Available online at http://jfpt.scik.org
J. Fixed Point Theory, 2018, 2018:9
ISSN: 2052-5338

SOME FIXED POINT THEOREMS OF INVERSE C− CLASS FUNCTION UNDER WEAK SEMI COMPATIBILITY

NAEEM SALEEM[1], ARSLAN HOJAT ANSARI[2], MUKESH KUMAR JAIN[3,*]

[1]Department of Mathematics, University of Management and Technology, C-II Johar Town, Lahore - Pakistan

[2]Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran

[3]J. N. V. Udalguri (BTAD) Assam, India

Abstract. In the present paper, we extend the work of recent notion C-class function by introducing a new notion inverse C-class function and obtain some fixed point theorems under weak semi compatibility in metric space. We also have proved some corollaries in supports of proved theorems.

Keywords: common fixed point; compatible mappings of type (E); inverse C-class function; E.A. property; coincidence point; weak semi compatibility; expansion mappings.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction and preliminaries

In the year of 1922, the important and very fruitful concepts of contraction mapping was given by Banach [1]. In 1968, Kannan [2] proved some fixed point theorems for a map satisfying contraction condition that did not require continuity at each point. In 1976, Jungck [3] generalized

*Corresponding author

E-mail address: naeem.saleem2@gmail.com

Received January 8, 2018

the Banach contraction principle by introducing the concept of commuting mappings and put the open problem for scholars for next some decades. Since then many fixed point theorems were proved and found many weaker forms of commutativity and continuity that contains the existence of common fixed points. In 1982, Sessa [4] introduced the notion of weak commutativity and obtained some common fixed point theorems of mappings. Jungck [5, 20] generalized the notion of weak commutativity by introducing compatible mappings and then weakly compatible mappings. Since then it carried a lot of unique common fixed point theorems of pair of maps. Possibly the first common fixed point theorem, without continuity conditions was proved by Pant [6, 7] by introducing reciprocal continuity. Recently, Pant et al. [9], Pant and Bisht [10] and Bisht and shahzad [11] generalized the notion of reciprocal continuity by introducing weak reciprocal continuity, conditionally reciprocal continuity and faintly compatible mappings respectively and obtained some fixed point theorems. In 2008, Al thagafi et al. [15] introduced the weaker form of weak compatible mapping by occasionally weakly compatible mappings (owc). One another generalization of notion of compatible mappings (says semi compatiable mappings) is introduced by Singh et al. [8]. They proved that the concept of semi compatible mappings is equivalent to concept of compatible mappings under some conditions of mappings. Recently Salooja et al. [12, 16] generalized the notion of semi compatibility by introducing weak semi compatible mapping and conditional semi compatible mapping and obtained some fixed point theorems by using these notions (for further details see Saluja et. al. [13]). They also proved that one of these notions named conditional semi compatible mapping is necessary condition for existence of common fixed point. In 2014, the concept of $C-$class function was introduced by Ansari [14]. In this note we shall prove some fixed point theorems by using the new notion of inverse $C-$class function under weak semi compatible mapping.

Next, we discuss some relevant definitions and results.

Definition 1.1. [5] - *Two self maps f and g of a metric space (X,d) are called compatible if* $\lim d(fgx_n, gfx_n) = 0$*, whenever x_n is a sequence in X such that* $\lim fx_n = \lim gx_n = t$ *for some t in X.*

Definition 1.2. [8] - *Two self maps f and g of a metric space (X,d) are called semi compatible if* $\lim fgx_n = gx$ *holds when* $\lim fx_n = \lim gx_n = x$ *for some $x \in X$.*

Definition 1.3. [17] - *Two self maps f and g of a metric space (X,d) are called R- weak commuting of type A_g if there exist some positive real number R such that*

$$d(gfx, ffx) \leq Rd(fx, gx)$$

for all x in X.

Definition 1.4. [17] - *Two self maps f and g of a metric space (X,d) are called R- weak commuting of type A_f if there exist some positive real number R such that*

$$d(fgx, ggx) \leq Rd(fx, gx)$$

for all x in X.

Definition 1.5. - *Let X be any set. If f and g are self maps of X. A point x in X is called coincidence point of f and g if and only if $fx = gx$.*

Definition 1.6. [15] - *A pair (f,g) of self mappings define on a nonempty set X is said to be occasionally weakly compatible mappings (in short owc) if there exists a point x in X , which is coincidence point of f and g at which f and g commute.*

Definition 1.7. - *Let X be a set. A symmetric on X is a mapping $d : X \times X \to [0,\infty)$ such that*

$d(x,y) = 0$ iff $x = y$ and $d(x,y) = d(y,x)$ for all $x,y \in X$.

Definition 1.8. [18] - *Two self maps f and g of metric space (X,d) are said to be $f-$compatible of type (E) if $\lim ffx_n = \lim fgx_n = gt$, whenever $\{x_n\}$ is a sequence in X such that $\lim fx_n = \lim gx_n = t$ for some t in X.*

Definition 1.9. [18] - *Two self maps f and g of metric space (X,d) are said to be $g-$compatible of type (E) if $\lim ggx_n = \lim gfx_n = ft$, whenever $\{x_n\}$ is a sequence in X such that $\lim fx_n = \lim gx_n = t$ for some t in X.*

Definition 1.10. [19] - *Let f and g are two self mappings of metric space (X,d) . Mappings f and g satisfy E.A. property if there exists a sequence $\{x_n\}$ in X such that $\lim fx_n = \lim gx_n = t$ for some $t \in X$.*

Definition 1.11. [12] - *Two self maps f and g of a metric space (X,d) are called weak semi compatible mappings if* $\lim fgx_n = gt$, *or* $\lim gfx_n = ft$, *whenever* $\{x_n\}$ *is a sequence in X such that* $\lim fx_n = \lim gx_n = t$ *for some t in X.*

In 2014 the concept of C-class functions was introduced by A. H. Ansari [14], defined as:

Definition 1.12. [14] *A mapping $f : [0,\infty)^2 \to \mathbb{R}$ is called C-class function if it is continuous and satisfies following axioms:*

(1): $f(s,t) \leq s$;

(2): $f(s,t) = s$ implies that either $s = 0$ or $t = 0$; for all $s,t \in [0,\infty)$.

Note for some F we have that $F(0,0) = 0$.

We denote all collections of $C-$class functions as $\mathscr{C}$.

Example 1.13. [14] *The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathscr{C}$, for all $s,t \in [0,\infty)$:*

- $F(s,t) = s - t$, $F(s,t) = s \Rightarrow t = 0$;
- $F(s,t) = ms$, $0<m<1$, $F(s,t) = s \Rightarrow s = 0$;
- $F(s,t) = \frac{s}{(1+t)^r}$; $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$ *or* $t = 0$;
- $F(s,t) = \log(t + a^s)/(1+t)$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ *or* $t = 0$;
- $F(s,t) = \ln(1 + a^s)/2$, $a > e$, $F(s,1) = s \Rightarrow s = 0$;
- $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1, r \in (0,\infty)$, $F(s,t) = s \Rightarrow t = 0$;
- $F(s,t) = s\log_{t+a} a$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ *or* $t = 0$;
- $F(s,t) = s\beta(s)$, $\beta : [0,\infty) \to [0,1)$ *and is continuous,* $F(s,t) = s \Rightarrow s = 0$;
- $f(s,t) = \vartheta(s)$; $\vartheta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is a generalized Mizoguchi-Takahashi type function,* $f(s,t) = s \Rightarrow \quad s = 0$;
- $f(s,t) = \frac{s}{\Gamma(1/2)} \int_0^\infty \frac{e^{-x}}{\sqrt{x}+t} dx$, *where Γ is the Euler Gamma function.*

After the motivation of $C-$Functions we define the following:

Definition 1.14. *A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called inverse-C-class function if it is continuous and satisfies following axioms:*

(1): $F(s,t) \geq s$;

(2): $F(s,t) = s$ *implies that either* $s = 0$ *or* $t = 0$; *for all* $s,t \in [0,\infty)$.

Note that for some F we have that $F(0,0) = 0$.

We denote collection of all inverse $C-$class functions as $\mathscr{C}_{inv}$.

Example 1.15. *The following functions* $F : [0,\infty)^2 \to \mathbb{R}$ *are elements of* $\mathscr{C}_{inv}$, *for all* $s,t \in [0,\infty)$:

(1) $F(s,t) = s+t$, $F(s,t) = s \Rightarrow t = 0$;

(2) $F(s,t) = ms$, $1<m<\infty$, $F(s,t) = s \Rightarrow s = 0$;

(3) $F(s,t) = s(1+t)^r$; $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$ *or* $t = 0$;

(4) $F(s,t) = \log_a(t+a^s)(1+t)$, $a > 1$, $F(s,t) = s \Rightarrow t = 0$;

(5) $F(s,t) = \phi(s), F(s,t) = s \Rightarrow s = 0$,*here* $\phi : [0,\infty) \to [0,\infty)$ *is a upper semicontinuous function such that* $\phi(0) = 0$, *and* $\phi(t) > t$ *for* $t > 0$,

(6) $f(s,t) = \vartheta(s)$; $\vartheta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ *is a generalized Mizoguchi-Takahashi type function,* $f(s,t) = s \Rightarrow \quad s = 0$;

For further details readers are referred to [14].

We will use the following control functions, defined as:

Let Φ denote the set of all functions $\varphi : [0,+\infty) \to [0,+\infty)$ that satisfy the following conditions:

(1) φ is lower semi-continuous on $[0,+\infty)$,

(2) $\varphi(0) = 0$,

(3) $\varphi(s) > 0$ for each $s > 0$.

Let Φ_1 denote the set of all functions $\varphi : [0,+\infty) \to [0,+\infty)$ that satisfy the following conditions:

(1) φ is lower semi-continuous on $[0,+\infty)$,

(2) $\varphi(0) \geq 0$,

(3) $\varphi(s) > 0$ for each $s > 0$

Let Ψ denote all the functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy:

(1) $\psi(t) = 0$ if and only if $t = 0$,

(2) ψ is continuous and increasing.

We use the following notations:

$$\min_0\{0, d(fx, gx), d(fy, gy)\} = \min\{d(fx, gx), d(fy, gy)\}.$$

For example:

$$\begin{aligned}\min_0\{0,1,2,3,4\} &= \min\{1,2,3,4\} = 1,\\ \min_0\{1,2,3,4\} &= \min\{1,2,3,4\}\end{aligned}$$

We need the following lemma in sequel.

Lemma 1.16. [21] *Suppose (X,d) be a metric space and $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m_{(k)}\}$ and $\{n_{(k)}\}$ with*

$$m_{(k)} > n_{(k)} > k \text{ such that } d(x_{m_{(k)}}, x_{n_{(k)}}) \geq \varepsilon, d(x_{m_{(k)}-1}, x_{n_{(k)}}) < \varepsilon$$

and

(i): $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \varepsilon$;

(ii): $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon$;

(iii): $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \varepsilon$.

We note that

$$\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \varepsilon \text{ and } \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)-1}) = \varepsilon.$$

Theorem 1.17. *Let X be a set with a symmetric d. Suppose that f and g are owc self maps of X satisfying:*

$$\psi(d(fx, fy)) \geq F(\psi(m(x,y)), \varphi(m(x,y))), \tag{1}$$

where $m(x,y) = \min_0\{d(gx, gy), d(fx, gx), d(fy, gy)\}$. Then f and g have common fixed point in X.

Proof. Since the mappings f and g are owc, then there exist a point $x \in X$ such that $fx = gx$ and $fgx = gfx$. Now by 1, we have

$$\begin{aligned}
& \psi(d(fx, fgx)) \\
\geq\ & F(\psi(\min_0\{d(gx, ggx), d(fx, gx), d(fgx, ggx)\}), \\
& \varphi(\min_0\{d(gx, ggx), d(fx, gx), d(fgx, ggx)\})), \\
\psi(d(fx, gfx)) \geq\ & F(\psi(\min_0\{d(fx, gfx), 0, 0\}), \varphi(\min_0\{d(fx, gfx), 0, 0\})) \\
\geq\ & F(\psi(\min\{d(fx, gfx\})), \varphi(\min\{d(fx, gfx)\})) \\
\geq\ & F(\psi(d(fx, gfx\}), \varphi(d(fx, gfx))) \geq \psi(d(fx, gfx))
\end{aligned}$$

This implies $\psi(d(fx, gfx)) = 0$ or $\varphi(d(fx, gfx)) = 0$. This yields $gfx = fx$. Therefore $ffx = fx$ and hence fx is a common fixed point of f and g. □

2. Main results

Theorem 2.1. *Let f and g are weak semi compatible $R-$weakly commuting type of A_f self mappings of a complete metric space (X, d) satisfying the following;*

- **a):** $f(X) \subseteq g(X)$;
- **b):** $\psi(d(fx, fy)) \geq F(\psi(m(x, y)), \varphi(m(x, y)))$;
  where $m(x, y) = \min_0\{d(gx, gy), d(fx, gx), d(fy, gy)\}$;
- **c):** *f and g are either f compatible of type (E) or $g-$compatible of type (E),*

where $\psi \in \Psi$ and $\varphi \in \Phi$.

Then f and g have a common fixed point in X .

Proof. Let x_0 be any point in X. Since $f(X) \subseteq g(X)$ there exist $x_1 \in X$ such that

$$fx_1 = gx_0 = y_0.$$

Similarly we can have a sequence satisfying

$$fx_{n+1=}gx_n = y_n$$

Now by condition (b)

$$\begin{aligned}\psi(d((fx_n, fx_{n+1})) &\geq F(\psi(\min_0\{d(gx_n, gx_{n+1}), d(fx_n, gx_n), d(fx_{n+1}, gx_{n+1})\}),\\ &\quad \varphi(\min_0\{d(gx_n, gx_{n+1}), d(fx_n, gx_n), d(fx_{n+1}, gx_{n+1})\}))\end{aligned}$$

Since $fx_n = y_{n-1}$ and $fx_{n+1} = y_n$, then the above inequality can be written as:

$$\begin{aligned}\psi(d((y_{n-1}, y_n)) &\geq F(\psi(\min_0\{d(y_n, y_{n+1}), d(y_{n-1}, y_n), d(y_n, y_{n+1})\}),\\ &\quad \varphi(\min_0\{d(y_n, y_{n+1}), d(y_{n-1}, y_n), d(y_n, y_{n+1})\}))\\ &= F(\psi(\min_0\{d(y_n, y_{n+1}), d(y_{n-1}, y_n)\}), \varphi(\min_0\{d(y_n, y_{n+1}), d(y_{n-1}, y_n)\}))\\ &\geq \psi(\min_0\{d(y_n, y_{n+1}), d(y_{n-1}, y_n)\}) \qquad (2a)\end{aligned}$$

Now suppose that

$$0 \leq r = d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n)$$

then

$$\min_0\{d(y_n, y_{n+1}), d(y_{n-1}, y_n)\} = d(y_n, y_{n+1}),$$

taking $n \to \infty$,in inequality 2a, using the continuity of all the functions involved in inequality, which becomes

$$\psi(r) \geq F(\psi(r), \varphi(r)) \geq \psi(r). \qquad (3)$$

Implies

$$F(\psi(r), \varphi(r)) = \psi(r).$$

Implies that either $\psi(r) = 0$ or $\varphi(r) = 0$, hence $r = 0$, in other words

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \qquad (4)$$

Now, we have to show that $\{y_n\}$ is a Cauchy sequence, on contrary that sequence $\{y_n\}$ is not a Cauchy sequence, then from lemma 1.16 there exists $\varepsilon>0$ such that we can find subsequences

$\{x_{n_k}\}$and $\{x_{m_k}\}$ of $\{x_n\}$ with $n_k > m_k > k$ such that

$$\varepsilon = \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) \tag{5}$$
$$= \lim_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) \tag{6}$$
$$= \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) \tag{7}$$
$$= \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}), \tag{8}$$

then sequence $\{x_n\}$ for all $n_k > m_k > k$, the inequality 3 becomes

$$\psi(\varepsilon) \geq F(\psi(\varepsilon), \varphi(\varepsilon)) \geq \psi(\varepsilon).$$

Implies either $\psi(\varepsilon) = 0$ or $\varphi(\varepsilon) = 0$, therefore $\varepsilon = 0$. Which is a contradiction. Therefore $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exist a point t in X such that $\{y_n\} \to t$ as $n \to \infty$ moreover

$$y_n = fx_{n+1} = gx_n \to t \text{ for } t \in X.$$

Now we have following possibilities:

Case-1: f and g are f -compatible of type (E). Since f and g are weak semi compatible, yields

$$\lim fgx_n = gt \text{ or } \lim gfx_n = ft.$$

Firstly, take

$$\lim gfx_n = ft.$$

Since f and g are f-compatible of type (E), yields

$$\lim ffx_n = \lim fgx_n = gt.$$

Now by (b)

$$\begin{aligned}\psi(d(ffx_n, ft)) \geq\ & F(\psi(\min_0\{d(gfx_n, gt), d(ffx_n, gfx_n), d(ft, gt)\}),\\ & \varphi(\min_0\{d(gfx_n, gt), d(ffx_n, gfx_n), d(ft, gt)\})).\end{aligned}$$

Now apply limit $n \to \infty$ on above inequality, yields

$$\begin{aligned}
\psi(d(gt,ft)) &\geq F(\psi(\min_0\{d(ft,gt),d(gt,ft),d(ft,gt)\}),\\
&\quad \varphi(\min_0\{d(ft,gt),d(gt,ft),d(ft,gt)\}))\\
&= F(\psi(d(ft,gt)),\varphi(d(ft,gt)))\\
&\geq \psi(d(ft,gt)).
\end{aligned}$$

Hence

$$\psi(d(gt,ft)) \geq F(\psi(d(ft,gt)),\varphi(d(ft,gt))) \geq \psi(d(ft,gt)),$$

implies

$$F(\psi(d(ft,gt)),\varphi(d(ft,gt))) = \psi(d(ft,gt)).$$

Which further implies that

$$\psi(d(ft,gt)) = 0 \text{ or } \varphi(d(ft,gt)) = 0,$$

in either case, we have $ft = gt$.

Since f and g are $R-$weak commuting of type A_f, we have

$$d(fgt,ggt) \leq Rd(ft,gt),$$

yields

$$fgt = gft \text{ or } fgt = gft = fft = ggt.$$

Now form (b)

$$\begin{aligned}
\psi(d(fft,ft)) &\geq F(\psi(\min_0\{d(gft,gt),d(fft,gft),d(ft,gt)\}),\\
&\quad \varphi(\min_0\{d(gft,gt),d(fft,gft),d(ft,gt)\}))\\
&= F(\psi(d(fft,ft),\varphi(d(fft,ft)).\\
&\geq \psi(d(fft,ft)).
\end{aligned}$$

With same arguments used above, we have

$$\psi(d(fft,ft) = 0 \text{ or } \varphi(d(fft,ft) = 0$$

Implies

$$fft = ft \text{ or } gt = fft = gft = ft.$$

Hence ft is common fixed point of f and g.

Now, suppose that we have

$$\lim fgx_n = gt.$$

Since f and g are $f-$compatible of type (E), this yields,

$$\lim ffx_n = \lim fgx_n = gt.$$

Also f and g are $R-$weakly commuting type A_f, implies

$$d(fgx_n, ggx_n) \leq Rd(fx_n, gx_n).$$

Now, apply limit $n \to \infty$, then we have

$$\lim ggx_n = gt.$$

From condition (b), we have

$$\begin{aligned} \psi(d(fx_n, fgx_n)) \quad \geq \quad & F(\psi(\min_0\{d(gx_n, ggx_n), d(fx_n, gx_n), d(fgx_n, ggx_n)\}), \\ & \varphi(\min_0\{d(gx_n, ggx_n), d(fx_n, gx_n), d(fgx_n, ggx_n)\})). \end{aligned}$$

Now, apply limit $n \to \infty$, then the above inequality becomes

$$\begin{aligned} \psi(d(t, gt)) \quad \geq \quad & F(\psi(\min_0\{d(t, gt), d(t,t), d(gt, gt)\}), \\ & \varphi(\min_0\{d(t, gt), d(t,t), d(gt, gt)\})) \\ = \quad & F(\psi(d(t, gt)), \varphi(d(t, gt)) \geq \psi(d(t, gt)). \end{aligned}$$

Implies, either

$$\psi(d(t, gt) = 0 \text{ or } \varphi(d(t, gt) = 0,$$

which further implies that

$$gt = t.$$

Now from (b), we have

$$\begin{aligned}\psi(d(fx_n,ft)) &\geq F(\psi(\min_0\{d(gx_n,gt),d(fx_n,gx_n),d(ft,gt)\}),\\ &\quad \varphi(\min_0\{d(gx_n,gt),d(fx_n,gx_n),d(ft,gt)\})),\end{aligned}$$

take limit $n \to \infty$, then above inequality becomes

$$\begin{aligned}\psi(d(t,ft)) &\geq F(\psi(\min_0\{d(t,t),d(t,t),d(ft,t)\}),\\ &\quad \varphi(\min_0\{d(t,t),d(t,t),d(ft,t)\})\\ &= F(\psi(d(ft,t)),\varphi(d(ft,t)) \geq \psi(d(ft,t)).\end{aligned}$$

Then either

$$\psi(d(ft,t)) = 0 \text{ or } \varphi(d(ft,t)) = 0,$$

yields $ft = t$ and hence $ft = gt = t$ implies that t is common fixed point of f and g.

Now, we consider the second possibility as:

Case 2- Let f and g are g-compatible of type (E). Since f and g are weak semi compatible, this yields either

$$\lim fgx_n = gt \text{ or } \lim gfx_n = ft.$$

First we take

$$\lim gfx_n = ft.$$

Since f and g are g-compatible of type (E), yields

$$\lim ggx_n = \lim gfx_n = ft.$$

Also f and g are $R-$weakly commuting type of A_f, implies

$$d(fgx_{n,}ggx_n) \leq Rd(fx_n,gx_n).$$

Now, apply limit $n \to \infty$, We have

$$\lim fgx_n = ft.$$

From the condition (b), we have

$$\begin{aligned}\psi(d(fgx_n,ft)) &\geq F(\psi(\min_0\{d(ggx_n,gt),d(fgx_n,ggx_n),d(ft,gt)\}),\\ &\quad \varphi(\min_0\{d(ggx_n,gt),d(fgx_n,ggx_n),d(ft,gt)\})).\end{aligned}$$

Now, apply limit $n\to\infty$ we have

$$\begin{aligned}0 &= \psi(d(ft,ft)) \geq F(\min_0\{d(ft,gt),d(ft,ft),d(ft,gt)\}),\\ &\quad \varphi(\min_0\{d(ft,gt),d(ft,ft),d(ft,gt)\})),\\ &= F(\psi(d(ft,gt)),\varphi(d(ft,gt))) \geq \psi(d(ft,gt))\end{aligned}$$

As a result, we have, either

$$\psi(d(ft,gt)) \text{ or } \varphi(d(ft,gt)) = 0,$$

this yields

$$ft = gt.$$

Since f and g are $R-$weak commuting of type A_f, we have

$$d(fgt,ggt) \leq Rd(ft,gt),$$

yields

$$fgt = gft \text{ or } fgt = gft = fft = ggt.$$

Now from (b), we have

$$\begin{aligned}\psi(d(fft,ft)) &\geq F(\psi(\min_0\{d(gft,gt),d(fft,gft),d(ft,gt)\}),\\ &\quad \varphi(\min_0\{d(gft,gt),d(fft,gft),d(ft,gt)\})),\\ &= F(\psi(d(fft,ft)),\varphi(d(fft,ft))) \geq \psi(d(fft,ft)),\end{aligned}$$

hence, either

$$\psi(d(fft,ft)) \text{ or } \varphi(d(fft,ft)) = 0,$$

implies

$$fft = ft \text{ or } fft = gft = ft.$$

Hence ft is common fixed point f and g .

Now, we take

$$\lim fgx_n = gt.$$

Since f and g are g-compatible of type (E), this yields

$$\lim ggx_n = \lim gfx_n = ft.$$

Now from (b), we have

$$\begin{aligned}\psi(d(fgx_n, ft)) &\geq F(\psi(\min_0\{d(ggx_n, gt), d(fgx_n, ggx_n), d(ft, gt)\}),\\ &\quad \varphi(\min_0\{d(ggx_n, gt), d(fgx_n, ggx_n), d(ft, gt)\})).\end{aligned}$$

Apply limit $n \to \infty$ on both side of above inequality, we have

$$\begin{aligned}\psi(d(gt, ft)) &\geq F(\psi(\min_0\{d(ft, gt), d(gt, ft), d(ft, gt)\}),\\ &\quad \varphi(\min_0\{d(ft, gt), d(gt, ft), d(ft, gt)\})),\\ &= F(\psi(d(fft, ft)), \varphi(d(fft, ft))) \geq \psi(d(fft, ft)).\end{aligned}$$

In this case, either

$$\psi(d(fft, ft)) \text{ or } \varphi(d(fft, ft)) = 0,$$

hence, we have

$$ft = gt.$$

Since f and g are $R-$weak commuting of type A_f, we have

$$d(fgt, ggt) \leq Rd(ft, gt),$$

yields

$$fgt = gft \text{ or } fgt = gft = fft = ggt.$$

From condition (b), we have

$$\begin{aligned}\psi(d(fft, ft)) &\geq F(\psi(\min_0\{d(gft, gt), d(fft, gft), d(ft, gt)\}),\\ &\quad \varphi(\min_0\{d(gft, gt), d(fft, gft), d(ft, gt)\})),\\ &= F(\psi(d(fft, ft)), \varphi(d(fft, ft))) \geq \psi(d(fft, ft)).\end{aligned}$$

In this case we have, either

$$\psi(d(fft, ft)) \text{ or } \quad \varphi(d(fft, ft)) = 0,$$

implies

$$fft = ft \text{ or } fft = gft = ft.$$

Hence ft is common fixed point f and g. □

Corollary 2.2. *Let f and g are weak semi compatible, $R-$weakly commuting type of A_f, self mappings of a complete metric space (X,d), which satisfies the following:*

- **a):** $f(X) \subseteq g(X)$
- **b):** $\psi(d(fx, fy)) \geq F(\psi(m_f(x,y)), \varphi(m_f(x,y)))$; where $m_f(x,y) = ad(gx, gy) + bd(fx, gx) + cd(fy, gy)$,
- **:** where $a, c > 1$, $a + c > 0$ and $b \in R$ such that $a + b + c > 1$.
- **c):** f and g are either f compatible of type (E) or $g-$compatible of type (E),

then f and g have a common fixed point in X .

Proof. From the condition (b) of corollary, it is simple to show that

$$\psi(d(fx, fy)) \geq F(\psi(m_f(x,y)), \varphi(m_f(x,y))) \geq F(\psi(m(x,y)), \varphi(m(x,y)));$$

where $m(x,y) = \min_0\{d(gx, gy), d(fx, gx), d(fy, gy)\}$.Remaining proof is on the same lines as the proof of Theorem 2.1. □

Lemma 2.3. [13] *If f and g are f-compatible of type (E) or g-compatible of type (E), then they are owc, but the converse is not true in general.*

Corollary 2.4. *Let X be a set and let d be the symmetric on X. Let self maps f and g on X satisfy:*

- **a):** $f(X) \subseteq g(X)$;
- **b):** $\psi(d(fx, fy)) \geq F(\psi(m(x,y)), \varphi(m(x,y)))$;
  where $m(x,y) = \min_0\{d(gx, gy), d(fx, gx), d(fy, gy)\}$;
- **c):** f and g are either f compatible of type (E) or $g-$compatible of type (E),

where $\psi \in \Psi$ and $\varphi \in \Phi$.

Then f and g have a common fixed point in X .

Proof. Let f and g are either $f-$compatible of type (E) or $g-$compatible of type (E), therefore the conclusion follows from theorem 1.17. □

Theorem 2.5. *Let f and g are weak semi compatible $R-$weakly commuting type of A_f self mappings of a complete metric space (X,d) such that,*

a): $f(X) \subseteq g(X)$;

b): $\psi(d(fx,fy)) \geq F(\psi(m_g(x,y)),\varphi(m_g(x,y)))$;

where $m_g(x,y) = \min_0\{d(gx,gy),k\dfrac{d(fx,gx)+d(fy,gy)}{2},k\dfrac{d(fx,gy)+d(fy,gx)}{2}\}$ with $k \geq 2$, $\psi \in \Psi$ and $\varphi \in \Phi$.

c): Mapping f and g are either $f-$compatible of type (E) or $g-$compatible of type (E).

If mappings f and g satisfy EA property, then f and g have common fixed point in X.

Proof. Since mappings f and g satisfy EA property, then there exists a sequence $\{x_n\}$ in X such that $\lim fx_n = \lim gx_n = t$ for some $t \in X$.

Case-1: f and g are f -compatible of type (E).

Since f and g are weak semi compatible, this yields

$$\lim fgx_n = gt \text{ or } \lim gfx_n = ft.$$

First we take

$$\lim gfx_n = ft.$$

Since f and g are f-compatible of type (E), yields

$$\lim ffx_n = \lim fgx_n = gt.$$

Now by (b)

$$\begin{aligned}
& \psi(d(ffx_n,ft)) \\
\geq\ & F(\psi(\min_0\{d(gfx_n,gt),k\frac{d(ffx_n,gfx_n)+d(ft,gt)}{2},k\frac{d(ffx_n,gt)+d(ft,gfx_n)}{2}\}), \\
& \varphi(\min_0\{d(gfx_n,gt),k\frac{d(ffx_n,gfx_n)+d(ft,gt)}{2},k\frac{d(ffx_n,gt)+d(ft,gfx_n)}{2}\})).
\end{aligned}$$

Now apply limit $n \to \infty$ on above inequality, yields

$$\begin{aligned}\psi(d(gt,ft)) &\geq F(\psi(\min_0\{d(ft,gt),k\frac{d(gt,ft)+d(ft,gt)}{2},k\frac{d(gt,gt)+d(ft,ft)}{2}\}),\\ &\quad \varphi(\min_0\{d(ft,gt),k\frac{d(gt,ft)+d(ft,gt)}{2},k\frac{d(gt,gt)+d(ft,ft)}{2}\})).\\ &\geq F(\psi(\min\{d(ft,gt),kd(ft,gt)\}),\varphi(\min\{d(ft,gt),kd(ft,gt)\}))\\ &\geq F(\psi(d(ft,gt)),\varphi(d(ft,gt)))\geq\psi(d(ft,gt)).\end{aligned}$$

This implies $\psi(d(ft,gt)) = 0$ or $\varphi(d(ft,gt)) = 0$ in either case, we have $ft = gt$.

Since f and g are $R-$weak commutating of A_f, we have

$$d(fgt,ggt) \leq Rd(ft,gt).$$

This yields

$$fgt = ggt \text{ and further } fgt = gft = fft = ggt.$$

Now form (b)

$$\begin{aligned}\psi(d(ft,fft)) &\geq F(\psi(\min_0\{d(gt,gft),k\frac{d(ft,gt)+d(fft,gft)}{2},k\frac{d(ft,gft)+d(fft,gt)}{2}\}),\\ &\quad \varphi(\min_0\{d(gt,gft),k\frac{d(ft,gt)+d(fft,gft)}{2},k\frac{d(ft,gft)+d(fft,gt)}{2}\}))\\ &\geq F(\psi(\min_0\{d(ft,fft),0,kd(fft,ft\}),\varphi(\min_0\{d(ft,fft),0,kd(fft,ft\}))\\ &\geq F(\psi(\min\{d(ft,fft),kd(fft,ft\}),\varphi(\min\{d(ft,fft),kd(fft,ft\}))\\ &\geq F(\psi(d(ft,fft)),\varphi(d(ft,fft))\geq\psi(d(ft,fft))\end{aligned}$$

With same arguments used above, we have

$$\psi(d(fft,ft) = 0 \text{ or } \varphi(d(fft,ft) = 0$$

Implies

$$fft = ft \text{ or } fft = gft = ft.$$

Hence ft is common fixed point of f and g.

Now, suppose that we have

$$\lim fgx_n = gt.$$

Since f and g are $f-$compatible of type (E), this yields,

$$\lim ffx_n = \lim fgx_n = gt.$$

Also f and g are $R-$weakly commuting type of A_f, implies

$$d(fgx_n, ggx_n) \leq Rd(fx_n, gx_n).$$

Now, apply limit $n \to \infty$, then we have

$$\lim ggx_n = gt.$$

From condition (b), we have

$$\begin{aligned}\psi(d(fx_n, fgx_n)) \quad &\geq \quad F(\psi(\min_0\{d(gx_n, ggx_n), d(fx_n, gx_n), d(fgx_n, ggx_n)\}),\\ &\quad\quad \varphi(\min_0\{d(gx_n, ggx_n), d(fx_n, gx_n), d(fgx_n, ggx_n)\})).\end{aligned}$$

Now, apply limit $n \to \infty$, then the above inequality becomes

$$\begin{aligned}\psi(d(t, gt)) \quad &\geq \quad F(\psi(\min_0\{d(t, gt), d(t, t), d(gt, gt)\}),\\ &\quad\quad \varphi(\min_0\{d(t, gt), d(t, t), d(gt, gt)\}))\\ &= \quad F(\psi(d(t, gt)), \varphi(d(t, gt)) \geq \psi(d(t, gt)).\end{aligned}$$

Implies, either

$$\psi(d(t, gt) = 0 \text{ or } \varphi(d(t, gt) = 0,$$

which further implies that

$$gt = t.$$

Now from (b), we have

$$\begin{aligned}&\psi(d(ft, fgx_n))\\ \geq \quad &F(\psi(\min_0\{d(gt, ggx_n), k\frac{d(ft, gt) + d(fgx_n, ggx_n)}{2}, k\frac{d(ft, ggx_n) + d(fgx_n, gt)}{2}\}),\\ &\varphi(\min_0\{d(gt, ggx_n), k\frac{d(ft, gt) + d(fgx_n, ggx_n)}{2}, k\frac{d(ft, ggx_n) + d(fgx_n, gt)}{2}\})).\end{aligned}$$

take limit $n \to \infty$, then above inequality becomes

$$\begin{aligned}\psi(d(ft,gt)) &\geq F(\psi(\min_0\{d(gt,gt),k\frac{d(ft,gt)+d(gt,gt)}{2},k\frac{d(ft,gt)+d(gt,gt)}{2}\}),\\ &\quad \varphi(\min_0\{d(gt,gt),k\frac{d(ft,gt)+d(gt,gt)}{2},k\frac{d(ft,gt)+d(gt,gt)}{2}\})).\\ &\geq F(\psi(\min_0\{0,k\frac{d(ft,gt)}{2},k\frac{d(ft,gt)}{2}\}),\varphi(\min_0\{0,k\frac{d(ft,gt)}{2},k\frac{d(ft,gt)}{2}\}))\\ &\geq F(\psi(\min\{k\frac{d(ft,gt)}{2},k\frac{d(ft,gt)}{2}\}),\varphi(\min\{k\frac{d(ft,gt)}{2},k\frac{d(ft,gt)}{2}\}))\\ &\geq F(\psi(k\frac{d(ft,gt)}{2}),\varphi(k\frac{d(ft,gt)}{2})) \qquad (9)\end{aligned}$$

Since $\psi(k\frac{d(ft,gt)}{2}) \geq \psi(d(ft,gt))$. By 9 it yields

$$\psi(k\frac{d(ft,gt)}{2}) \geq F(\psi(k\frac{d(ft,gt)}{2}),\varphi(k\frac{d(ft,gt)}{2})) \geq \psi(k\frac{d(ft,gt)}{2})$$

Then either

$$\psi(k\frac{d(ft,gt)}{2}) = 0 \text{ or } \varphi(k\frac{d(ft,gt)}{2})) = 0,$$

yields $ft = gt$. Now by preceeding work it can be easily shown that ft is common fixed point of f and g.

Now, we consider the second possibility as:

Case 2- Let f and g are g-compatible of type (E). Since f and g are weak semi compatible, this yields either

$$\lim fgx_n = gt \text{ or } \lim gfx_n = ft.$$

First we take

$$\lim gfx_n = ft.$$

Since f and g are g-compatible of type (E), yields

$$\lim ggx_n = \lim gfx_n = ft.$$

Also f and g are $R-$weakly commuting type of A_f, implies

$$d(fgx_{n,}ggx_n) \leq Rd(fx_n,gx_n).$$

Now, apply limit $n \to \infty$, We have

$$\lim fgx_n = ft.$$

From the condition (b), we have

$$\begin{aligned}
&\psi(d(fgx_n, fx_n)) \\
\geq\ & F(\psi(\min_0\{d(ggx_n, gx_n), k\frac{d(fgx_n, ggx_n)+d(fx_n, gx_n)}{2}, k\frac{d(fgx_n, gx_n)+d(fx_n, ggx_n)}{2}\}), \\
&\varphi(\min_0\{d(ggx_n, gx_n), k\frac{d(fgx_n, ggx_n)+d(fx_n, gx_n)}{2}, k\frac{d(fgx_n, gx_n)+d(fx_n, ggx_n)}{2}\})).
\end{aligned}$$

Now, apply limit $n \to \infty$ we have

$$\begin{aligned}
\psi(d(ft,t)) \ \geq\ & F(\psi(\min_0\{d(ft,t), k\frac{d(ft,ft)+d(t,t)}{2}, k\frac{d(ft,t)+d(t,ft)}{2}\}), \\
& \varphi(\min_0\{d(ft,t), k\frac{d(ft,ft)+d(t,t)}{2}, k\frac{d(ft,t)+d(t,ft)}{2}\})), \\
\geq\ & F(\psi(\min_0\{d(ft,t), 0, kd(ft,t)\}), \varphi(\min_0\{d(ft,t), 0, kd(ft,t)\})), \\
\geq\ & F(\psi(\min\{d(ft,t), kd(ft,t)\}), \varphi(\min\{d(ft,t), kd(ft,t)\})), \\
\geq\ & F(\psi(d(ft,t)), \varphi(d(ft,t)) \geq \psi(d(ft,t)).
\end{aligned}$$

This implies either

$$\psi(d(ft,t)) = 0 \text{ or } \varphi(d(ft,t)) = 0,$$

this yields

$$ft = t.$$

Since $f(X) \subseteq g(X)$ then there exist a point $u \in X$ such that $ft = gu$. Now by (b), we have

$$\begin{aligned}
&\psi(d(fgx_n, fu)) \\
\geq\ & F(\psi(\min_0\{d(ggx_n, gu), k\frac{d(fgx_n, ggx_n)+d(fu, gu)}{2}, k\frac{d(fgx_n, gu)+d(fu, ggx_n)}{2}\}), \\
&\varphi(\min_0\{d(ggx_n, gu), k\frac{d(fgx_n, ggx_n)+d(fu, gu)}{2}, k\frac{d(fgx_n, gu)+d(fu, ggx_n)}{2}\})).
\end{aligned}$$

Now limiting $n \to \infty$ yields

$$\begin{aligned}
\psi(d(ft,fu)) &\geq F(\psi(\min_0\{d(ft,gu),k\frac{d(ft,ft)+d(fu,gu)}{2},k\frac{d(ft,gu)+d(fu,ft)}{2}\}),\\
&\quad \varphi(\min_0\{d(ft,gu),k\frac{d(ft,ft)+d(fu,gu)}{2},k\frac{d(ft,gu)+d(fu,ft)}{2}\}))\\
\psi(d(t,fu)) &\geq F(\psi(\min_0\{0,k\frac{d(fu,t)}{2},k\frac{d(fu,t)}{2}\}),\varphi(\min_0\{0,k\frac{d(fu,t)}{2},k\frac{d(fu,t)}{2}\}))\\
&\geq F(\psi(k\frac{d(fu,t)}{2}),\varphi(k\frac{d(fu,t)}{2})) \qquad (10)
\end{aligned}$$

Since $\psi(k\frac{d(fu,t)}{2}) \geq \psi(d(fu,t))$ as $k \geq 2$. Then 10 implies

$$\psi(k\frac{d(fu,t)}{2}) \geq F(\psi(k\frac{d(fu,t)}{2}),\varphi(k\frac{d(fu,t)}{2})) \geq \psi(k\frac{d(fu,t)}{2})$$

This implies $\psi(k\frac{d(fu,t)}{2}) = 0$ or $\varphi(k\frac{d(fu,t)}{2}) = 0$. This implies $fu = t$. Hence $fu = gu$. Now, by preceeeding work it can be easily shown that fu is common fixed point f and g .

Now, we take

$$\lim fgx_n = gt.$$

Since f and g are g-compatible of type (E), this yields

$$\lim ggx_n = \lim gfx_n = ft.$$

Now from (b), we have

$$\begin{aligned}
&\psi(d(fgx_n,ft))\\
\geq\ & F(\psi(\min_0\{d(ggx_n,gt),k\frac{d(fgx_n,ggx_n)+d(ft,gt)}{2},k\frac{d(fgx_n,gt)+d(ft,ggx_n)}{2}\}),\\
&\varphi(\min_0\{d(ggx_n,gt),k\frac{d(fgx_n,ggx_n)+d(ft,gt)}{2},k\frac{d(fgx_n,gt)+d(ft,ggx_n)}{2}\})).
\end{aligned}$$

Apply limit $n \to \infty$ on both side of above inequality, we have

$$\begin{aligned}\psi(d(gt,ft)) &\geq F(\psi(\min_0\{d(ft,gt),k\frac{d(gt,ft)+d(ft,gt)}{2},k\frac{d(gt,gt)+d(ft,ft)}{2}\}),\\ &\quad \varphi(\min_0\{d(ft,gt),k\frac{d(gt,ft)+d(ft,gt)}{2},k\frac{d(gt,gt)+d(ft,ft)}{2}\}),\\ &\geq F(\psi(\min_0\{d(ft,gt),kd(gt,ft),0\}),\varphi(\min_0\{d(ft,gt),kd(gt,ft),0\}))\\ &\geq F(\psi(\min\{d(ft,gt),kd(gt,ft)\}),\varphi(\min\{d(ft,gt),kd(gt,ft)\}))\\ &\geq F(\psi(d(ft,gt)),\varphi(d(ft,gt)))\geq\psi(d(ft,gt)).\end{aligned}$$

This implies, either

$$\psi(d(gt,ft)) \text{ or } \varphi(d(gt,ft)) = 0,$$

hence, we have

$$ft = gt.$$

Now by preceeding work it can be easily shown that ft is a common fixed point of f and g. □

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Banach, S. Sur les Operations dans les Ensembles Abstraits et leur Applications aux Equations Integrales, Fund. Math., 3 (1922), 133-181.

[2] Kannan, R. Some results on fixed points, Bull. Cal. Math. Soc., 60 (1968), 71-76.

[3] Jungck, G. Commuting mappings and fixed points, Am. Math. Mon., 83 (1976), 261-263.

[4] Sessa, S. On a weak commutativity conditions in fixed point considerations, Publ. Inst. Math., 32, no.46 (1982), 149-153.

[5] Jungck, G. Compatible mappings and common fixed points, Int. J. Math. Math. Sci., 9, no.4 (1986), 771-779.

[6] Pant, R.P. Common fixed points of four mappings, Bull. Calcutta Math. Soc., 90, no.4 (1998), 281-286.

[7] Pant, R.P. Discontinuity and fixed points, J. Math. Anal. Appl., 240 (1999), 284-289.

[8] Singh,B. and Jain,S. Semi compatibility, compatibility and fixed point theorems in fuzzy metric spaces, J. Chungcheong Math. Soc., 18, no.1 (2005), 1-22.

[9] Pant,R.P., Bisht,R.K. and Arora,D. Weak reciprocal continuity and fixed point theorems, Ann. Univ. Ferrara, 57 (2011), 181-190.

[10] Pant, R.P. and Bisht, R.K., Common fixed point theorems under a new continuity condition, Ann. Univ. Ferrara, 58 (2012), 127-141.

[11] Bisht, R.K. and Shahzad, N., Faintly compatible mappings and common fixed points, Fixed Point Theory Appl., 2013 (2013), Article ID 156.

[12] Saluja, A.S., Jain, M.K., and Jhade, P.K., Weak semi compatibility and fixed point theorems, Bull. Int. Math. Virt. Inst., 2(2012), 205-217.

[13] Saluja, A.S., Jain, M.K., Some Fixed Point Theorems under Weak semicompatibility, Chinese J. Math., 2014 (2014), Article ID 618965, 8 pages.

[14] Ansari, A.H., Note on contractive type mappings and related fixed point, The 2nd Regional Conference on Mathematics and Applications, Payame Noor University, 2014, pages 377-380.

[15] M.A.Al-Thagafi and shahjad, N., Generalized I-nonexpansive self maps and invariant pproximations, Acta Math. Sin., 24(5) (2008), 867-876.

[16] Saluja, A.S., Jain, M.K., Fixed Point Theorems under Conditional semicompatibility with control function, Adv. Fixed Point Theory, 3(2013), 648-666.

[17] Pathak, H.K., Cho, Y.J. and Kang, S.M. Remarks on R-weakly commuting mappings and common fixed point theorems, Bull. Korean Math. Soc., 34, no.2 (1997), 247-257.

[18] Singh, M.R., Singh, Y.M., On various types of compatible maps and common fixed point theorems for non-continuous maps, Hacettepe J. Math. Stat., volume 40 (4) (2011), 203-213.

[19] Aamri, M., Moutawakil, D. EL., Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002) 181-188.

[20] Jungck, G., Common fixed points for noncontinuous nonself maps on nonmetric spaces, Far East J. Math. Sci. 4 (1996), 199-215.

[21] G.V.R. Babu and P.D. Sailaja,A Fixed Point Theorem of Generalized Weakly Contractive Maps in Orbitally Complete Metric Spaces,Thai J. Math. 9 (2011), 1–10.