [0,a]} \{|x(\omega;t) - y(\omega;t)|\},$$

for all $x(\omega), y(\omega) \in C([0,a], \mathbb{R})$ and $\omega \in \Omega$. It's known that $C([0,a], \mathbb{R}, \|.\|_\infty)$ is a Banach space under this norm.

Next, we consider the equation (4.1) under the following conditions:

(H_1) $h(t;\omega) \in C([0,a], \mathbb{R})$,

(H_2) A random Carathéodory function $f : \Omega \times [0,a] \times [0,a] \times \mathbb{R} \to \mathbb{R}$ satisfying

$$|f_1(\omega;t,s,u(\omega)) - f_2(\omega;t,s,v(\omega))| \leq \theta(\omega)\,|u(\omega) - v(\omega)|,$$

for all $t,s \in [0,a]$, $0 < \theta(\omega) < 1$ and $u(\omega), v(\omega) \in C([0,a], \mathbb{R})$,

(H_3) $u(\omega) : \Omega \to \mathbb{R}$ is a random fixed point for the random continuous operator S, i.e., $u(\omega) = S(\omega, u(\omega))$,

(H_4) For all $\theta(\omega) > 0$ and $t \in [0,a]$, the random value $t.\theta(\omega) < 1$.

The following theorem shows the convergence of random hybrid iterative process (2.6) to a solution of random integral equation (4.1).

Theorem 3.1. *Let (Ω, β, μ) be probability measure space and $\mathbb{R}$ is a separable Banach space. Assume that the axioms $(H_1) - (H_4)$ holds, then the random Jungck-Picard-Krasnoselskii hybrid iterative process (2.6) converges strongly to a random solution of the random nonlinear integral equation (4.1).*

Proof. For $x(\omega) \in C([0,a], \mathbb{R})$, $\omega \in \Omega$ and $t \in [0,a]$, we define the random integral operator $T : \Omega \times [0,a] \to \mathbb{R}$ by

$$T(x)(\omega;t) = h(t;\omega) + \int_0^t f(\omega;t,s,x(\omega;s))ds.$$

Let $S(\omega, x_n(\omega))$ be an iterative sequence generated by the random Jungck-Picard-Krasnoselskii hybrid iterative process (2.6) and $p(\omega) : \Omega \to \mathbb{R}$ be a unique random fixed point of the random operator T, which is a solution of a random nonlinear integral equation (4.1).

Now, we prove that $S(\omega, x_n(\omega))$ converges strongly to $p(\omega)$. For each $\omega \in \Omega$ and $t \in [0,a]$, we have

$$\begin{aligned}
& \|S(\omega, y_n(\omega)) - p(\omega)\|_\infty \\
\leq\ & (1-\lambda)\|S(\omega,x_n(\omega)) - p(\omega)\|_\infty + \lambda\|T(\omega,x_n(\omega)) - p(\omega)\|_\infty \\
=\ & (1-\lambda)\|S(\omega,x_n(\omega)) - p(\omega)\|_\infty + \lambda\|T(\omega,x_n(\omega)) - T(\omega,p(\omega))\|_\infty \\
=\ & (1-\lambda)\|S(\omega,x_n(\omega)) - p(\omega)\|_\infty + \lambda \max_{t\in[0,a]}\{|T(\omega,x_n(\omega;t)) - T(\omega,p(\omega;t))|\} \\
=\ & (1-\lambda)\|S(\omega,x_n(\omega)) - p(\omega)\|_\infty + \lambda \max_{t\in[0,a]}\left\{\left|\int_0^t f(\omega;t,s,x_n(\omega;s))ds - \int_0^t f(\omega;t,s,p(\omega;s))ds\right|\right\}
\end{aligned}$$

(4.2)
$$\leq (1-\lambda)\|S(\omega,x_n(\omega)) - p(\omega)\|_\infty + \lambda \max_{t\in[0,a]}\left\{\int_0^t |f(\omega;t,s,x_n(\omega;s)) - f(\omega;t,s,p(\omega;s))|\,ds\right\}.$$

Applying conditions (H_2) and (H_3) in (4.2), we get

$$\begin{aligned}
& \|S(\omega, y_n(\omega)) - p(\omega)\|_\infty \\
\leq\ & (1-\lambda)\|S(\omega,x_n(\omega)) - p(\omega)\|_\infty + \lambda \max_{t\in[0,a]}\left\{\int_0^t \theta(\omega)(|S(\omega,x_n(\omega;s)) - p(\omega;s)|)ds\right\} \\
\leq\ & (1-\lambda)\|S(\omega,x_n(\omega)) - p(\omega)\|_\infty + \lambda.\theta(\omega)\int_0^t \max_{t\in[0,a]}\{|S(\omega,x_n(\omega;s)) - p(\omega;s)|\}ds \\
=\ & (1-\lambda)\|S(\omega,x_n(\omega)) - p(\omega)\|_\infty + (\lambda.\theta(\omega).t)\|S(\omega,x_n(\omega)) - p(\omega)\|_\infty
\end{aligned}$$

(4.3)
$$= [1-\lambda(1-t.\theta(\omega))]\|S(\omega,x_n(\omega)) - p(\omega)\|_\infty.$$

By (2.6) and (4.2), we can write

$$\begin{aligned}
\|S(\omega,x_{n+1}(\omega))-p(\omega)\|_\infty &= \|T(\omega,y_n(\omega))-T(\omega,p(\omega))\|_\infty \\
&= \max_{t\in[0,a]}\left\{\left|\int_0^t f(\omega;t,s,y_n(\omega;s))ds-\int_0^t f(\omega;t,s,p(\omega;s))ds\right|\right\} \\
&\leq \max_{t\in[0,a]}\left\{\int_0^t |f(\omega;t,s,y_n(\omega;s))-f(\omega;t,s,p(\omega;s))|\,ds\right\}
\end{aligned}$$

$$\leq t.\theta(\omega)\,\|S(\omega,y_n(\omega))-p(\omega)\|_\infty. \tag{4.4}$$

It follows from (4.3) and (4.4) that,

$$\begin{aligned}
\|S(\omega,x_{n+1}(\omega))-p(\omega)\|_\infty &\leq t.\theta(\omega)[1-\lambda(1-t.\theta(\omega))]\,\|S(\omega,x_n(\omega))-p(\omega)\|_\infty \\
&\leq [1-\lambda(1-t.\theta(\omega))]\,\|S(\omega,x_n(\omega))-p(\omega)\|_\infty.
\end{aligned}$$

Taking $\lambda(1-t.\theta(\omega))=\rho_n<1$ and $k_n=\|S(\omega,x_n(\omega))-p(\omega)\|_\infty$, thus, $k_{n+1}\leq(1-\rho_n)k_n$. Therefore all conditions of Lemma 1.1 are satisfied. Hence $\lim_{n\to\infty}\|S(\omega,x_n(\omega))-p(\omega)\|_\infty=0$. So our random iterative hybrid process converges strongly to a unique random solution $p(\omega)$ of problem (4.1). This completes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

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