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A NEW APPROXIMATION METHOD FOR COMMON FIXED POINTS OF NONEXPANSIVE MAPPINGS IN BANACH SPACES AND APPLICATION

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Abstract. In this paper, we propose a new iterative method for fixed point problem of a finite family of nonexpansive mappings in settings of a real Banach spaces having a weakly continous duality map. We prove that the sequence generated by the proposed method converges strongly to a common fixed point of a finite family of nonexpansives mappings which is also the solution of some variational inequality problems.There is no compactness assumption . The results obtained in this paper are significant improvement on important recent results.

Keywords: iterative method; nonexpansive mapping; variational inequality; common fixed point.

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1. Introduction

Let E be a Banach space with norm $\cdot\|$ and dual E^*. For any $x \in E$ and $x^* \in E^*$, $\langle x^*, x\rangle$ is used to refer to $x^*(x)$. Let $\varphi : [0,+\infty) \to [0,\infty)$ be a stricly increasing continuous function such that $\varphi(0) = 0$ and $\varphi(t) \to +\infty$ as $t \to \infty$ such a function φ is called gauge. Associed to a gauge φ is a duality map $J_\varphi : E \to 2^{E^*}$ defined by:

(1) $$J_\varphi(x) := \{x^* \in E^* : \langle x, x^*\rangle = ||x||\varphi(||x||), ||x^*|| = \varphi(||x||)\}, x \in E.$$

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If the gauge is defined by $\varphi(t)=t$, then the corresponding duality map is called the *normalized duality map* and is denoted by J. Hence the normalized duality map is given by

$$J(x) := \{x^* \in E^* : \langle x, x^* \rangle = ||x||^2 = ||x^*||^2 =\}, \forall x \in E.$$

Notice that

$$J_\varphi(x) = \frac{\varphi(||x||)}{||x||} J(x), x \neq 0.$$

Recall that a Banach space E is said to be smooth if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2}$$

exist for each $x, y \in S_E$ (Here $S_E := \{x \in E : ||x|| = 1\}$ is the unit sphere of E.) A real Banach space E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S_E$, and E is frechet differentiable if it is smooth and the limit is attained uniformly for $y \in S_E$. It is known that E is smooth if only if each duality map J_φ is single-valued, that E is Frechet differentiable if and only if each duality map J_φ is norm-to-norm continuous in E, and that E is uniformly smooth if and only if each duality map J_φ is is norm-to-norm uniformly continuous on bounded subsets of E. Following Browder [4], we say that a Banach space has a weakly continuous duality map if there exists a gauge φ such that J_φ is a single-valued and is weak-to-weak* sequentially continous, i.e., if $(x_n) \subset E$, $x_n \xrightarrow{w} x$, then $J_\varphi(x_n) \xrightarrow{w^*} J_\varphi(x)$). It is know that l^p $(1 < p < \infty)$ has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$ (see [13] fore more detais on duality maps). Finally recall that a Banach space E satisfies Opial property (see, e.g., [20]) if $\limsup_{n\to+\infty} \|x_n - x\| < \limsup_{n\to+\infty} \|x_n - x\|$ whenever $x_n \xrightarrow{w} x$, $x \neq y$. A Banach space E that has a weakly continuous duality map satisfies Opial's property. Let E be a real Banach space and K be a nonempty subset of E. A map $T : K \to E$ is said to be *Lipschitz* if there exists an $L \geq 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\| \quad \text{for all } x, y \in K; \tag{3}$$

if $L < 1$, T is called *contraction* and if $L = 1$, T is called nonexpansive. For $T : K \to K$ nonexpansive with a fixed point, where K is a closed convex nonempty subset of a real Banach space E, unlike in the case of Banach contraction mapping principle, trivial examples show that the sequence $\{x_n\}$ generated by the *Picard iterates*, $x_{n+1} := Tx_n, n \geq 0$, may fail to converge to such a fixed point even when such a fixed point is unique. More precisely, let B be the closed unit ball of $\mathbb{R}^2$ and let T be the anticlockwise rotation of $\frac{\pi}{4}$ about the origin of coordinates. Then, T is nonexpansive with the origin as the only fixed point. Moreover, the sequence $\{x_n\}$ defined by $x_{n+1} := Tx_n, n \geq 0$ with $x_0 = (0,1) \in B$, does not converge to (0,0) (see, e.g., Chidume[15]). Krasnoselskii [23], however, showed that in this example, one can obtain convergent sequence of successive approximations if $\frac{1}{2}(I+T)$ is used instead of T where I denotes the identity map on $\mathbb{R}^2$, that is, if the sequence of successive approximations is defined by $x_0 \in K$,

$$x_{n+1} = \frac{1}{2}\Big(x_n + Tx_n\Big), \quad n \geq 0, \tag{4}$$

instead of the usual Picard iterates, $x_{n+1} = Tx_n, x_0 \in K, n \geq 0$. Clearly, the fixed point sets of T and $\frac{1}{2}(I+T)$ are the same so that the limit of a convergent sequence defined by (4) is necessarily a fixed point T. A generalization of equation (4) which has proved successful in the approximation of fixed points of nonexpansive maps $T : K \to K$ (when they exist), K is a closed convex subset of a normed linear space, is the following scheme: $x_0 \in K$,

$$x_{n+1} = (1-\lambda)x_n + \lambda Tx_n, \quad n \geq 0;\ \lambda \in (0,1), \tag{5}$$

(see, e.g., Schaefer [33]). However, the most general iterative scheme now studied for approximating fixed point of *nonexpansive* mappings is the following: $x_0 \in K$,

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, \quad n \geq 0, \tag{6}$$

where $\{\alpha_n\} \subset (0,1)$ is a real sequence satisfying appropriate conditions (see, e.g., Chidume [16], Eldestein and O'Brain [19], Ishikawa [21]). Under the conditions that; $\lim \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$; the sequence $\{x_n\}$ generated by (6) is generally referred to as *Mann* sequence in the light of Mann [26].

This Manns method is remarkably useful for finding fixed points of a nonexpansive mappings and provides a unified framework for some kinds of algorithms from various different fields. In this respect, the following result is basic and important.

Theorem 1.1. *Let X be an Opial space and $T : K \to K$ be a nonexpansive self-mapping of a weakly compact convex subset K of X. For any $x_0 \in K$, let $\{x_n\} \subset K$ be the sequence given by*

(6) *where* $\{\alpha_n\} \subset (0,1)$ *is a non-increasing real sequence satisfying:* $0 < a \leq \alpha_n < 1$ *for all* $n \geq 1$*. Then* $\{x_n\}$ *converges weakly to a fixed point of* T*.*

However, as in Theorem 1.1, Manns method for nonexpansive mappings has only weak convergence. Thus a natural question rises: could we obtain a strong convergence theorem by using the well-known Mann method for non-expansive mappings? In this connection, in 1975, Genel and Lindenstrass[24] gave a counter-example. Hence the modification is necessary in order to guarantee the strong convergence of Manns method. Some attempts to construct iteration algorithm so that strong convergence is guaranteed have been made. Let E be a real Banach space, K a closed convex subset of E and $T : K \to K$ a nonexpansive mapping. For fixed $t \in (0,1)$ and arbitrary $u \in K$, let $z_t \in K$ denote the unique fixed point of T_t defined by $T_t x := tu + (1-t)Tx, x \in K$. Assume $F(T) := \{x \in K : Tx = x\} \neq \emptyset$. Browder [6] proved that if $E = H$, a Hilbert space, then $\lim_{t \to 0} z_t$ exists and is a fixed point of T. Reich [7] extended this result to uniformly smooth Banach spaces. Kirk [8] obtained the same result in arbitrary Banach spaces under the additional assumption that T has pre-compact range. For a sequence $\{\alpha_n\}$ of real numbers in $[0,1]$ and an arbitrary $u \in K$, let the sequence $\{x_n\}$ in K be iteratively defined by $x_0 \in K$,

$$x_{n+1} := \alpha_n u + (1-\alpha_n)Tx_n, n \geq 0. \tag{7}$$

Concerning this process, Reich [7] posed the following question.

Question. *Let E be a Banach space. Is there a sequence* $\{\alpha_n\}$ *such that whenever a weakly compact convex subset K of E has the fixed point property for nonexpansive mappings, then the sequence* $\{x_n\}$ *defined by (7) converges to a fixed point of T for arbitrary fixed* $u \in K$ *and all nonexpansive* $T : K \to K$*?*

Halpern [9] was the first to study the convergence of the algorithm (7) in the framework of Hilbert spaces. He proved the following theorem.

Theorem 1.2. [Halpern, [9]] *Let K be a bounded closed convex subset of a Hilbert space H and* $T : K \to K$ *be a nonexpansive mapping. Let* $u \in K$ *be arbitrary. Define a real sequence* $\{\alpha_n\}$ *in* $[0,1]$ *by* $\alpha_n = n^{-\theta}$, $\theta \in (0,1)$*. Define a sequence* $\{x_n\}$ *in K by* $x_1 \in K$, $x_{n+1} = \alpha_n u +$

$(1-\alpha_n)Tx_n,\ n \geq 1$. *Then,* $\{x_n\}$ *converges strongly to the element of* $F(T) := \{x \in K : Tx = x\}$ *nearest to u.*

An iteration method with recursion formula of the form (7) is referred to as a *Halpern-type iteration method*. Lions [10] improved Theorem 1.2, still in Hilbert spaces, by proving strong convergence of $\{x_n\}$ to a fixed point of T if the real sequence $\{\alpha_n\}$ satisfies the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$; and (iii) $\lim_{n\to\infty} \frac{|\alpha_n - \alpha_{n-1}|}{\alpha_n^2} = 0$. Reich [7] gave an affirmative answer to the above question in the case when E is uniformly smooth and $\alpha_n = n^{-a}$ with $0 < a < 1$. It was observed that both Halpern's and Lions' conditions on the real sequence $\{\alpha_n\}$ excluded the natural choice, $\alpha_n := (n+1)^{-1}$.

Recently, several theorems have been proved on the approximation of common fixed points of finite *nonexpansive* mappings (see for example [7], [11] , [2], and the references therein).

We observe that in all theses theorems have been proved in Hilbert spaces or compactness assumptions are needed on K or on the operators to get strong convergence.

It is our purpose in this paper to construct a new iterative algorithm and prove strong convergence theorems for approximating fixed points of finite family of nonexpansive mappings in reflexive real Banach spaces having weakly continuous duality maps. No compactness assumption is made. Then, we apply our results to variational inequality problems. Finally, our method of proof is of independent interest.

2. Preliminaires

We start with the following *demiclosedness principle* for nonexpansive mappings.

Lemma 2.1. [demiclosedness principle, Browder [17]] *Let E be a real Banach space satisfying Opial's property, K be a closed convex subset of E, and* $T : K \to K$ *be a nonexpansive mapping such that* $F(T) \neq \emptyset$. *Then* $I - T$ *is demiclosed; that is,*

$$\{x_n\} \subset K,\ x_n \rightharpoonup x \in K \text{ and } (I-T)x_n \to y \text{ implies that } (I-T)x = y.$$

Lemma 2.2. [Xu, [34]] *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that* $a_{n+1} \leq (1-\alpha_n)a_n + \sigma_n$ *for all* $n \geq 0$, *where* $\{\alpha_n\}$ *is a sequence in* $(0,1)$ *and* $\{\sigma_n\}$ *is a sequence in* $\mathbb{R}$ *such that*
(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, (b) $\limsup_{n\to\infty} \frac{\sigma_n}{\alpha_n} \leq 0$ *or* $\sum_{n=0}^{\infty} |\sigma_n| < \infty$. *Then* $\lim_{n\to\infty} a_n = 0$.

3. Main results

Lemma 3.1. *Let* K *be a nonempty, closed convex cone of a real Banach space* E, $m \geq 1$ *and* $\{T_i : i = 1,2,...m\}$ *a family of nonexpansive selfmappings on* K, λ *be a constant in* $(0,1)$. *Then, for each* $t \in (0,1)$, *there exists a unique* $x_t \in K$ *such that*

$$x_t = t(\lambda x_t) + (1-t)\sum_{i=1}^{m} \beta_i T_i x_t,$$

where $\sum_{i=1}^{m} \beta_i = 1$.

Proof. For each $t \in (0,1)$, define the mapping $T_t : K \to K$ by:

$$T_t x = t(\lambda x) + (1-t)\sum_{i=1}^{m} \beta_i T_i x, \forall x \in K.$$

We show that T_t is a contraction. For this, let $x, y \in K$. We have

$$\|T_t x - T_t y\| = \|[t(\lambda x) + (1-t)\sum_{i=1}^{m} \beta_i T_i x] - [t(\lambda y) + (1-t)\sum_{i=1}^{m} \beta_i T_i y]\| \leq [1-(1-\lambda)t]\|x-y\|.$$

Therefore, T_t is a contraction. Using Banach's contraction principle, there exists a unique fixed point x_t of T_t in K, i.e,

$$x_t = t(\lambda x_t) + (1-t)\sum_{i=1}^{m} \beta_i T_i x_t. \tag{8}$$

□

Proposition 3.2. *Let* K *be a nonempty, closed convex cone of a real Banach space* E, $m \geq 1$ *and* $\{T_i : i = 1,2,...m\}$ *a family of nonexpansive selfmappings on* K. *Then* $\sum_{i=1}^{m} \beta_i T_i$ *is a nonexpansive selfmappings on* K.

Proof. Let $x, y \in K$, we have:

$$\|\sum_{i=1}^{m}\beta_i T_i x - \sum_{i=1}^{m}\beta_i T_i y\| \leq \sum_{i=1}^{m}\beta_i\|T_i x - T_i y\| \leq \|x-y\|.$$

Hence, $\sum\limits_{i=1}^{m}\beta_i T_i$ is a nonexpansive selfmappings on K. □

We now prove the following theorem.

Theorem 3.3. *Let K be a nonempty, closed convex cone of reflexive real Banach space E which has uniformly Gâteaux differentiable norm, $m \geq 1$ and $\{T_i : i = 1,2,...m\}$ a family of nonexpansive selfmappings on K. Assume that E has a weakly continuous duality map and $F\left(\sum_{i=1}^{m}\beta_i T_i\right) = \bigcap\limits_{i=1}^{m} F(T_i) \neq \emptyset$. Then as $t \to 0$, the net $\{x_t\}$ defined by* (8) *converges strongly to a fixed point of T_i, $1 \leq i \leq m$.*

Proof. The proof is given in 3 steps.

Step 1. We prove that $\{x_t\}$ is bounded. Let $u \in \bigcap\limits_{i=1}^{m} F(T_i)$. From (8), we have

$$\begin{aligned}\|x_t - u\| &= \|t(\lambda x_t) + (1-t)\sum_{i=1}^{m}\beta_i T_i x_t - u\| \\ &\leq t\lambda\|x_t - u\| + (1-t)\|\sum_{i=1}^{m}\beta_i T_i x_t - u\| + (1-\lambda)t\|u\| \\ &\leq [1-(1-\lambda)t]\|x_t - u\| + (1-\lambda)t\|u\|,\end{aligned}$$

which implies that

$$\|x_t - u\| \leq \|u\|.$$

Hence,$\{x_t\}$ is bounded.

Step 2. We show that $\{x_t\}$ is relatively norm compact as $t \to 0$. Using (8), we have

$$\|x_t - \sum_{i=1}^{m}\beta_i T_i x_t\| = t\|\lambda x_t - \sum_{i=1}^{m}\beta_i T_i x_t\| \to 0, \text{ as } t \to 0. \tag{9}$$

Now, let $\{t_n\} \subset (0,1)$ be a sequence such that $t_n \to 0$ as $n \to +\infty$. Set $x_n := x_{t_n}$. We show that $\{x_n\}$ has a convergence subsequence. To this end, from (9), we have

$$\|x_n - \sum_{i=1}^{m}\beta_i T_i x_n\| \to 0. \tag{10}$$

Let φ be a jauge such that the corresponding duality map J_φ is single valued and weak-to-weak* seqentially continuous from E to E^*. Let $u \in \bigcap_{i=1}^{m} F(T_i)$. From (1) and (8), we have

$$\begin{aligned}\|x_t-u\|\varphi(\|x_t-u\|) &= \langle t(\lambda x_t)+(1-t)\sum_{i=1}^{m}\beta_i T_i x_t-u, J_\varphi(x_t-u)\rangle \\ &= t\lambda\langle x_t-u, J_\varphi(x_t-u)\rangle+(1-t)\langle\sum_{i=1}^{m}\beta_i T_i x_t-u, J_\varphi(x_t-u)\rangle \\ &\quad -(1-\lambda)t\langle u, J_\varphi(x_t-u)\rangle \\ &\le [1-(1-\lambda)t]\|x_t-u\|\varphi(\|x_t-u\|)-(1-\lambda)t\langle u, J_\varphi(x_t-u)\rangle.\end{aligned}$$

So,

$$\|x_t-u\|\varphi(\|x_t-u\|)\le\langle u, J_\varphi(u-x_t)\rangle.$$

In particular,

$$\|x_n-u\|\varphi(\|x_n-u\|)\le\langle u, J_\varphi(u-x_n)\rangle\ \forall u\in\bigcap_{i=1}^{m}F(T_i), \tag{11}$$

which implies that

$$\|x_n-u\|\le\|u\|.$$

Therefore, $\{x_n\}$ is bounded.

Since E is reflexive and K is closed and convex, there exists $\{x_{n_k}\}$ a subsequence of $\{x_n\}$ that converges weakly to $x^*\in K$. Using Lemma 2.1, it follows that $x^*\in\bigcap_{i=1}^{m}F(T_i)$. Replacing u by x^* in (11), we have:

$$\|x_{n_k}-x^*\|\varphi(\|x_{n_k}-x^*\|)\le\langle x^*, J_\varphi(x^*-x_{n_k})\rangle\ \forall k\ge 1. \tag{12}$$

Using (12), the fact that $x_{n_k}\rightharpoonup x^*$ as $k\to\infty$ and J_φ is weakly continuous, it follows that

$$\|x_{n_k}-x^*\|\varphi(\|x_{n_k}-x^*\|)\to 0 \text{ as } k\to\infty. \tag{13}$$

Using (13), the fact that $\{x_n\}$ is bounded, and φ is continuous and satisfies $\varphi(t)=0$ if and only if $t=0$, we deduce that $\|x_{n_k}-x^*\|\to 0$ as $k\to\infty$. Hence $x_{n_k}\to x^*$. This proves the relatively compactness of the net $\{x_t\}$.

Step 3. We show that the entire net $\{x_t\}$ converges to a common fixed point of T_i, $1\le i\le m$. We claim that the net $\{x_t\}$ has a unique cluster point. From **Step2**, the net $\{x_t\}$ has a cluster point. Now suppose that $x^*\in E$ and $x^{**}\in E$ are two cluster points of $\{x_t\}$. Let $\{t_n\}\subset(0,1)$

such that $t_n \to 0$ and $x_{t_n} \to x^*$, as $n \to \infty$ and $\{s_n\} \subset (0,1)$ such that $s_n \to 0$ and $x_{s_n} \to x^{**}$, as $n \to \infty$. Set $x_n = x_{t_n}$ and $z_n = x_{s_n}$.

Following the same arguments as in **step2**, it follows that x^* , $x^{**} \in \bigcap_{i=1}^{m} F(T_i)$, and the following estimates hold:

$$\|x_n - x^{**}\| \varphi(\|x_n - x^{**}\|) \leq \langle x^{**}, J_\varphi(x^{**} - x_n) \rangle, \tag{14}$$

and

$$\|z_n - x^*\| \varphi(\|z_n - x^*\|) \leq \langle x^*, J_\varphi(x^* - z_n) \rangle. \tag{15}$$

Letting $n \to \infty$ in (14) and (15) gives

$$\|x^* - x^{**}\| \varphi(\|x^* - x^{**}\|) \leq \langle x^{**}, J_\varphi(x^{**} - x^*) \rangle \tag{16}$$

and

$$\|x^{**} - x^*\| \varphi(\|x^{**} - x^*\|) \leq \langle x^*, J_\varphi(x^* - x^{**}) \rangle. \tag{17}$$

Adding up (16) and (17) yields

$$2\|x^* - x^{**}\| \varphi(\|x^* - x^{**}\|) \leq \|x^* - x^{**}\| \varphi(\|x^* - x^{**}\|),$$

which implies that $x^* = x^{**}$.This complete the proof. □

We now apply Theorem 3.3 to approximate fixed points of common finite family of nonexpansive mappings.

Theorem 3.4. *Let K be a nonempty, closed convex cone of reflexive real Banach space E which has uniformly Gâteaux differentiable norm, $m \geq 1$ and $\{T_i : i = 1, 2, ...m\}$ a family of nonexpansive selfmappings on K. Assume that E has a weakly continuous duality map and $F\left(\sum_{i=1}^{m} \beta_i T_i\right) = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\alpha_n\}$ be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_0 \in K$ by:*

$$x_{n+1} = \alpha_n(\lambda_n x_n) + (1 - \alpha_n) \sum_{i=1}^{m} \beta_i T_i x_n \; n \geq 0. \tag{18}$$

Suppose the following conditions hold :

$(a) \lim_{n\to\infty} \alpha_n = 0$; $\quad (b) \lim_{n\to\infty} \lambda_n = 1; \sum_{i=1}^{m} \beta_i = 1$;
$(c) \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty, \sum_{n=0}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty} (1-\lambda_n)\alpha_n = \infty.$
Then, the sequence $\{x_n\}$ *converges to a common fixed point of* $T_i, \quad 1 \le i \le m.$

Proof. First, we prove that the sequence $\{x_n\}$ is bounded. Let $u \in \bigcap_{i=1}^{m} F(T_i)$. From (18), we have

$$\begin{aligned}
\|x_{n+1} - u\| &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)\sum_{i=1}^{m} \beta_i T_i x_n - u\| \\
&\le \alpha_n \lambda_n \|x_n - u\| + (1-\lambda_n)\alpha_n \|u\| + (1-\alpha_n)\|\sum_{i=1}^{m} \beta_i T_i x_n - u\| \\
&= [1-(1-\lambda_n)\alpha_n]\|x_n - u\| + (1-\lambda_n)\alpha_n \|u\| \le \max\{\|x_n - u\|, \|u\|\}.
\end{aligned}$$

Hence, $\{x_n\}$ is bounded and so is $\{T_i x_n\}$.

From (18), it follows that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n(\lambda_n x_n) + (1-\alpha_n)\sum_{i=1}^{m} \beta_i T_i x_n - \alpha_{n-1}(\lambda_{n-1}x_{n-1}) + (1-\alpha_{n-1})\sum_{i=1}^{m} \beta_i T_i x_{n-1}\| \\
&= \|\alpha_n \lambda_n (x_n - x_{n-1}) + \alpha_n(\lambda_n - \lambda_{n-1})x_{n-1} + (\alpha_n - \alpha_{n-1})(\lambda_{n-1}x_{n-1}) \\
&\quad + (1-\alpha_n)(\sum_{i=1}^{m} \beta_i T_i x_n - \sum_{i=1}^{m} \beta_i T_i x_{n-1}) + (\alpha_{n-1} - \alpha_n)\sum_{i=1}^{m} \beta_i T_i x_{n-1}\| \\
&\le \alpha_n \lambda_n \|x_n - x_{n-1} + (1-\alpha_n)\|\sum_{i=1}^{m} \beta_i T_i x_n - \sum_{i=1}^{m} \beta_i T_i x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\lambda_{n-1}\|x_{n-1}\| \\
&\quad + \|\sum_{i=1}^{m} \beta_i T_i x_{n-1}\|) + \alpha_n |\lambda_n - \lambda_{n-1}|\|x_{n-1}\| \\
&\le [1-(1-\lambda_n)\alpha_n]\|x_n - x_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + \alpha_n|\lambda_n - \lambda_{n-1}|)M_1;
\end{aligned}$$

Hence,

$$\|x_{n+1} - x_n\| \le [1-(1-\lambda_n)]\|x_n - x_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + \alpha_n|\lambda_n - \lambda_{n-1}|)M_1, \tag{19}$$

where $M_1 > 0$ is such that $\sup_n\{\|x_{n-1}\| + \|\sum_{i=1}^{m} \beta_i T_i x_{n-1}\|\} \le M_1$. Hence, from (19) and Lemma 2.2, we deduce

$$\lim_{n\to+\infty} \|x_{n+1} - x_n\| = 0.$$

At the same time, we note that

$$\|x_{n+1}-\sum_{i=1}^{m}\beta_i T_i x_n\| = \alpha_n\|(\lambda_n x_n)-\sum_{i=1}^{m}\beta_i T_i x_n\| \to 0.$$

Therefore, we have

$$\lim_{n\to+\infty}\|x_n-\sum_{i=1}^{m}\beta_i T_i x_n\| = 0. \tag{20}$$

Next, we prove that $\limsup_{n\to+\infty}\langle x^*, J(x^*-x_n)\rangle \leq 0$, where $x^*=\lim_{t\to 0} x_t$ $\{x_t\}$ is the net defined by (8). From (1), (8), the fact that T_i, $1\leq i\leq m$, is nonexpansive and $\{x_t\}$ and $\{x_n\}$ are bounded, we have the following estimates

$$\begin{aligned}
\|x_t-x_n\|^2 &= \langle x_t-x_n, J(x_t-x_n)\rangle = t\langle x_t-x_n, J(x_t-x_n)\rangle-(1-\lambda)t\langle x_t, J(x_t-x_n)\rangle \\
&+ (1-t)\langle\sum_{i=1}^{m}\beta_i T_i x_t-\sum_{i=1}^{m}\beta_i T_i x_n, J(x_t-x_n)\rangle+(1-t)\langle\sum_{i=1}^{m}\beta_i T_i x_n-x_n, J(x_t-x_n)\rangle \\
&\leq \|x_t-x_n\|^2-(1-\lambda)t\langle x_t, x_t-x_n\rangle+(1-t)\langle\sum_{i=1}^{m}\beta_i T_i x_n-x_n, J(x_t-x_n)\rangle \\
&\leq \|x_t-x_n\|^2-(1-\lambda)t\langle x_t, J(x_t-x_n)\rangle+M_2\|\sum_{i=1}^{m}\beta_i T_i x_n-x_n\|,
\end{aligned}$$

Where $M_2>0$ such that $\sup\{\|x_t-x_n\|, t\in(0,1), n\geq 0\}\leq M_2$. Therefore, we have

$$\langle x_t, J(x_t-x_n)\rangle \leq \frac{M_2}{(1-\lambda)t}\|\sum_{i=1}^{m}\beta_i T_i x_n-x_n\|. \tag{21}$$

From (21) and (20), we obtain

$$\limsup_{n\to+\infty}\langle x_t, J(x_t-x_n)\rangle \leq 0. \tag{22}$$

Letting $t\to 0$, noting the fact that $x_t\to x^*$ in norm and the fact that the duality map J is norm-to-norm uniformly continuous on bounded sets on E, we get

$$\limsup_{n\to+\infty}\langle x^*, J(x^*-x_n)\rangle \leq 0.$$

Finally, we show that $x_n \to x^*$. From (18), we have

$$
\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \langle x_{n+1}-x^*, J(x_{n+1}-x^*)\rangle = \alpha_n\lambda_n\langle x_n-x^*, J(x_{n+1}-x^*)\rangle \\
&+ (1-\lambda_n)\alpha_n\langle x^*, J(x^*-x_{n+1})\rangle + (1-\alpha_n)\langle\sum_{i=1}^{m}\beta_i T_i x_n - x^*, J(x_{n+1}-x^*)\rangle \\
&\leq [1-(1-\lambda_n)\alpha_n]\|x_n-x^*\|\|x_{n+1}-x^*\| + (1-\lambda_n)\alpha_n\langle x^*, J(x^*-x_{n+1})\rangle \\
&\leq \frac{1-(1-\lambda_n)\alpha_n}{2}(\|x_n-x^*\|^2+\|x_{n+1}-x^*\|^2) + (1-\lambda_n)\alpha_n\langle x^*, J(x^*-x_{n+1})\rangle,
\end{aligned}
$$

which implies that

$$
\|x_{n+1}-x^*\|^2 \leq [1-(1-\lambda_n)\alpha_n]\|x_n-x^*\| + 2(1-\lambda_n)\alpha_n\langle x^*, J(x^*-x_{n+1})\rangle.
$$

We can check that all the assumptions of Lemma 2.2 are satisfied. Therefore, we deduce $x_n \to x^*$. This complete the proof. □

Corollary 3.5. *Assume that $E = l_p$, $1 < p < \infty$. Let K be a nonempty, closed and convex cone of E and $T : K \to K$ be a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\alpha_n\}$ be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_0 \in K$ by:*

$$
x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)Tx_n \ n \geq 0. \tag{23}
$$

Suppose the following conditions hold :

(a) $\lim_{n\to\infty}\alpha_n = 0$; (b) $\lim_{n\to\infty}\lambda_n = 1$;

(c) $\sum_{n=0}^{\infty}|\alpha_n-\alpha_{n-1}| < \infty$, $\sum_{n=0}^{\infty}|\lambda_n-\lambda_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n = \infty$.

Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.

Proof. Since $E = l_p$, $1 < p < \infty$ have weakly continuous duality map, the proof follows from Theorem 3.4 with $m = 1$. □

Corollary 3.6. *Assume that $E = l_p$, $1 < p < \infty$. Let K be a nonempty, closed and convex cone of E, $m \geq 1$ and $\{T_i : i = 1,2,...m\}$ a family of nonexpansive selfmappings on K. Assume that $F\left(\sum_{i=1}^{m}\beta_i T_i\right) = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\alpha_n\}$ be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_0 \in K$ by:*

$$
x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)\sum_{i=1}^{m}\beta_i T_i x_n \ n \geq 0. \tag{24}
$$

Suppose the following conditions hold :

$(a)\lim\limits_{n\to\infty}\alpha_n=0;\quad (b)\lim\limits_{n\to\infty}\lambda_n=1;\sum\limits_{i=1}^{m}\beta_i=1$

$(c)\ \sum\limits_{n=0}^{\infty}|\alpha_n-\alpha_{n-1}|<\infty,\ \sum\limits_{n=0}^{\infty}|\lambda_n-\lambda_{n-1}|<\infty$ *and* $\sum\limits_{n=0}^{\infty}(1-\lambda_n)\alpha_n=\infty.$

Then, the sequence $\{x_n\}$ *converges strongly to a common fixed point of* $T_i,\ \ 1\le i\le m.$

Proof. Since $E=l_p,\ 1<p<\infty$ have weakly continuous duality map, the proof follows from Theorem 3.4. □

Corollary 3.7. *Let* H *be a real Hilbert space and* K *a nonempty, closed and convex cone of* E, $m\ge 1$ *and* $\{T_i : i=1,2,...m\}$ *a family of nonexpansive selfmappings on* K. *Assume that* $F\left(\sum_{i=1}^{m}\beta_iT_i\right)=\bigcap\limits_{i=1}^{m}F(T_i)\neq\emptyset$. *Let* λ_n *and* α_n *be two sequences in* $(0,1)$. *Let* $\{x_n\}$ *be a sequence generated iteratively from arbitrary* $x_0\in K$ *by:*

$$x_{n+1}=\alpha_n(\lambda_nx_n)+(1-\alpha_n)\sum_{i=1}^{m}\beta_iT_ix_n\ n\ge 0. \tag{25}$$

Suppose the following conditions hold :

$(a)\lim\limits_{n\to\infty}\alpha_n=0;\quad (b)\lim\limits_{n\to\infty}\lambda_n=1;\sum\limits_{i=1}^{m}\beta_i=1$

$(c)\ \sum\limits_{n=0}^{\infty}|\alpha_n-\alpha_{n-1}|<\infty,\ \sum\limits_{n=0}^{\infty}|\lambda_n-\lambda_{n-1}|<\infty$ *and* $\sum\limits_{n=0}^{\infty}(1-\lambda_n)\alpha_n=\infty.$

Then the sequence $\{x_n\}$ *converges strongly to a common fixed point of* $T_i,\ \ 1\le i\le m.$

Proof. The proof follows from Theorem 3.4. □

Remark 1. The Mann algorithm (see, [37]) for nonexpansive mappings, without any compactness assumptions, gives only weak convergence of the associated sequence. Here, we prove strong convergence thereom without any compactness assumptions . Therefore, our results improve many recent results using Manns method to approximate fixed points of nonexpansive mappings.

We know give example of space E, set K and mapping T_i satisfying the assumptions of Theorem 3.4.

Let $E=l_p,\ 1<p<\infty$, and K be a subset of E defined by:

$$K:=\Big\{x=(x_n)\in E\ :x_n\ge 0,\ \forall n\ge 1\Big\}.$$

Finally, let $T_i : K \to K$,$1 \le i \le m$, be the mapping defined by:

$$T_i x = \Big(x_1, x_2, x_3, x_{i-1}, x_{i+1}, ..., x_n, ...\Big), x = (x_n), \in K,\ n \ge 1.$$

It is well known (see, e.g., [13]) that $l_p,\ 1 < p < \infty$, has weakly continuous duality map. The set K is a nonempty, closed, convex cone in l_p and the map T_i is nonexpansive. Therefore, the spaces E, the set K and te map T_i satisfies all the assumptions of Theorem 3.4.

Remark 2. In our theorems, we assume that K is a cone. But, in some cases, for example, if K is the closed unit ball, we can weaken this assumption to the following: $\lambda x \in K$ for all $\lambda \in (0,1)$ and $x \in K$. Therefore, in the case where E is a real Hilbert space or $E = l_p,\ 1 < p < \infty$, our results can be used to approximated fixed points of nonexpansive mappings from the closed unit ball to itself.

In fact, we have the following.

Corollary 3.7. *Assume that $E = l_p,\ 1 < p < \infty$ or E is a real Hilbert space. Let B be the closed unit ball of E, $m \ge 1$ and $\{T_i : i = 1,2,...m\}$ a family of nonexpansive selfmappings on B. Assume that $F\Big(\sum_{i=1}^m \beta_i T_i\Big) = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$. Let λ_n and α_n be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_0 \in B$ by:*

$$x_{n+1} = \alpha_n(\lambda_n x_n) + (1-\alpha_n)\sum_{i=1}^{m} \beta_i T_i x_n\ n \ge 0. \tag{26}$$

Suppose the following conditions hold :

$(a) \lim_{n\to\infty} \alpha_n = 0;\quad (b) \lim_{n\to\infty} \lambda_n = 1; \sum_{i=1}^{m} \beta_i = 1$

$(c) \sum_{n=0}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty,\ \sum_{n=0}^{\infty} |\lambda_n - \lambda_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty} (1-\lambda_n)\alpha_n = \infty.$

Then the sequence $\{x_n\}$ converges strongly to a common fixed point of T_i, $1 \le i \le m$.

Remark 3. Real sequences that satisfy conditions a(i), (ii) and (iii) re given by: $\alpha_n = \dfrac{1}{\sqrt{n}}$ and $\lambda_n = 1 - \dfrac{1}{\sqrt{n}}$, $\beta_i = \dfrac{1}{p}$, $1 \le i \le p$,

4. Application to Variational inequality problems

Let K be a nonempty,closed convex subset of a smooth real Banach E, $m \geq 1$ and $\{T_i : i = 1,2,...m\}$ a family of nonexpansive selfmappings on K, such that $\overset{m}{\underset{i=1}{\cap}} F(T_i) \neq \emptyset$, let $f : K \to K$ be a contraction mapping. The point $u \in \overset{m}{\underset{i=1}{\cap}} F(T_i)$ is said to be a solution of the variational inequality problem $V(T, f, K)$ provided that

$$\langle u - f(u), J(u - v)\rangle \leq 0 \ \ \forall v \in \overset{m}{\underset{i=1}{\cap}} F(T_i). \tag{27}$$

lemma 4.1 *Let E be a smooth real Banach space, K a nonempty,closed convex of E, $m \geq 1$ and $\{T_i : i = 1,2,...m\}$ a family of nonexpansive selfmappings on K, such that $\overset{m}{\underset{i=1}{\cap}} F(T_i) \neq \emptyset$. Then*

$$\langle x - \sum_{i=1}^{m} \beta_i T_i x, J(x - p)\rangle \geq 0 \ \forall x \in K, \ p \in \overset{m}{\underset{i=1}{\cap}} F(T_i).$$

Proof. Using Schwartz inequality, property of J and T_i, we obtain

$$\begin{aligned}
\langle x - \sum_{i=1}^{m} \beta_i T_i x, J(x - p)\rangle &= \langle x - \sum_{i=1}^{m} \beta_i T_i x + p - p, J(x - p)\rangle \\
&= \|x - p\|^2 - \langle \sum_{i=1}^{m} \beta_i T_i x - p, J(x - p)\rangle \\
&\geq \|x - p\|^2 - \sum_{i=1}^{m} \beta_i \|T_i x - T_i p\| \|x - p\| \\
&\geq \|x - p\|^2 - \|x - p\|^2 \geq 0.
\end{aligned}$$

Hence, $\langle x - \sum_{i=1}^{m} \beta_i T_i x, J(x - p)\rangle \geq 0$. □

We prove the following result.

Theorem 4.2 *Let K be a nonempty, closed convex cone of reflexive real Banach space E which has uniformly Gâteaux differentiable norm, $m \geq 1$ and $\{T_i : i = 1,2,...m\}$ a family of nonexpansive selfmappings on K. Assume that E has a weakly continuous duality map and $F\left(\sum_{i=1}^{m} \beta_i T_i\right) = \overset{m}{\underset{i=1}{\cap}} F(T_i) \neq \emptyset$. Let $\{\lambda_n\}$ and $\{\alpha_n\}$ be two sequences in $(0,1)$. Let $\{x_n\}$ be a sequence generated iteratively from arbitrary $x_0 \in K$ by:*

$$x_{n+1} = \alpha_n(\lambda_n x_n) + (1 - \alpha_n) \sum_{i=1}^{m} \beta_i T_i x_n \ n \geq 0. \tag{28}$$

Suppose the following conditions hold :

$(a)\lim_{n\to\infty}\alpha_n=0;\quad (b)\lim_{n\to\infty}\lambda_n=1;\sum_{i=1}^{m}\beta_i=1;$

$(c)\ \sum_{n=0}^{\infty}|\alpha_n-\alpha_{n-1}|<\infty,\ \sum_{n=0}^{\infty}|\lambda_n-\lambda_{n-1}|<\infty$ *and* $\sum_{n=0}^{\infty}(1-\lambda_n)\alpha_n=\infty.$

Then the sequence $\{x_n\}$ *converges strongly to a unique solution of* $V(T,0,K)$.

Proof. It follows from (8),

$$x_t=-\frac{1-t}{(1-\lambda)t}(x_t-\sum_{i=1}^{m}\beta_iT_ix_t).$$

Using Lemma 4.1, then for any $p\bigcap_{i=1}^{m}F(T_i)$, we have

$$\langle x_t,J(x_t-p)\rangle=-\frac{1-t}{(1-\lambda)t}\langle x_t-\sum_{i=1}^{m}\beta_iT_ix_t,J(x_t-p\rangle\le 0.$$

Letting $t\to 0$, noting the fact that $x_t\to x^*$ in norm and the fact that the duality map J is norm-to-norm uniformly continuous on bounded sets on E, we obtain

$$\langle x^*,J(x^*-p)\rangle\le 0. \tag{29}$$

Using the fact that x_n and x_t have the same limit (Theorem 3.4) , therefore, we deduce $\{x_n\}$ converges to the solution of a variational inequality $V(T,0,K)$. Using (29) and the properties of J, we get the uniqueness of the solution of the variational inequality $V(T,0,K)$. This complete this proof. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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