Available online at http://jfpt.scik.org
J. Fixed Point Theory, 2018, 2018:6
ISSN: 2052-5338

COMMON FIXED POINTS FOR WEAKLY COMPATIBLE MAPPINGS IN MULTIPLICATIVE CONE METRIC SPACES

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Abstract. In this paper we introduces the concept of cone multiplicative metric space and proved some fixed point results in cone multiplicative metric space. Also, we establish a theorem postulating a unique common fixed point for four self maps through weak compatibility satisfying a more generalized contractive condition in a non-normal cone multiplicative metric space. An example illustrates the main result of this paper.

Keywords: cone multiplicative metric space; normal cones; fixed point; weakly compatible mappings.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

The study of fixed point theory become a subject of great interest due to its applications in Mathematics as well as in other areas of research. There are many researchers who have worked in fixed point theory of contractive mapping see, [4, 6]. In [4], Banach presented a most outstanding result concerning to contraction mapping, this famous result is known as Banach contraction principle.

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Received November 5, 2017

There has been a number of generalizations of metric space. One such generalization is that of a cone metric space initiated by Huang and Zhang [7]. In this space they replaced the set of real numbers of a metric space by an ordered Banach space and gave some fundamental results for a self map satisfying a contractive condition. In [1] Abbas and Jungck, generalized the result of [7] for two self maps through weak compatibility in a normal cone metric space. Along the same lines, Vetro [13] proved some fixed point theorems for two self maps satisfying a contractive condition through weak compatibility. In [9] the authors introduce the concept of compatibility in cone metric space. Recently, Rezapour and Hamlbarani [11] were able to omit the assumption of normality in a cone metric space, which is a milestone in developing fixed point theory. Also in [2] Arshad et al. proved a fixed point theorem for three self map adopting the contractive condition of [12] through weak compatibility.

Ozavsar and Cevikel [10] introduced the concept of multiplicative contraction mappings and proved some fixed point theorems of such mappings on a complete multiplicative metric space. They also gave some topological properties of the relevant multiplicative metric space. Hxiaoju et al. [8] studied common fixed points for weak commutative mappings on a multiplicative metric space. For further details about multiplicative metric space and related concepts, we refer the reader to [5, 10].

In this paper we introduces the concept of cone multiplicative metric space and proved some fixed point results in cone multiplicative metric space. Also, we establish a theorem postulating a unique common fixed point for four self maps through weak compatibility satisfying a more generalized contractive condition in a non-normal cone multiplicative metric space. An example illustrates the main result of this paper.

2. Multiplicative Cone Metric Spaces

In this section we shall define cone multiplicative metric spaces and prove some properties. Let E always be a positive real Banach space and P a subset of E. P is called a cone if and only if

(P1) P is closed, non-empty, and $P \neq \{1\}$;

(P2) $a, b \in \mathbb{R}$, $x, y \in P$ implies $x^a y^b \in P$;

(P3) $x \in P$ and $\frac{1}{x} \in P$ implies $x = 1$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $\frac{y}{x} \in P$. We shall write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $\frac{y}{x} \in intP$, $intP$ denotes the interior of P.

The cone P is said to be normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$1 \leq x \leq y \text{ implies } \|x\| \leq \|y\|^K.$$

The Least positive number satisfying above is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$$

for some $y \in E$, then there is $x \in E$ such that $\left\|\frac{x_n}{x}\right\| \to 1$ $(n \to \infty)$. Equivalently, the cone P is regular if and only if every decreasing sequence which is bounded below is convergent. It is well known that a regular cone is a normal cone.

In the following we always suppose E is a positive real Banach space, P is a cone in E with $intP \neq \emptyset$ and $\leq$ is partial ordering with respect to P.

Remark 2.1. Let P be a cone in a positive real Banach space E. We have

(i) for every $x \in intP$ and $\lambda > 0$, we have $x^\lambda \in intP$.

(ii) if $x, y \in intP$, then $xy \in intP$.

Definition 2.2. Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(CMM1) $d(x,y) \geq 1$ for all $x, y \in X$ and $d(x,y) = 1$ if and only if $x = y$;

(CMM2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(CMM3) $d(x,y) \leq d(x,z).d(z,y)$ for all $x, y, z \in X$ (Multiplicative triangle inequality).

Then d is called a cone multiplicative metric on X, and (X,d) is called a cone multiplicative metric space.

It is obvious that cone multiplicative metric space generalize multiplicative metric space.

Example 2.3. Let $\mathbb{R}^n_+$ be the collection of all n-tuples of positive real numbers. Let $X = \mathbb{R}^n_+, E = \mathbb{R}_+$ and $P = [1,\infty)$ and let $d : X \times X \to E$ be defined by

$$d(x,y) = \left(\left|\frac{x_1}{y_1}\right| . \left|\frac{x_2}{y_2}\right| \cdots \left|\frac{x_n}{y_n}\right|\right),$$

where $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in X$ and $|.| : E \to E$ is defined by

$$|a| = \begin{cases} a, & \text{if } a \geq 1; \\ \frac{1}{a}, & \text{if } a < 1. \end{cases}, \tag{1}$$

see [3]. Then it is obvious that all conditions of multiplicative cone metric are satisfied.

Example 2.4. Let $X = \mathbb{R}$, $E = [1,\infty)$ and $P = [1,\infty)$ and let $d : X \times X \to E$ be defined by $d(x,y) = a^{|x-y|}$, where $x,y \in X$ and $a > 1$. Then d is cone multiplicative metric and hence (X,d) is a cone multiplicative metric space.

Example 2.5. Let $E = \mathbb{R}^2_+$, $P = \{(x,y) \in E : x,y \geq 1\}$, $X = \mathbb{R}$ and $d : X \times X \to E$ such that $d(x,y) = \left(\left|\frac{x}{y}\right|, \left|\frac{x}{y}\right|^{\alpha}\right)$, where $\alpha \geq 0$ is a constant. Then (X,d) is a cone multiplicative metric space.

Definition 2.6. Let (X,d) be a cone multiplicative metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then $\{x_n\}$ in X is said to be

(i) a multiplicative convergent to x, if for every $c \in E$ with $1 \ll c$ there is N such that for all $n > N$, $d(x_n,x) \ll c$ (or equivalently, if for every multiplicative open ball $B_c(x) = \{y : d(x,y) \ll c\}$, $c \gg 1$, there exists a natural number N such that for all $n > N$, then $x_n \in B_c(x)$), that is, $d(x_n,x) \to 1$ as $n \to \infty$. In this case, we write $x_n \to x (n \to \infty)$ or $\lim_{n\to\infty} x_n = x$.

(ii) a multiplicative Cauchy sequence if for all $c \gg 1$, there exists $N \in \mathbb{N}$ such that $d(x_n,x_m) \ll c$ for all $m,n \in \mathbb{N}$, that is, $d(x_n,x_m) \to 1$ as $n,m \to \infty$.

(ii) We call a multiplicative cone metric space is complete if every multiplicative Cauchy sequence in it is multiplicative convergent to $x \in X$.

Lemma 2.7. *Let (X,d) be a cone multiplicative metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. Then $x_n \to x\,(n \to \infty)$ if and only if $d(x_n,x) \to 1\,(n \to \infty)$.*

Remark 2.8. The set of positive real number $\mathbb{R}_+$ is not complete according to the usual metric. Let $X = E = P = \mathbb{R}_+$ and the sequence $\{x_n = \frac{1}{n}\}$. It is obvious $\{x_n\}$ is a Cauchy sequence in X with respect to the usual cone metric space and X is not a complete cone metric space since $0 \notin \mathbb{R}_+$. In case of a multiplicative cone metric space, we take a sequence $\{x_n = a^{\frac{1}{n}}\}$, where $a > 1$. Then $\{x_n\}$ is a Cauchy sequence since for $n \geq m$,

$$d(x_n,x_m) = \left|\frac{x_n}{x_m}\right| = \left|\frac{a^{\frac{1}{n}}}{a^{\frac{1}{m}}}\right| = |a^{\frac{1}{n}-\frac{1}{m}} < a^{\frac{1}{m}-\frac{1}{n}} < a^{\frac{1}{m}} \ll c \text{ if } m > \frac{\log a}{\log c},$$

where $|a|$ is defined by Equation (1). Also, $\{x_n\} \to 1$ as $n \to \infty$ and $1 \in \mathbb{R}_+$. Hence (X,d) is a complete multiplicative cone metric space.

Lemma 2.9. *Let (X,d) be a multiplicative cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ multiplicative convergent to x and $\{x_n\}$ multiplicative convergent to y, then $x = y$. That is, the limit of $\{x_n\}$ is unique.*

Proof. For any $c \in E$ with $c \gg 1$, there is $N \in \mathbb{N}$ such that for all $n > N$, $d(x_n,x) \ll \sqrt{c}$ and $d(x_n,y) \ll \sqrt{c}$. We have

$$d(x,y) \leq d(x_n,x).d(x_n,y) \leq c.$$

Hence $\|d(x,y)\| \leq \|c\|^K$. Since c is an arbitrary $d(x,y) = 1$ and so $x = y$.

Lemma 2.10. *Let (X,d) be a multiplicative cone metric space, $\{x_n\}$ be a sequence in X. If $\{x_n\}$ is a multiplicative convergent to x, then $\{x_n\}$ is a multiplicative Cauchy sequence.*

Proof. For any $c \in E$ with $c \gg 1$, there is $N \in \mathbb{N}$ such that for all $n,m > N$, $d(x_n,x) \ll \sqrt{c}$ and $d(x_m,x) \ll \sqrt{c}$. Hence $d(x_n,x_m) \leq d(x_n,x).d(x_m,x) \ll c$. Therefore, $\{x_n\}$ is a multiplicative Cauchy sequence.

Lemma 2.11. *Let (X,d) be a multiplicative cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $x_n \to x, y_n \to y$ $(n \to \infty)$. Then $d(x_n,y_n) \to d(x,y)$ $(n \to \infty)$.*

Proof. For every $\varepsilon > 1$, choose $c \in E$ with $c \gg 1$. From $x_n \to x$ and $y_n \to y$, there is N such that for all $n > N$, $d(x_n, x) \ll \sqrt{c}$ and $d(y_n, y) \ll \sqrt{c}$. We have

$$d(x_n, y_n) \leq d(x_n, x) d(x, y) d(y_n, y)$$

$$d(x, y) \leq d(x_n, x) d(x_n, y_n) d(y_n, y).$$

which imply

$$\frac{d(x_n, y_n)}{d(x, y)} \leq d(x_n, y).d(y_n, y)$$
$$\frac{d(x, y)}{d(x_n, y_n)} \leq d(x_n, y).d(y_n, y)$$

Hence, we have

$$\left\| \frac{d(x_n, y_n)}{d(x, y)} \right\| \leq \|d(x_n, y).d(y_n, y)\| \ll \|c\| \text{ for all } n > n,$$

where $\|.\|$ is the multiplicative Euclidean norm. Therefore $d(x_n, y_n) \to d(x, y)$ $(n \to \infty)$.

Proposition 2.12. *Let (X, d) be a multiplicative cone metric space, P be a normal cone with normal constant K. If $\{x_{2n}\}$ is a subsequence of a multiplicative Cauchy sequence $\{x_n\}$ and $x_{2n} \to z$, then $x_n \to z$.*

Proof. As $x_{2n} \to z$ and $\{x_n\}$is multiplicative Cauchy sequence, for any $c \in E$ with $c \gg 1$ there is a positive integer N such that for all $n > N$,

$$d(x_n, x_{2n}) \ll \sqrt{c} \quad \text{and} \quad d(x_{2n}, z) \ll \sqrt{c}.$$

Now,

$$d(x_n, z) \leq d(x_n, x_{2n}).d(x_{2n}, z) \ll \sqrt{c}\sqrt{c} = c.$$

Therefore, $d(x_n, z) \ll c$ for all $n > N$ and so $x_n \to z$.

The continuity of the self-maps in the cone multiplicative metric space is, in fact, the sequentially continuity. If $f : X \to X$, where (X, d) is a cone multiplicative metric space, then f is continuous at the point $a \in X$ if, for every sequence $x_n \in X$, which converges in the cone multiplicative metric d to a, the sequence $fx_n \to fa$, i.e.,

$$d(x_n, x) \ll c \Longrightarrow d(fx_n, fa) \ll c.$$

In the case of the normal cone this is equivalent to

$$d(x_n,x) \to 1 \Longrightarrow d(fx_n, fa) \to 1,$$

in the norm of the Banach space E with the cone P.

The following lemmas will be useful in what follows

Lemma 2.13. *If $c \in intP$ and $1 \leq a_n$, $a_n \to 1$, then there is a natural number N, for all $n > N$, we have $a_n \ll c$.*

Proof. Let $c \gg 1$ be given. Choose a symmetric neighborhood V such that $cV \subset P$. Since $a_n \to 1$ then there is an N such that $a_n \in V$ and $\frac{1}{a_n} \in V$ for all $n > N$. This means that $ca_n, c\frac{1}{a_n} \in cV \subset P$ for all $n > N$, that is, $a_n \ll c$.

Lemma 2.14. *Let u, v, w be vectors from Banach space E.*

(i) If $u \leq v$ and $v \ll w$, then $u \ll w$.

(ii) If $1 \leq u \ll c$, for each $c \in intP$, then $u = 1$.

Lemma 2.15. *If $1 \leq d(x_n,x) \leq b_n$ and $b_n \to 1$, then $d(x_n,x) \ll c$, where x_n and x are a sequence and given point in X, respectively.*

Proof. The proof follows from Lemmas 2.13 and 2.14(i).

Lemma 2.16. *If $1 \leq a_n \leq b_n$ and $a_n \to a$, $b_n \to b$, then $a \leq b$ for each cone P.*

Lemma 2.17. *If E is a positive real Banach space with a cone P, and if $a \leq a^{\lambda}$, where $a \in P$ and $0 \leq \lambda < 1$, then $a = 1$.*

Proof. The condition $a \leq a^{\lambda}$ means that $a^{\lambda-1} \in P$, i.e., $a^{-(1-\lambda)} \in P$. Since $a \in P$ and $1-\lambda \geq 0$, then also $a^{1-\lambda} \in P$. Thus we have $a^{\lambda-1}, a^{1-\lambda} \in P$, and therefore $a = 1$.

Lemma 2.18. *Let (X,d) be a cone multiplicative metric space with respect to a cone P in a positive real Banach space E, and let $k_1, k_2, k > 0$ be constants. Suppose that $x_n \to x$ and $y_n \to y$*

in X and

$$a^k \le [d(x_n,x)]^{k_1}[d(y_n,y)]^{k_2} \tag{2}$$

Then $a = 1$.

Proof. As $x_n \to x$ and $y_n \to y$, there exists a positive integer N_c such that

$$\frac{c^{\frac{1}{k_1+k_2}}}{d(x_n,x)}, \frac{c^{\frac{1}{k_1+k_2}}}{d(y_n,y)} \in intP, \quad \text{for all } n > N_c.$$

Therefore, by Remark 2.1, we have

$$\frac{c^{\frac{k_1}{k_1+k_2}}}{d^{k_1}(x_n,x)}, \frac{c^{\frac{k_2}{k_1+k_2}}}{d^{k_2}(y_n,y)} \in intP, \quad \text{for all } n > N_c.$$

Again by multiplication and Remark 2.1, we have $\frac{c}{d^{k_1}(x_n,x)d^{k_2}(y_n,y)} \in intP$ for all $n > N_c$. From (2) and Lemma 2.14, we have $\frac{c}{a^k} \in intP$, i.e. $a^k \ll c$ for each $c \in intP$ and so, $a = 1$.

3. Fixed Point Theorems

In this section we shall prove some fixed points theorems of contractive mappings.

Definition 3.1. Let T be a mapping of a multiplicative cone metric space (X,d) into itself. Then T is called a multiplicative contraction if there exists a real number $\lambda \in [0,1)$ such that $d(Tx,Ty) \le [d(x,y)]^\lambda$ for all $x,y \in X$.

Now, based on the definition of multiplicative contraction, we introduce the following Banach contraction principle for cone multiplicative metric spaces.

Theorem 3.2. *Let (X,d) be a complete cone multiplicative cone space, P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx,Ty) \le d^\lambda(x,y), \qquad \textit{for all } x,y \in X,$$

where $\lambda \in [0,1)$ is a constant. Then T has a unique fixed point in X, and for any $x \in X$, iterative sequence $\{T^n\}$ converges to the fixed point.

Proof. Choose $x_0 \in X$. Set $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \cdots, x_{n+1} = Tx_n = T^{n+1}x_0, \cdots$. We have

$$d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1}) \leq [d(x_n,x_{n-1})]^{\lambda} \leq [d(x_{n-1},x_{n-2})]^{\lambda^2} \leq \cdots \leq [d(x_1,x_0)]^{\lambda^n}.$$

So for $n > m$,

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n-1})d(x_{n-1},x_{n-2})\cdots d(x_{m+1},x_m) \\ &\leq d^{n-1}(x_1,x_0)d^{n-2}(x_1,x_0)\cdots d^m(x_1,x_0) \\ &= [d(x_1,x_0)]^{(\lambda^{n-1}+\lambda^{n-2}+\cdots+\lambda^m)} \leq [d(x_1,x_0)]^{\frac{\lambda^m}{1-\lambda}}. \end{aligned}$$

We get $\|d(x_n,x_m)\| \leq \|d(x_1,x_0)\|^{\frac{K\lambda^m}{1-\lambda}}$. This implies $d(x_n,x_m) \to 1$ $(n,m \to \infty)$. Hence $\{x_n\}$ is a multiplicative Cauchy sequence. By the completeness of X, there is $x^* \in X$ such that $x_n \to x^*$ $(n \to \infty)$. Since

$$d(Tx^*,x^*) \leq d(Tx_n,Tx^*)d(Tx_n,x^*) \leq [d(x_n,x^*)]^{\lambda} d(x_{n+1},x^*),$$

$$\|d(Tx^*,x^*)\| \leq \|d(x_n,x^*)\|^{K\lambda} \|d(x_{n+1},x^*)\| \to 1.$$

Hence $\|d(Tx^*,x^*)\| = 1$. This implies $Tx^* = x^*$. So, x^* is a fixed point of T.

Now if y^* is another fixed point of T, then

$$d(x^*,y^*) = d(Tx^*,Ty^*) \leq d^{\lambda}(x^*,y^*).$$

Hence it follows from Lemma 2.17 that $d(x^*,y^*) = 1$ and so $x^* = y^*$. Therefore the fixed point of T is unique.

Corollary 3.3. *Let (X,d) be a complete cone multiplicative cone space, P be a normal cone with normal constant K. For $c \in E$ with $c \gg 1$ and $x_0 \in X$, set $B(x_0,c) = \{x \in X : d(x_0,x) \leq c\}$. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx,Ty) \leq d^{\lambda}(x,y), \qquad \text{for all } x,y \in B(x_0,c),$$

where $\lambda \in [0,1)$ is a constant and $d(Tx_0,x_0) \leq c^{1-\lambda}$. Then T has a unique fixed point in $B(x_0,c)$.

Proof. We only need to prove that $B(x_0,c)$ is complete and $Tx \in B(x_0,c)$ for all $x \in B(x_0,c)$.

Suppose $\{x_n\}$ is a multiplicative Cauchy sequence in $B(x_0,c)$. Then $\{x_n\}$ is a multiplicative Cauchy sequence in X. By completeness of X, there is $x \in X$ such that $x_n \to x (n \to \infty)$. We have

$$d(x_0,x) \le d(x_n,x_0)d(x_0,x) \le cd(x_n,x).$$

Since $x_n \to x$, $d(x_n,x) \to 1$. Hence $d(x_0,x) \le c$, and $x \in B(x_0,c)$. Therefore $B(x_0,c)$ is complete.

For every $x \in B(x_0,c)$,

$$d(x_0,Tx) \le d(Tx_0,x_0)d(Tx_0,Tx) \le c^{1-\lambda}d^{\lambda}(x_0,x) \le c.$$

Hence $Tx \in B(x_0,c)$.

Corollary 3.4. *Let (X,d) be a complete cone multiplicative cone space, P be a normal cone with normal constant K. Suppose a mapping $T : X \to X$ satisfies for some positive integer n,*

$$d(T^n x, T^n y) \le d^{\lambda}(x,y), \qquad \text{for all } x,y \in X,$$

where $\lambda \in [0,1)$ is a constant. Then T has a unique fixed point in X.

Proof. From Theorem 3.2, T^n has a unique fixed point x^*. But $T^n(Tx^*) = T(T^n x^*) = Tx^*$, so Tx^* is also a fixed point of T^n. Hence $Tx^* = x^*$, x^* is a fixed point of T. Since the fixed point of T is also fixed point of T^n, the fixed point of T is unique.

Definition 3.5 Let (X,d) be a cone multiplicative metric space. If for any sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ is multiplicative convergent in X. Then X is called sequentially compact cone multiplicative metric space.

Theorem 3.6. *Let (X,d) be sequentially compact cone multiplicative metric space, P be a regular cone. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx,Ty) < d(x,y), \qquad \text{for all } x,y \in X.$$

Then T has a unique fixed point in X.

Proof. Choose $x_0 \in X$. Set $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \cdots, x_{n+1} = Tx_n = T^{n+1}x_0, \cdots$. If for some n, $x_{n+1} = x_n$, then x_n is a fixed point of T, the proof is complete. So, we assume that for all n, $x_{n+1} \neq x_n$. Set $d_n = d(x_{n+1}, x_n)$, then

$$d_{n+1} = d(x_{n+2}, x_{n+1}) = d(Tx_{n+1}, Tx_n) < d(x_{n+1}, x_n) = d_n.$$

Therefore d_n is a decreasing sequence bounded below by 1. Since P is regular, there is $d^* \in E$ such that $d_n \to d^*$ $(n \to \infty)$. From the sequentially compact of X, there are subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and $x^* \in X$ such that $x_{n_i} \to x^*(i \to \infty)$. We have

$$d(Tx_{n_i}, Tx^*) \leq d(x_{n_i}, x^*), \quad i = 1, 2, \cdots.$$

So

$$\|d(Tx_{n_i}, Tx^*)\| \leq \|d(x_{n_i}, x^*)\|^K \to 1 \quad (i \to \infty).$$

where K is the normal constant of E. Hence $Tx_{n_i} \to Tx^*$ $(i \to \infty)$. Similarly $T^2x_{n_i} \to T^2x^*$ $(i \to \infty)$. By using Lemma 2.11, we have $d(Tx_{n_i}, x_{n_i}) \to d(Tx^*, x^*)$ $(i \to \infty)$ and $d(T^2x_{n_i}, Tx_{n_i}) \to d(T^2x^*, Tx^*)$ $(i \to \infty)$. It is obvious that $d(Tx_{n_i}, x_{n_i}) = d_{n_i} \to d^* = d(Tx^*, x^*)$ $(i \to \infty)$. Now we shall prove that $Tx^* = x^*$. If $Tx^* \neq x^*$, then $d^* \neq 1$. We have

$$d^* = d(Tx^*, x^*) > d(T^2x^*, Tx^*) = \lim_{i\to\infty} d(T^2x_{n_i}, Tx_{n_i}) = \lim_{i\to\infty} d_{n_i+1} = d^*.$$

Hence, we have a contradiction, so $Tx^* = x^*$. That is, x^* is a fixed point of T. The uniqueness of fixed point is obvious.

Theorem 3.7. *Let (X, d) be a complete cone multiplicative cone space, P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leq [d(Tx, x)d(Ty, y)]^\lambda, \qquad \textit{for all } x, y \in X,$$

where $\lambda \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X and for any $x \in X$, iterative sequence $\{T^nx\}$ multiplicative converges to the fixed point.

Proof. Choose $x_0 \in X$. Set $x_1 = Tx_0, x_2 = Tx_1 = T^2 x_0, \cdots, x_{n+1} = Tx_n = T^{n+1}x_0, \cdots$. We have

$$\begin{aligned} d(x_{n+1},x_n) = d(Tx_n,Tx_{n-1}) &\leq [d(Tx_n,x_n)d(Tx_{n-1},x_{n-1})]^{\lambda} \\ &= [d(x_{n+1},x_n)d(x_n,x_{n-1})]^{\lambda}. \end{aligned}$$

So

$$d(x_{n+1},x_n) \leq [d(x_n,x_{n-1})]^{\frac{\lambda}{1-\lambda}} = [d(x_n,x_{n-1})]^h,$$

where $h = \frac{\lambda}{1-\lambda}$. For $n > m$,

$$\begin{aligned} d(x_n,x_m) &\leq d(x_n,x_{n-1})d(x_{n-1},x_{n-2})\cdots d(x_{m+1},x_m) \\ &\leq [d(x_1,x_0)]^{h^{n-1}+h^{n-2}+\cdots+h^m} \leq d^{\frac{h^m}{1-h}}(x_1,x_0). \end{aligned}$$

We get $\|d(x_n,x_m)\| \leq \|d(x_1,x_0)\|^{\frac{Kh^m}{1-h}}$. This implies that $d(x_n,x_m) \to 1$ $(n,m \to \infty)$. Hence $\{x_n\}$ is a multiplicative Cauchy sequence. By the completeness of X, there is $x^* \in X$ such that $x_n \to x^*$ $(n \to \infty)$. Since

$$\begin{aligned} d(Tx^*,x^*) &\leq d(Tx_n,Tx^*)d(Tx_n,x^*) \\ &\leq [d(Tx_n,x_n)d(Tx^*,x^*)]^{\lambda} d(x_{n+1},x^*) \\ d(Tx^*,x^*) &\leq [d(Tx_n,x_n)]^{\frac{\lambda}{1-\lambda}}[d(x_{n+1},x^*)]^{\frac{1}{1-\lambda}}, \\ \|d(Tx^*,x^*)\| &\leq \|d(Tx_n,x_n)\|^{K\frac{\lambda}{1-\lambda}}\|d(x_{n+1},x^*)\|^{\frac{1}{1-\lambda}} \to 1. \end{aligned}$$

Hence $\|d(Tx^*,x^*)\| = 1$. This implies that $Tx^* = x^*$. So x^* is a fixed point of T. Now if y^* is another point of T, then

$$d(x^*,y^*) = d(Tx^*,Ty^*) \leq d^{\lambda}(Tx^*,x^*)d^{\lambda}(Ty^*,y^*) = 1.$$

Hence $x^* = y^*$. Therefore the fixed point of T is unique.

Theorem 3.8. *Let (X,d) be a complete cone multiplicative cone space, P be a normal cone with normal constant K. Suppose the mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx,Ty) \leq [d(Tx,y)d(Ty,x)]^{\lambda}, \qquad \textit{for all } x,y \in X,$$

where $\lambda \in [0, \frac{1}{2})$ is a constant. Then T has a unique fixed point in X and for any $x \in X$, iterative sequence $\{T^n x\}$ multiplicative converges to the fixed point.

Proof. Choose $x_0 \in X$. Set $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \cdots, x_{n+1} = Tx_n = T^{n+1}x_0, \cdots$. We have

$$\begin{aligned} d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1}) &\leq [d(Tx_n, x_{n-1})d(Tx_{n-1}, x_n)]^\lambda \\ &= [d(x_{n+1}, x_n)d(x_n, x_{n-1})]^\lambda. \end{aligned}$$

So

$$d(x_{n+1}, x_n) \leq [d(x_n, x_{n-1})]^{\frac{\lambda}{1-\lambda}} = [d(x_n, x_{n-1})]^h,$$

where $h = \frac{\lambda}{1-\lambda}$. For $n > m$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n-1})d(x_{n-1}, x_{n-2}) \cdots d(x_{m+1}, x_m) \\ &\leq [d(x_1, x_0)]^{h^{n-1}+h^{n-2}+\cdots+h^m} \leq d^{\frac{h^m}{1-h}}(x_1, x_0). \end{aligned}$$

We get $\|d(x_n, x_m)\| \leq \|d(x_1, x_0)\|^{\frac{Kh^m}{1-h}}$. This implies that $d(x_n, x_m) \to 1$ $(n, m \to \infty)$. Hence $\{x_n\}$ is a multiplicative Cauchy sequence. By the completeness of X, there is $x^* \in X$ such that $x_n \to x^*$ $(n \to \infty)$. Since

$$\begin{aligned} d(Tx^*, x^*) &\leq d(Tx_n, Tx^*)d(Tx_n, x^*) \\ &\leq [d(Tx^*, x_n)d(Tx_n, x^*)]^\lambda d(x_{n+1}, x^*) \\ d(Tx^*, x^*) &\leq [d(Tx_n, x^*)]^{\frac{\lambda}{1-\lambda}}[d(x_{n+1}, x^*)]^{\frac{1}{1-\lambda}} d(x_{n+1}, x^*), \\ \|d(Tx^*, x^*)\| &\leq \|d(Tx_n, x_n)\|^{\frac{K\lambda}{1-\lambda}} \|d(x_{n+1}, x^*)\|^{\frac{1}{1-\lambda}} \|d(x_{n+1}, x^*)\| \to 1. \end{aligned}$$

Hence $\|d(Tx^*, x^*)\| = 1$. This implies that $Tx^* = x^*$. So x^* is a fixed point of T. Now if y^* is another point of T, then

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leq [d(Tx^*, y^*)d(Ty^*, x^*)]^\lambda = [d(x^*, y^*)]^{2\lambda}.$$

Hence $d(x^*, y^*) = 1$ and so $x^* = y^*$. Therefore the fixed point of T is unique.

Remark 3.9. Theorems 3.2, 3.6, 3.7 and 3.8 generalize the fixed point theorems of contractive mappings in multiplicative metric spaces to cone multiplicative metric spaces.

Example 3.10. Let $E=\mathbb{R}_+$, and $P=[1,\infty$ a normal cone in P. Let $X=\{(x,1)\in\mathbb{R}^2_+ : 1\leq x\leq 2\}\cup\{(1,y)\in\mathbb{R}^2_+ : 1\leq y\leq 2\}$. Consider a mapping $d: X\times X\to E$ defined by

$$d((a,b),(c,d))=\left(\left|\frac{a}{c}\right|\left|\frac{b}{d}\right|\right)^{\frac{1}{3}},$$

where $|a|$ is defined by Equation (1). Then (X,d) is a complete cone multiplicative metric space.

Let the mapping $T: X\to X$ defined by

$$T((x,1))=(1,\sqrt{x}) \quad \text{and} \quad T((1,y))=(\sqrt{y},1)$$

Thus the mapping holds the following multiplicative contraction condition

$$d(T(a,b),T(c,d))\leq [d((a,b).(c,d))]^{\frac{1}{2}}, \quad \text{for all } (a,b),(c,d)\in X.$$

It is obvious that T has a unique fixed point $(1,1)$.

Example 3.11. We can make use of Theorem 3.2 in this section to the given first order boundary problem of periodic type

$$b^* = F(t,b(t)) \qquad b(1)=b_0 \tag{3}$$

We can write (3) as, $b(t)=b_0\int_1^t F(s,b(s))^{ds}$.

Consider the boundary problem (3) with the continuous function F and let $F(a,b)$ satisfies the local multiplicative Lipschitz condition, that is,

$$\left|\frac{G(a,b)}{G(a,c)}\right|\leq L^{|\frac{b}{c}|}$$

Set $N=\max\limits_{[h,h]\times[a,b]}|F(a,b)|$, then it must have a unique solution of (3).

Let $X=E=C^*([-h,h])$ and $P=\{a\in E : a\geq 0\}$. Define $d^*: X\times X\to E$ by $d^*(a,b)=\max\limits_{t\in[-h,h]}\left|\frac{a(t)}{b(t)}\right|$ and $f: X\to\mathbb{R}$. Clearly (X,d^*) is a complete cone multiplicative metric space. We can write (3) as,

$$x(t)=y_0\int_1^t F(s,x(s))^{ds}.$$

A mapping is defined as $T : C([h,h]) \to \mathbb{R}$ by $Ta(t) = b0 \int_1^t F(s,a(s))^{ds}$.

If $a(t), b(t) \in B(\xi,\delta) = \{\phi(t) \in C^*([-h,h]) : d(\phi,\xi) \leq \delta\}$ then from

$$\begin{aligned}
d^*(Ta,Tb) &= \max_{-h\leq t\leq h} \left|\frac{Ta}{Tb}\right| \leq \max_{-h\leq t\leq h} \int_1^t \left[\left|\frac{F(s,a(s))}{F(s,b(s))}\right|\right]^{ds} \\
&\leq \max_{-h\leq t\leq h} \int_1^t \left[L^{|\frac{a(s)}{b(s)}|}\right]^{ds} \\
&\leq \max_{-h\leq t\leq h} \left(\int_1^t 1^{ds}\right)^{L^{d(a,b)}} \\
&= \max_{-h\leq t\leq h} (|t-1|)^{L^{d(a,b)}} \\
&\leq (K^L)^{d(a,b)} \leq [d(a,b)]^{\lambda}
\end{aligned}$$

and

$$\begin{aligned}
d^*(Ta,b_0) &= \max_{-h\leq t\leq h} \left|\frac{Ta}{b_0}\right| = \max_{-h\leq t\leq h} \int_1^t F(s,a(s))^{ds} \\
&\leq \max_{-h\leq t\leq h} |F(s,a(s))| \leq \delta.
\end{aligned}$$

we calculate $T : B(\xi,\delta) \to B(\xi,\delta)$ is a contractive mapping. At last, we will show that $(B(\xi,\delta),d)$ is complete, because $\{a_n\}$ is Cauchy sequence in $B(\xi,\delta)$, for all $a \in X$ such that $d^*(a_n,a) \ll C$. Thus

$$d^*(\xi,a) \leq d^*(a_n,\delta).d(a_n,\delta) \leq \delta.C$$

Hence $d^*(\xi,a) \leq \delta$, which means that $a \in B(\xi,\delta)$, that is $(B(\xi,\delta),d)$ is complete. Thus the fixed point of T is unique i.e., $a(t) \in B(\xi,\delta)$. Thus, we conclude that, there exist a unique solution of (3).

4. Weakly Compatible Maps

In this section, we establish a theorem postulating a unique common fixed point for four self maps through weak compatibility satisfying a more generalized contractive condition in a non-normal cone multiplicative metric space.

Definition 4.1. ([1]) Let f and g be self maps of a set X. If $w = fx = gx$, for some $x \in X$, then w is called a point of coincidence of f and g.

Definition 4.2. ([11]) Let X be any set. A pair of self-maps (f,g) is said to be weakly compatible if $u \in X$ and $fu = gu$ imply $gfu = fgu$.

Proposition 4.3.([1]) *Let (f,g) be a pair of weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Theorem 4.4. *Let (X,d) be a complete cone multiplicative metric space with respect to a cone P contained in a positive real Banach space E. Suppose that A,B,S and T are self mappings $X \to X$ satisfying:*

(C1) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$;
(C2) the pairs (A,S) and the (B,T) are weakly compatible;
(C3) One of $A(X), S(X), B(X), T(X)$ is complete;
(C4) for some $\lambda, \mu, \delta, \gamma \in [0,1)$ with $\lambda + \mu + \delta + 2\gamma < 1$ we have

$$d(Ax,By) \leq [d(Ax,Sx)]^{\lambda}.[d(By,Ty)]^{\mu}[d(Sx,Ty)]^{\delta}[d(Ax,Ty).d(Sx,By)]^{\gamma}$$

for all $x,y \in X$.

Then A,B,S and T have a unique common fixed point.

Proof. Let x_0 be any point in X. Using (C4) we construct sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$y_{2n} = Ax_{2n} = Tx_{2n+1}, \qquad y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}, \tag{4}$$

for all $n \in \mathbb{N}$. Our aim is to show that $\{y_n\}$ is a multiplicative Cauchy in X.

Step1. Taking $x = x_{2n}$ and $y = x_{2n+1}$ in (C4) we get

$$d(Ax_{2n},Bx_{2n+1}) \leq [d(Ax_{2n},Sx_{2n})]^{\lambda}.[d(Bx_{2n+1},Tx_{2n+1})]^{\mu}.[d(Sx_{2n},Tx_{2n+1})]^{\delta}.$$
$$[d(Ax_{2n},Tx_{2n+1}).d(Sx_{2n},Bx_{2n+1})]^{\gamma}.$$

Using (4) we get

$$d(y_{2n},y_{2n+1}) \leq [d(y_{2n},y_{2n-1})]^{\lambda}.[d(y_{2n+1},y_{2n}]^{\mu}.[d(y_{2n-1},y_{2n})]^{\delta}.$$
$$[d(y_{2n},y_{2n}).d(y_{2n-1},y_{2n+1})]^{\gamma}$$
$$\leq [d(y_{2n},y_{2n-1})]^{\lambda}.[d(y_{2n+1},y_{2n}]^{\mu}.[d(y_{2n-1},y_{2n})]^{\delta}.$$
$$[d(y_{2n-1},y_{2n}).d(y_{2n},y_{2n+1})]^{\gamma}.$$

By writing $d(y_n,y_{n+1}) = d_n$ we have

$$d_{2n} \leq [d_{2n-1}]^{\lambda}.[d_{2n}]^{\mu}[d_{2n-1}]^{\delta}[d_{2n}.d_{2n-1}]^{\gamma}$$

i.e. $[d_{2n}]^{1-\mu-\gamma} \leq [d_{2n-1}]^{\lambda+\delta+\gamma}$, which implies

$$d_{2n} \leq d_{2n-1}^{h}, \tag{5}$$

where $h = (\lambda+\delta+\gamma)/(1-\mu-\gamma)$. In view of (C4), we deduce that $h < 1$.

Taking $x = x_{2n+2}$ and $y = x_{2n+1}$ in (C4), we get

$$d(Ax_{2n+2},Bx_{2n+1}) \leq [d(Ax_{2n+2},Sx_{2n+2})]^{\lambda}.[d(Bx_{2n+1},Tx_{2n+1})]^{\mu}$$
$$[d(Sx_{2n+2},Tx_{2n+1})]^{\delta}[d(Ax_{2n+2},Tx_{2n+1}).d(Sx_{2n+2},Bx_{2n+1})]^{\gamma}$$

Using (4) we get

$$d(y_{2n+2},y_{2n+1}) \leq [d(y_{2n+2},y_{2n+1})]^{\lambda}.[d(y_{2n+1},y_{2n})]^{\mu}$$
$$[d(y_{2n+1},y_{2n})]^{\delta}[d(y_{2n+2},y_{2n}).d(y_{2n+1},y_{2n+1})]^{\gamma}$$
$$\leq [d(y_{2n+2},y_{2n+1})]^{\lambda}.[d(y_{2n+1},y_{2n})]^{\mu}$$
$$[d(y_{2n+1},y_{2n})]^{\delta}[d(y_{2n+2},y_{2n+1}).d(y_{2n+1},y_{2n})]^{\gamma}.$$

So, we have

$$d_{2n+1} \leq [d_{2n+1}]^{\lambda}[d_{2n}]^{\mu}[d_{2n}]^{\delta}[d_{2n+1}d_{2n}]^{\gamma}$$

that is, $[d_{2n+1}]^{1-\lambda-\gamma} \leq [d_{2n}]^{\mu+\delta+\gamma}$ which implies

$$d_{2n+1} \leq d_{2n}^{k}, \tag{6}$$

where $k=(\mu+\delta+\gamma)/(1-\lambda-\gamma)$. By condition (C4), we have $k<1$. In view of (5) and (6) we have

$$d_{2n+1}\leq d_{2n}^{k}\leq d_{2n-1}^{kh}\leq d_{2n-2}^{k^2h}\leq\cdots\leq d_0^{k^{n+1}h^n},$$

where $d_0=d(y_0,y_1)$, and

$$d_{2n}\leq d_{2n-1}^{h}\leq d_{2n-2}^{hk}\leq\cdots\leq d_0^{k^nh^n}.$$

Therefore, $d_{2n+1}\leq d_0^{k^{n+1}h^n}$ and $d_{2n}\leq d_0^{k^nh^n}$. Also,

$$d(y_n,y_{n+p})\leq d(y_n,y_{n+1}).d(y_{n+1},y_{n+2})\cdots d(y_{n+p-1},y_{n+p}),$$

that is,

$$d(y_n,y_{n+p})\leq d_n.d_{n+1}\cdots d_{n+p-1}. \tag{7}$$

We next suppose that $n=2m$ is even. By (7) we have

$$\begin{aligned}
d(y_{2m},y_{2m+p}) &\leq d_0^{[h^mk^m+h^mk^{m+1}+h^{m+1}k^{m+1}+h^{m+1}k^{m+2}+\cdots]}\\
&=d_0^{h^mk^m(!=k+kh+hk^2+h^2k^2+\cdots)}\\
&=d_0^{h^mk^m[(1+hk+h^2k^2+\cdots)+(k+hk^2+h^2k^3+\cdots)]}\\
&=d_0^{h^mk^m(1+k)[!=hk+h^2k^2+\cdots]}.
\end{aligned}$$

Since $hk<1$ and P is closed, we conclude that

$$d(y_{2m},y_{2m+p})\leq d_0^{(hk)^m\frac{1+k}{1-kh}}. \tag{8}$$

Now for $c\in intP$, there exists $r>0$ such that $\frac{c}{y}\in intP$ if $\|y\|<r$. Choose a positive integer N_c such that $\left\|d_0^{(hk)^m\frac{1+k}{1-kh}}\right\|<r$ for all $m>N_c$, which implies that

$$\frac{c}{d_0^{(hk)^m\frac{1+k}{1-kh}}}\in intP.$$

On the other hand, (8) means that

$$\frac{d_0^{(hk)^m\frac{1+k}{1-kh}}}{d(y_{2m},y_{2m+p})}\in intP.$$

So we have $\dfrac{c}{d(y_{2m},y_{2m+p})}\in intP$ for all $m>N_c$ and for p. The same argument applies if $n=2m+1$ is odd. Thus $d(y_{n+p},y_n)\ll c$, for all p and for all $n>N_c$. Hence $\{y_n\}$ is a multiplicative

Cauchy in X.

Case 1: $S(X)$ is complete. Since $\{y_n\}$ is a multiplicative Cauchy in X, it follows $y_{2n+1} = Sx_{2n+2}$ is a multiplicative Cauchy in X, which is complete. So $y_{2n+1} \to z = Su$ for some $u \in X$. Now

$$d(Au,Su) \leq d(Au,Bx_{2n+1}).d(Bx_{2n+1},Su)$$
$$= d(y_{2n+1},Su).d(Au,Bx_{2n+1}).$$

Using (C4) with $x = u$ and $y = x_{2n+1}$, we have

$$\begin{aligned} d(Au,Su) \leq\ & d(y_{2n+1},Su).[d(Au,Su)]^{\lambda}.[d(Bx_{2n+1},Tx_{2n+1})]^{\mu}[d(Su,Tx_{2n+1})]^{\delta} \\ & [d(Au,Tx_{2n+1}).d(Su,Bx_{2n+1})]^{\gamma} \\ & = d(y_{2n+1},Su).[d(Au,Su)]^{\lambda}.[d(y_{2n+1},y_{2n})]^{\mu}[d(Su,y_{2n})]^{\delta} \\ & [d(Au,y_{2n}).d(y_{2n+1},Su)]^{\gamma} \\ & \leq d(y_{2n+1},Su).[d(Au,Su)]^{\lambda}.[d(y_{2n+1},Su).d(Su,y_{2n})]^{\mu}[d(Su,y_{2n})]^{\delta} \\ & [d(Au,Su).d(Su,y_{2n}).d(y_{2n+1},Su)]^{\gamma}. \end{aligned}$$

Thus

$$[d(Au,Su)]^{(1-\lambda-\gamma)} \leq [d(y_{2n},Su)]^{(\mu+\delta+\gamma)}.[d(y_{2n+1},Su)]^{(1+\mu+\gamma)}.$$

As $y_{2n} \to Su$, $y_{2n+1} \to Su$ and $1-\lambda-\gamma > 0$, using Lemma 2.18, we have $d(Au,Su) = 1$, and we get $Au = Su$. Thus $Au = Su = z$. Therefore z is a point of coincidence of the pair (A,S). Since (A,S) is weakly compatible, $Az = Sz$.

Step 2. As $A(X) \subseteq T(X)$, there exists $v \in X$ such that $z = Au = Tv$. So

$$z = Au = Su = Tv. \tag{9}$$

Taking $x = u$ and $y = v$ in (C4) we have

$$d(Au,By) \leq [d(Au,Su)]^{\lambda}.[d(Bv,Tv)]^{\mu}[d(Su,Tv)]^{\delta}[d(Au,Tv).d(Su,Bv)]^{\gamma}.$$

Using (9) we have

$$d(z,Bv) \leq [d(z,Bv)]^{\mu+\gamma}.$$

As $\mu+\gamma<1$, using Lemma 2.17, it follows that $d(Bv,z)=1$ and we get $Bv=z$. As the pair (B,T) is weak compatible we get $Bz=Tz$. Taking $x=z$, $y=z$ in (C4) and using $Az=Sz, Bz=Tz$ we get

$$d(Az,Bz)\leq[d(Az,Bz)]^{\delta+2\gamma}.$$

As $0\leq\delta+2\gamma<1$ we get $Az=Bz$, by Lemma 2.17 and we have $Az=Sz=Bz=Tz$. Thus z is a point of coincidence of the four self maps A,B,S,T.

Case 2. $T(X)$ is complete. The proof of this case is similar to Case 1.

Case 3. $A(X)$ is complete. $\{y_n\}$ is a multiplicative Cauchy sequence in X. Hence $\{y_{2n}=Ax_{2n}\}$ is a Cauchy sequence in $A(X)$, which is complete. Hence $y_{2n}\to z=Aw$ for some $w\in X$. As $A(X)\subseteq T(X)$ there exists $p\in X$ such that $z=Aw=Tp$. It follows from Case 2 that $Az=Bz=Sz=Tz$. Thus, also in this case, the maps A,B,S,T have a common point of coincidence.

Case 4. $B(X)$ is complete. The proof of this case is similar to Case 3.

Step 3. We have $z=Bz=Sz$. Let $Au=Su$ be another point of coincidence of the pair (A,S). Now

$$\begin{aligned}d(z,Au)&\leq d(z,Bx_{2n+1}).d(Bx_{2n+1},Au)\\&=d(z,y_{2n+1}).d(u,Bx_{2n+1}).\end{aligned}$$

Taking $x=u$ and $y=x_{2n+1}$ in (C4), we get

$$\begin{aligned}d(z,Au)\leq{}& d(z,y_{2n+1}).[d(Au,Su)]^{\lambda}.[d(Bx_{2n+1},Tx_{2n+1})]^{\mu}[d(Su,Tx_{2n+1})]^{\delta}\\&[d(Au,Tx_{2n+1}).d(Su,Bx_{2n+1})]^{\gamma}\\&=d(z,y_{2n+1}).[d(y_{2n+1},y_{2n})]^{\mu}[d(Au,y_{2n})]^{\delta}\\&[d(Au,y_{2n}).d(y_{2n+1},Au)]^{\gamma}\\&\leq d(z,y_{2n+1}).[d(y_{2n+1},z).d(z,y_{2n})]^{\mu}[d(Au,z).d(z,y_{2n}]^{\delta}\\&[d(Au,z).d(z,y_{2n}).d(y_{2n+1},z).d(z,Au)]^{\gamma}.\end{aligned}$$

Thus

$$[d(z,Au)]^{(1-\delta-2\gamma)}\leq[d(z,y_{2n+1})]^{(1+\mu+\gamma)}.[d(z,y_{2n})]^{(\mu+\delta+\gamma)}.$$

Since $y_{2n+1} \to z$ and $y_{2n} \to z$, and we have $1-\delta-2\gamma > 0$, using Lemma 2.18 we obtain $d(z,Au) = 1$ and so $Au = z$. Hence the point of coincidence of (A,S) is unique. As the pair (A,S) is weakly compatible by Proposition 4.3, z is the unique common fixed point of A and S. Hence $z = Az = Bz = Sz = Tz$ is the unique fixed point of A,B,S and T.

Taking and $T = S$ in Theorem 4.4 we have the following corollary for three self mappings:

Corollary 4.5. *Let (X,d) be a complete cone multiplicative metric space with respect to a cone P contained in a positive real Banach space E. Let A,B and S be self mappings on X satisfying:*

(i) $A(X) \cap B(X) \subseteq S(X)$*;*

(ii) the pairs (A,S) and (B,S) are weakly compatible;

(iii) one of $S(X)$ or $A(X) \cup B(X)$ is complete;

(iv) for some $\lambda,\mu,\delta,\gamma \in [0,1)$ with $\lambda+\mu+\delta+2\gamma < 1$, we have

$$d(Ax,By) \leq [d(Ax,Sx)]^{\lambda}.[d(By,Sy)]^{\mu}.[d(Sx,Sy)]^{\delta}.[d(Ax,Sy).d(By,Sy)]^{\gamma},$$

for all $x,y \in X$.

Then A,B and S have a unique common fixed point in X.

Taking $B = A$ and $T = S$ in Theorem 4.4, we obtain

Corollary 4.6. *Let (X,d) be a complete cone multiplicative metric space with respect to a cone P contained in a positive real Banach space E. Let A and S be self mappings on X satisfying:*

(i) $A(X) \subseteq S(X)$*;*

(ii) the pair (A,S) is weakly compatible;

(iii) one of $S(X)$ or $A(X)$ is complete;

(iv) for some $\lambda,\mu,\delta,\gamma \in [0,1)$ with $\lambda+\mu+\delta+2\gamma < 1$, we have

$$d(Ax,Ay) \leq [d(Ax,Sx)]^{\lambda}.[d(Ay,Sy)]^{\mu}.[d(Sx,Sy)]^{\delta}.[d(Ax,Sy).d(Sx,Ay)]^{\gamma},$$

for all $x,y \in X$.

Then A,B and S have a unique common fixed point in X.

Taking $S = I$, the identity map on X, in Corollary 4.6 we have

Corollary 4.7. *Let (X,d) be a complete cone multiplicative metric space. Let A be self mapping on X satisfying the following:*

(C) for some $\lambda,\mu,\delta,\gamma \in [0,1)$ with $\lambda+\mu+\delta+2\gamma < 1$, we have

$$d(Ax,Ay) \leq [d(Ax,x)]^{\lambda}.[d(Ay,y)]^{\mu}.[d(x,y)]^{\delta}.[d(Ax,y).d(x,Ay)]^{\gamma},$$

for all $x,y \in X$.

Then the map A has the unique fixed point in X and for any $x \in X$, the iterative sequence $\{A^n x\}$ converges to the fixed point.

Proof. Existence and uniqueness of the fixed point follows from Corollary 4.6, by taking $S = I$ there. Taking $T = S = I$, $B = A$ and $x_0 = x$ in Theorem 4.4 we have $y_0 = Ax$, $y_1 = A^2x, \cdots y_{n+1} = A^n x$, etc. Thus for each x, the sequence $\{A^n x\}$ converges to the fixed point z.

Taking $\gamma = 0$ in Corollary 4.7 we have

Corollary 4.8. *Let (X,d) be a complete cone multiplicative metric space. Let A be self mapping on X satisfying the following:*

(C) for some $\lambda,\mu,\delta \in [0,1)$ with $\lambda+\mu+\delta < 1$, we have

$$d(Ax,Ay) \leq [d(Ax,x)]^{\lambda}.[d(Ay,y)]^{\mu}.[d(x,y)]^{\delta},$$

for all $x,y \in X$.

Then the map A has the unique fixed point in X and for any $x \in X$, the iterative sequence $\{A^n x\}$ converges to the fixed point.

The following example illustrates the main theorem of this section.

Example 4.9 Let $X = \mathbb{R}_+$, $E = \mathbb{R}_+^2$, and consider the cone

$$P = \{(x,y) \in \mathbb{R}_+^2 : x \geq 1, y \geq 1\} \subset E.$$

Fix a real number $\alpha > 0$ and define a cone multiplicative metric $d : X \times X \to E$ by

$$d(x,y) = \left(\left| \frac{x}{y} \right|, \left| \frac{x}{y} \right|^{\alpha} \right),$$

where $|.|$ is defined by Equation (1). Then (X,d) is a complete cone multiplicative metric space. Define self maps A,B,S and T on X by

$$Ax = Bx = x^{\frac{2}{3}} \text{ and } Sx = Bx = x^2,$$

for all $x \in \mathbb{R}_+$. Conditions (C1),(C2),(C3) of Theorem 4.4 hold trivially. Condition (C4) is equivalent to

$$\left| \frac{x}{y} \right| \leq x^{2\lambda} y^{2\mu} \left| \frac{x}{y} \right|^{3\delta} \left| \frac{x}{y^3} \right|^{\gamma} \left| \frac{x^3}{y} \right|. \tag{10}$$

Consider it in the following four cases:

(a) $y \leq x \leq y^3$, in which (10) becomes

$$x^{(2\lambda+3\mu+4\gamma-1)} y^{(2\mu-3\delta+2\gamma+1)} \geq 1.$$

(b) $y^3 \leq x$, in which (10) becomes

$$x^{(2\lambda+3\delta+4\gamma-1)} y^{(2\mu-3\delta+4\gamma+1)} \geq 1.$$

(c) $x^3 \leq y$, in which (10) becomes

$$x^{(2\lambda+3\delta+4\gamma-1)} y^{(2\mu+3\delta+4\gamma-1)} \geq 1.$$

(d) $x \leq y \leq x^3$, in which (10) becomes

$$x^{(2\lambda-3\delta+2\gamma-1)} y^{(2\mu+3\delta+2\gamma-1)} \geq 1.$$

Thus condition (C4) also holds good for $\lambda = \frac{1}{4}$, $\mu = \gamma = \frac{1}{5}$ and $\delta = \frac{1}{15}$ and 1 is the unique common fixed point of the maps A,B,S and T.

5. Conclusion

In this manuscript, we introduces for the first time the notion of cone multiplicative metric space and presented some fixed point theorems in setting of cone multiplicative metric space. We also give an application to support our main result.

Conflict of Interests

The authors declare that there is no conflict of interests.

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