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CONTROL PROBLEMS ASSOCIATED WITH SUBDIFFERENTIAL OPERATORS AND APPLICATIONS TO VISCOSITY

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Abstract. The paper provides in the finite dimensional setting, the existence of viscosity subsolutions (expressed in terms of value function) of Hamilton-Jacobi-Bellman equations related to control problems subject to evolution inclusions involving time-dependent subdifferential operators and Young measures.

Keywords: Young measure; viscosity subsolution; value function; relaxed control; evolution inclusion; subdifferential operator.

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1. Introduction

We present in this paper some applications to control and viscosity theory that we obtain from the study of a class of evolution problems governed by time-dependent subdifferential operators.

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In the first part, we study some variational properties of a value function V_J defined on $I \times H$ by

$$V_J(\tau, z) := \sup_{\mu \in \mathscr{R}} \int_\tau^T \int_U J(t, x_{z,\mu}(t), u)\mu_t(du)\, dt,$$

where $I := [0,T]$ is an interval and H stands for a real separable Hilbert space. The cost function $J : I \times H \times U \to \mathbb{R}$ is an integrand and the control space U is compact metric. The control measure μ belongs to the space of Young measures $\mathscr{R} := \{\nu : I \to \mathscr{M}^1_+(U),\ t \mapsto \nu_t$ is λ-measurable$\}$, that is, the set of all λ-measurable mappings from I into the space $\mathscr{M}^1_+(U)$ of all probability Radon measures on $(U, \mathscr{B}(U))$. The absolutely continuous function $x_{z,\mu}(\cdot)$ is the unique solution on I of the problem

$$(P_{z,\mu}) \qquad \begin{cases} -\dot{x}_{z,\mu}(t) \in \partial\varphi(t, x_{z,\mu}(t)) + \int_{\Gamma(t)} g(t, x_{z,\mu}(t), u)\mu_t(du) \\ \text{a.e. } t \in I, \\ x_{z,\mu}(0) = z \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

proved in our previous results [28]. Here, the (set-valued) operator $\partial\varphi(t,\cdot)$, $t \in I$, is the subdifferential of a time-dependent proper lower semi-continuous (lsc) convex function $\varphi(t,\cdot)$ of H into $\mathbb{R}_+ \cup \{+\infty\}$, $\operatorname{dom}\varphi(t,\cdot)$ denotes the effective domain of the function $\varphi(t,\cdot)$, and the perturbation $g : I \times H \times U \longrightarrow H$ is a mapping that satisfies appropriate conditions. The set-valued map $\Gamma : I \to U$ is λ-measurable with nonempty compact values.

In the second part, we show in the finite dimensional setting ($H = \mathbb{R}^d$), that under some restrictions imposed on the cost functional and the dynamics, the value function associated to the continuous cost function above, is a viscosity subsolution of the corresponding Hamilton-Jacobi-Bellman equation

$$\frac{\partial V_J}{\partial t}(t,x) + H(t, x, \nabla V_J(t,x)) = 0.$$

Several results related to the Hamilton-Jacobi-Bellman equation have been developed by many authors in various cases. For ordinary differential equations, with nonconvex sweeping processes and m-accretive operators, see, e.g., [6] and with the subdifferential of a primal lower nice "pln for short" functions $\varphi : H \longrightarrow \mathbb{R} \cup \{+\infty\}$, we refer to [8]. The work in [11] concerned the problem when the dynamic is governed by the subdifferential of an integral functional (see also [9], for ordinary differential equations). These papers involve two Young measures (controls), and replace the maximization problem associated with V_J above by a sup inf one. Earlier results in viscosity theory can be found in [17, 18, 19, 20, 21], see also [22], dealing with sup inf and inf sup problems from differential games theory (with two players), where the controls are measurable mappings. For further details about control and viscosity on sweeping process, evolution equations, some classes of evolution inclusions of second order and related results see, e.g., [3, 4, 7, 10, 12, 14, 15, 23, 24, 27].

In this framework, as in [11], one relaxed control is considered, for simplicity reasons. The evolution is driven by differential inclusion governed by the subdifferential of a function depending on time and state variable ($\varphi : I \times H \longrightarrow \mathbb{R}_+ \cup \{+\infty\}$). Thanks to our result of this paper, we will present other sup inf type value functions (the case of two controls) associated with such control problems where the controls are Young measures, and their relation with the viscosity solution of the associated Hamilton-Jacobi-Bellman equation. This will be published in a forthcoming paper.

The paper consists of 4 sections. In section 2, we give some background material. We introduce in the next section, some results of [28] related to single-valued time-dependent perturbations, and concepts about Young measures. We study also the continuous dependence of the solution of $(P_{x_0,\mu})$ on the initial position x_0 and μ. In the last section, we make use of this continuous dependence result to point out some semi-continuity properties, and more generally some variational features of the value function of minimization problems submitted to $(P_{z,\mu})$. Then, we state an appropriate dynamic programming principle that entails the existence of a viscosity subsolution of some Hamilton-Jacobi-Bellman equations associated with a control problem submitted to $(P_{z,\mu})$, in the finite dimensional setting.

2. Preliminaries

We provide here, basic notions which will be needed in the development of the paper. Throughout the paper $I := [0,T]$ $(T > 0)$ is an interval of $\mathbb{R}$ and H is a real *separable* Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and the associated norm by $\|\cdot\|$.

We use the following definitions and notations. We denote by $B[x,r]$ the closed ball of center x

and radius r on H, and by $\mathbf{1}_A$ the characteristic function of a set A, that is, $\mathbf{1}_A(x) = 1$ if $x \in A$ and 0 otherwise. We denote by λ the Lebesgue measure. On the space $\mathscr{C}_H(I)$ of continuous maps $x : I \to H$, we consider the norm of uniform convergence defined by $\|x\|_\infty = \sup\limits_{t\in I}\|x(t)\|$. Let X be a metric space, we denote by $\mathscr{C}(X)$ the set of all continuous functions from X into $\mathbb{R}$. When X is compact, the topological dual space of $(\mathscr{C}(X), \|\cdot\|_\infty)$ corresponds to the space $\mathscr{M}(X)$ of all Radon measures on X. By $L^p_H(I)$ for $p \in [1,+\infty[$ (resp. $p = +\infty$), we denote the space of measurable maps $x : I \to H$ such that $\int_I \|f(t)\|^p dt < +\infty$ (resp. which are essentially bounded) endowed with the usual norm $\|x\|_{L^p_H(I)} = (\int_I \|x(t)\|^p dt)^{\frac{1}{p}}$, $1 \le p < +\infty$ (resp. endowed with the usual essential supremum norm $\| \ \|$). Recall that the topological dual of $L^1_H(I)$ is $L^\infty_H(I)$. For any subset S of H, $\delta^*(S,\cdot)$ represents the support function of S, that is, for all $y \in H$,

$$\delta^*(S,y) := \sup_{x\in S}\langle y,x\rangle.$$

Let φ be a lsc convex function from H into $\mathbb{R} \cup \{+\infty\}$ which is proper in the sense that its effective domain $\operatorname{dom}\varphi$ defined by

$$\operatorname{dom}\varphi := \{x \in H : \varphi(x) < +\infty\}$$

is nonempty and, as usual, its Fenchel conjugate is defined by

$$\varphi^*(v) := \sup_{x\in H}[\langle v,x\rangle - \varphi(x)].$$

The subdifferential $\partial\varphi(x)$ of φ at $x \in \operatorname{dom}\varphi$ is

$$\partial\varphi(x) = \{v \in H : \varphi(y) \ge \langle v, y-x\rangle + \varphi(x) \ \forall y \in \operatorname{dom}\varphi\}$$

and its effective domain is $\operatorname{Dom}\partial\varphi = \{x \in H : \partial\varphi(x) \neq \emptyset\}$. It is well known (see, e.g., [5]) that if φ is a proper lsc convex function, then its subdifferential operator $\partial\varphi$ is a maximal monotone operator. Concerning the properties of maximal monotone operators in Hilbert spaces, we refer to [1] and [5]. We also refer to [13] for details concerning convex analysis and measurable set-valued mappings.

We will close this section of preliminaries by recalling the two following straightforward consequences of Gronwall's lemma proved in [16].

Lemma 2.1. *Let $(x_n(\cdot))$ be a sequence of absolutely continuous maps from $I := [T_0, T]$ to H. Assume that $\lim_n x_n(T_0) = 0$ and, for any n,*

$$\frac{d}{dt}(\|x_n(t)\|^2) \leq \beta_n(t)\|x_n(t)\|^2 + \alpha_n(t) \quad \text{a.e. } t \in I,$$

where $\alpha_n(\cdot)$ and $\beta_n(\cdot)$ are non-negative functions in $L^1_{\mathbb{R}}(I)$. Assume moreover that the sequence $(\beta_n(\cdot))$ is bounded in $L^1_{\mathbb{R}}(I)$ and $\lim_n \int_{T_0}^{T} \alpha_n(t)\, dt = 0$. Then,

$$\lim_n \|x_n(\cdot)\|_\infty = 0.$$

Lemma 2.2. *Let $(\eta_n(\cdot))$ be a sequence of non-negative absolutely continuous functions from $I := [T_0, T]$ to $\mathbb{R}$. Assume that $\lim_n \eta_n(T_0) = 0$ and, for any n,*

$$\dot{\eta}_n(t) \leq \beta(t)\eta_n(t) + \alpha_n(t) \quad \text{a.e. } t \in I,$$

where $\alpha_n(\cdot), \beta(\cdot) \in L^1_{\mathbb{R}}(I)$ with $\beta(\cdot) \geq 0$. Assume further that the sequence $(\alpha_n(\cdot))$ is bounded in $L^1_{\mathbb{R}}(I)$ and, for any $t \in I$, one has $\lim_n \int_{T_0}^{t} \alpha_n(s)\, ds = 0$. Then, for all $t \in I$,

$$\lim_n \eta_n(t) = 0.$$

3. Control problems governed by functional evolution inclusions involving subdifferential operators with Young measures

We investigate in the present section the continuous dependence of the solution of the following problem

$$(P_{\mathscr{R}}) \qquad \begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + \displaystyle\int_{\Gamma(t)} g(t, x(t), u)\mu_t(du) \quad \text{a.e. } t \in I \\ x(0) = x_0 \in \operatorname{dom}\varphi(0, \cdot), \end{cases}$$

with respect to the initial data x_0 and the control measure μ. To begin with, it seems convenient to recall some results concerning a perturbed problem whose perturbation is a single-valued time-dependent map and its application to control theory.

3.1 Single-valued time-dependent perturbations

The following theorem proved recently by Saïdi-Thibault-Yarou [28], ensures the existence and the uniqueness of an absolutely continuous solution for a Lipschitz perturbation of an evolution inclusion governed by the subdifferential operator (see Theorem 4.1 [28]).

Theorem 3.1. *Let* $I := [T_0, T]$. *Let* $\varphi : I \times H \to \mathbb{R}_+ \cup \{+\infty\}$ *be such that*

(H_1) *for each* $t \in I$, *the function* $x \mapsto \varphi(t,x)$ *is proper, lsc, and convex;*

(H_2) *there exist a non-negative* ρ*-Lipschitz function* $k : H \longrightarrow \mathbb{R}_+$ *and an absolutely continuous function* $a : I \to \mathbb{R}$, *with a non-negative derivative* $\dot{a} \in L^2_{\mathbb{R}}(I)$, *such that*

$$\varphi^*(t,x) \leq \varphi^*(s,x) + k(x)|a(t) - a(s)|$$

for every $(t,s,x) \in I \times I \times H$, *where* $\varphi^*(t,\cdot)$ *is the conjugate function of* $\varphi(t,\cdot)$ *(recalled above).*

Let $f : I \times H \to H$ *be a map such that*

- f *is separately measurable on* I;
- *for every* $\eta > 0$, *there exists a non-negative function* $\gamma_\eta(\cdot) \in L^2_{\mathbb{R}}(I)$ *such that, for all* $t \in I$ *and for any* $x, y \in B[0,\eta]$
$$\|f(t,x) - f(t,y)\| \leq \gamma_\eta(t)\|x - y\|;$$
- *there exists a non-negative function* $\beta(\cdot) \in L^2_{\mathbb{R}}(I)$ *such that, for all* $t \in I$ *and for all* $x \in H$, *one has*
$$\|f(t,x)\| \leq \beta(t)(1 + \|x\|).$$

Then, for any $x_0 \in \operatorname{dom}\varphi(T_0,\cdot)$, *the problem*

$$\begin{cases} -\dot{x}(t) \in \partial\varphi(t,x(t)) + f(t,x(t)) & \text{a.e. } t \in I \\ x(T_0) = x_0, \end{cases} \tag{1}$$

has one and only one absolutely continuous solution $x(\cdot)$ *on* I.

The following proposition gives properties concerning the unique solution of the foregoing problem with the initial condition x_0 (see Proposition 4.2 [28]).

Proposition 3.1. *The absolutely continuous solution* $x(\cdot)$ *of* (1) *satisfies*

$$\int_{T_0}^{T} \|\dot{x}(t)\|^2 dt \leq \alpha + \sigma \int_{T_0}^{T} \|f(t,x(t))\|^2 dt, \tag{2}$$

where

$$\begin{aligned}\alpha &= (k^2(0)+3(\rho+1)^2)\int_{T_0}^{T}\dot{a}^2(t)dt+2[T-T_0+\varphi(T_0,x_0)-\varphi(T,x(T))]\\ \sigma &= k^2(0)+3(\rho+1)^2+4.\end{aligned}$$

Further,

$$\|x(\cdot)\|_\infty \le K \quad \text{and} \quad \int_{T_0}^{T}\|\dot{x}(t)\|^2\,dt \le \alpha+\sigma(1+K)^2\int_{T_0}^{T}\beta^2(t)\,dt,$$

with

$$K := \|x_0\| + [\xi(T)]^{\frac{1}{2}};$$

and where $\xi(\cdot)$ *is the increasing, continuous, and non-negative function defined on* $[T_0,T]$ *by*

$$\xi(s) := b(s)+2\sigma(s-T_0)\int_{T_0}^{s} b(\tau)\beta^2(\tau)\exp(2\sigma\int_{\tau}^{s}\theta\beta^2(\theta)\,d\theta)\,d\tau,$$

and for each $t\in[T_0,T]$

$$b(t) := (t-T_0)[\alpha+2\sigma(1+\|x_0\|)^2\int_{T_0}^{t}\beta^2(\tau)\,d\tau].$$

Remark 3.1. *Since* $-\varphi(T,x(T))\le 0$, *this term can be omitted and the constant* α *may be replaced by* α^0 *in* (2) *where*

$$\alpha^0 = (k^2(0)+3(\rho+1)^2)\int_{0}^{T}\dot{a}^2(t)\,dt+2[T+\varphi(0,x_0)].$$

From now on, we mean by $\sigma,\ \alpha,\ \alpha^0$ the same constants defined in the preceding proposition. In order to apply the theorem of existence and uniqueness (Theorem 3.1) to control theory, let's recall some preliminary results about Young measures (see, e.g., [2, 12, 16, 26]).

3.2. Young measures, Carathéodory integrands and narrow convergence

Consider a complete measure space $(S,\mathscr{S},\sigma)$ with a non-negative finite measure σ and a complete separable metric space U. We call a Young measure on $S\times U$, any non-negative finite measure ν on $(S\times U,\mathscr{S}\otimes\mathscr{B}(U))$ that satisfies $\forall A\in\mathscr{S},\ \nu(A\times U)=\sigma(A)$. In other words, a Young measure on $\mathscr{S}\otimes\mathscr{B}(U)$ is a non-negative measure whose projection on S (that is, its image by the map $(s,u) \mapsto s$) is equal to σ. We denote by $\mathscr{Y}(S,\sigma,U)$, the set of Young measures.

Let $\mathscr{M}^1_+(U)$ be the set of all probability measures on $(U,\mathscr{B}(U))$. We denote by $\mathscr{Y}_{dis}(S,\sigma,U)$ (as in [12]), the set of maps $\mu : S \to \mathscr{M}^1_+(U)$ (up to σ-almost everywhere equality) such that, for any $B \in \mathscr{B}(U)$, the function $s \mapsto \mu_s(B)$ is $\mathscr{S}$-measurable.

Remark 3.2. *If $\mu \in \mathscr{Y}_{dis}(S,\sigma,U)$, $A \in \mathscr{S} \otimes \mathscr{B}(U)$ and if $\mathbf{1}_A$ is the characteristic function of A, then the function $s \mapsto \int_U \mathbf{1}_A(s,u)\mu_s(du)$ is $\mathscr{S}$-measurable on S and for all $A \in \mathscr{S} \otimes \mathscr{B}(U)$, the set function ν defined by*

$$\nu(A) = \int_S \int_U \mathbf{1}_A(s,u)\,\mu_s(du)\sigma(ds), \tag{3}$$

is a Young measure on $S \times U$.
We call any member of $\mathscr{Y}_{dis}(S,\sigma,U)$, a disintegrable Young measure. We also have, under the conditions on S and U (above), any Young measure on $S \times U$ is associated with some $\mu \in \mathscr{Y}_{dis}(S,\sigma,U)$ in the way above.

Remark 3.3.

(1) *If ν is the Young measure corresponding to the member $\mu \in \mathscr{Y}_{dis}(S,\sigma,U)$, i.e., the Young measure defined by (3), then, for any function $\phi : S \times U \to \mathbb{R} \cup \{-\infty,+\infty\}$ which is $\mathscr{S} \otimes \mathscr{B}(U)$-measurable and non-negative (resp. ν-integrable), the function $s \mapsto \int_U \phi(s,u)\mu_s(du)$ is σ-measurable (resp. σ-integrable) and we have*

$$\int_{S\times U} \phi\, d\nu = \int_S \int_U \phi(s,u)\,\mu_s(du)\sigma(ds).$$

(2) *If ν is a Young measure associated with some $\mu \in \mathscr{Y}_{dis}(S,\sigma,U)$ we don't distinct between ν and μ, that is, for all $s \in S$, we write ν_s instead of μ_s.*

(3) *Any map $u(\cdot) : S \to U$ which is $\mathscr{S}$-measurable, defines a Young measure on $S \times U$ called the Young measure associated with $u(\cdot)$. This is the Young measure corresponding to the member $\mu \in \mathscr{Y}_{dis}(S,\sigma,U)$ defined by $\mu_s := \delta_{u(s)}$, where $\delta_{u(s)}$ is the Dirac mass at the point $u(s)$, i.e, for any $B \in \mathscr{B}(U)$, $\delta_{u(s)}(B) = 1$ if $u(s) \in B$ and 0 otherwise.*

We call integrand any function $\phi : S \times U \to \mathbb{R} \cup \{-\infty,+\infty\}$ that is $\mathscr{S} \otimes \mathscr{B}(U)$-measurable. An integrand is said to be of Carathéodory type (resp. normal) if, for any $s \in S$, the partial function $\phi(s,\cdot)$ is continuous and takes finite values on U (resp. lower semi-continuous on U).

An integrand ϕ is said to be L^1-bounded if there exists some non-negative function $\gamma \in L^1_{\mathbb{R}}(S,\sigma)$ such that $|\phi(s,u)| \leq \gamma(s)$ for all $(s,u) \in S \times U$.

We endow the set $\mathscr{Y}(S,\sigma,U)$ with the narrow topology. Now, we recall that a sequence (ν^n) of $\mathscr{Y}(S,\sigma,U)$ converges narrowly to ν in $\mathscr{Y}(S,\sigma,U)$ if, for any L^1-bounded Carathéodory integrand ϕ,

$$\lim_n \int_{S\times U} \phi \, d\nu^n = \int_{S\times U} \phi \, d\nu. \tag{4}$$

We also say that a sequence (μ^n) of $\mathscr{Y}_{dis}(S,\sigma,U)$ converges in $\mathscr{Y}_{dis}(S,\sigma,U)$ to μ if the sequence of the corresponding Young measures converges in $\mathscr{Y}(S,\sigma,U)$, that is, for any L^1-bounded Carathéodory integrand ϕ,

$$\lim_n \int_S \int_U \phi(s,u)\mu^n_s(du)\sigma(ds) = \int_S \int_U \phi(s,u)\mu_s(du)\sigma(ds). \tag{5}$$

The space $\mathscr{Y}_{dis}(I,\lambda,U)$ has the following compactness property (see [12]).

Proposition 3.2. *If U is a compact metric space, then any sequence in $\mathscr{Y}_{dis}(I,\lambda,U)$ has a subsequence that converges in $\mathscr{Y}_{dis}(I,\lambda,U)$.*

We need the following Fiber product theorem

Theorem 3.2. *Consider a compact interval of $\mathbb{R}$ I and two complete separable metric spaces X, Y. Let $(\mu^n)_n$ be a sequence in $\mathscr{Y}(I,\lambda,X)$, and $(\nu^n)_n$ be a sequence in $\mathscr{Y}(I,\lambda,Y)$. Consider a λ-measurable map $u : I \to X$. Suppose that*

- *(i) $(\mu^n)_n$ converges narrowly to $\delta_{u(\cdot)} \in \mathscr{Y}(I,\lambda,X)$,*
- *(ii) $(\nu^n)_n$ converges narrowly to $\nu^\infty \in \mathscr{Y}(I,\lambda,Y)$,*

Then, $(\mu^n \otimes \nu^n)_{n\in\mathbb{N}}$ converges narrowly to $\delta_{u(\cdot)} \otimes \nu^\infty$ in $\mathscr{Y}(I,\lambda,X \times Y)$, where

$$\forall s \in I,\ (\mu^n \otimes \nu^n)_s := \mu^n_s \otimes \nu^n_s \quad \text{and} \quad (\delta_{u(\cdot)} \otimes \nu^\infty)_s := \delta_{u(s)} \otimes \nu^\infty_s.$$

Note that convergence in probability is equivalent to narrow convergence in the sense of (i).

We give the following lemma (Lemma 2. [16])

Lemma 3.1. *Suppose further that U is a compact space. Consider λ-measurable maps $h_n(\cdot), h_\infty(\cdot)$: $I \to H$ $(n \geq 1)$ and ν^n, $\nu^\infty \in \mathscr{Y}(I,\lambda,U)$. Suppose that (ν^n) converges to ν^∞ in $\mathscr{Y}(I,\lambda,U)$ and $(h_n(t))$ converges weakly in H to $h_\infty(t)$ for all $t \in I$. Let $\theta^n, \theta^\infty \in \mathscr{Y}(I,\lambda,H \times U)$ be defined by $\theta^n_t := \delta_{h_n(t)} \otimes \nu^n_t$ and $\theta^\infty_t := \delta_{h_\infty(t)} \otimes \nu^\infty_t$. Consider an integrand $\phi : I \times (H \times U) \to \mathbb{R}$*

such that, for any $t \in I$, $\phi(t,\cdot,\cdot)$ *is sequentially continuous on* $H^w \times U$*, where* H^w *denotes the space* H *endowed with the weak topology. Suppose also that the measurable function* $t \mapsto \sup_{(n,u)\in(\mathbb{N}\cup\{\infty\})\times U} |\phi(t,h_n(t),u)|$ *is* λ*-integrable on* I*. Then, we have*

$$\lim_{n\to\infty} \int_{I\times H\times U} \phi \, d\theta^n = \int_{I\times H\times U} \phi \, d\theta.$$

3.3 Original controls and relaxed controls

From now on, the metric space U is assumed to be *compact*.
Let $\Gamma : I \to U$ be a λ-measurable set-valued map with nonempty compact values. Let us consider the set-valued map $\Sigma(\cdot)$ defined on I by

$$\Sigma(t) = \{P \in \mathscr{M}_+^1(U) : P(\Gamma(t)) = 1\}.$$

Denote by S_Γ (resp. S_Σ) the set of all λ-measurable selections (up to almost everywhere equality) of Γ (resp. Σ). The set S_Σ is nonempty. In fact, the following result holds (see, e.g., [13, 26]).

Proposition 3.3. *Let* $\Gamma : I \to U$ *be a* λ*-measurable set-valued map with nonempty compact values. The set-valued map* $\Sigma(\cdot)$ *defined on* I *by*

$$\Sigma(t) = \{P \in \mathscr{M}_+^1(U) : P(\Gamma(t)) = 1\}$$

is λ*-measurable with nonempty compact convex values and the set* S_Σ *is nonempty and sequentially closed in* $\mathscr{Y}_{dis}(I,\lambda,U)$.

The members of S_Γ are called original controls and those of S_Σ relaxed controls. Clearly, $S_\Gamma \subset S_\Sigma$ in the sense that, for any $\zeta \in S_\Gamma$, the Young measure μ with $(\mu_t := \delta_{\zeta(t)})_{t\in I}$ satisfies $\mu \in S_\Sigma$.

Let $g : I \times H \times U \longrightarrow H$ be a map satisfying:

(i) for any $t \in I$, $g(t,\cdot,\cdot)$ is continuous on $H \times U$;

(ii) for any $(x,u) \in H \times U$, $g(\cdot,x,u)$ is λ-measurable on I;

(iii) for any $\eta > 0$, there exists a non-negative function $\gamma_\eta(\cdot) \in L^2_{\mathbb{R}}(I)$ such that, for all $(t,u) \in I \times U$ and for all $x,y \in B[0,\eta]$

$$\|g(t,x,u) - g(t,y,u)\| \leq \gamma_\eta(t)\|x-y\|;$$

(iv) there exists a non-negative function $\beta(\cdot) \in L^2_{\mathbb{R}}(I)$ such that, for all $(t,x,u) \in I \times H \times U$, one has

$$\|g(t,x,u)\| \leq \beta(t)(1+\|x\|).$$

For an arbitrary μ in the set S_Σ of relaxed controls, let us study the evolution inclusion governed by the subdifferential operator with a perturbation containing Young measures, namely $(P_{\mathscr{R}})$.

Theorem 3.3. *Let $\mu \in S_\Sigma$. Assumptions (H_1)-(H_2) and (i)-(iv) above are made, for $h_\mu : I \times H \to H$*

$$h_\mu(t,x) := \int_{\Gamma(t)} g(t,x,u)\mu_t(du);$$

one has

(I) *for any $x \in H$, $h_\mu(\cdot,x)$ is λ-measurable on I;*

(II) *for any $\eta > 0$, for all $t \in I$ and for all $x,y \in B[0,\eta]$,*

$$\|h_\mu(t,x) - h_\mu(t,y)\| \leq \gamma_\eta(t)\|x-y\|;$$

(III) *for all $(t,x) \in I \times H$, one has*

$$\|h_\mu(t,x)\| \leq \beta(t)(1+\|x\|).$$

Moreover, the evolution problem $(P_{\mathscr{R}})$ admits a unique absolutely continuous solution $x_\mu : I \to H$ that satisfies

(a) *$\dot{x}_\mu \in L^2_H(I)$ and*

$$\int_0^T \|\dot{x}_\mu(t)\|^2 dt \leq \alpha^0 + \sigma \int_0^T \|h_\mu(t,x_\mu(t))\|^2 \, dt;$$

(b) *there exists a positive real number $\xi_{0,T}$ depending on x_0, $\varphi(0,x_0)$, $\beta(\cdot)$, T and independent of $\mu \in S_\Sigma$ such that*

$$\sup\{\|x_\mu(t)\| : t \in I,\ \mu \in S_\Sigma\} \leq \|x_0\| + [\xi(T)]^{\frac{1}{2}} := \xi_{0,T}$$

and

$$\|x_\mu(t)-x_\mu(s)\| \le (t-s)^{\frac{1}{2}}[\alpha^0+\sigma(1+\xi_{0,T})^2\int_0^T \beta^2(t)\,dt]^{\frac{1}{2}},$$

for any $s,t \in I$ $(s \le t)$.

Proof. According to Remark (1), the map

$$h_\mu(t,x) = \int_{\Gamma(t)} g(t,x,u)\mu_t(du) = \int_U g(t,x,u)\mu_t(du)$$

is separately λ-measurable on I. Moreover, thanks to the assumptions on g and the fact that $\mu_t(\Gamma(t)) = \mu_t(U) = 1$, (II)-(III) hold true. Thus, Theorem 3.1 applied to φ, h_μ and x_0 provides a unique absolutely continuous solution $x_\mu : I \to H$ of $(P_{\mathscr{R}})$, for which properties in (a) are immediate (in view of Proposition 3.1).

Another helpful conclusion of Proposition 3.1. ensures that for any $\mu \in S_\Sigma$ and any $t \in I$,

$$\|x_\mu(t)-x_0\| \le [\xi(t)]^{\frac{1}{2}},$$

where

$$\xi(t) := b(t) + 2\sigma t \int_0^t b(\tau)\beta^2(\tau)\exp(2\sigma\int_\tau^t \theta\beta^2(\theta)\,d\theta)\,d\tau,$$

with

$$b(t) := t[\alpha + 2\sigma(1+\|x_0\|)^2\int_0^t \beta^2(\tau)\,d\tau].$$

In particular, this reveals that

$$\sup\{\|x_\mu(t)\| : t \in I,\ \mu \in S_\Sigma\} \le \|x_0\| + [\xi(T)]^{\frac{1}{2}} := \xi_{0,T} \in \mathbb{R}.$$

Moreover, thanks to the absolute continuity of x_μ and (a), for $0 \le s \le t \le T$,

$$\begin{aligned}\|x_\mu(t)-x_\mu(s)\| &= \|\int_s^t \dot{x}_\mu(\tau)\,d\tau\| \le (t-s)^{\frac{1}{2}}(\int_0^T \|\dot{x}_\mu(\tau)\|^2\,d\tau)^{\frac{1}{2}} \\ &\le (t-s)^{\frac{1}{2}}[\alpha^0+\sigma(1+\xi_{0,T})^2\int_0^T \beta^2(t)\,dt]^{\frac{1}{2}},\end{aligned}$$

then, (b) holds true.

In the remaining of the paper, given $\mu \in S_\Sigma$, $x_\mu(\cdot)$ will automatically denote the absolutely continuous solution of the differential inclusion $(P_{\mathscr{R}})$.

3.4. Continuous dependence of the solution of $(P_{\mathscr{R}})$ with respect to the initial position x_0 and μ

Now, we address the important question of the continuous dependence of the solution of $(P_{\mathscr{R}})$ with respect to the initial data x_0 and μ. It will play a crucial rule in our development.

Theorem 3.4. *Suppose that hypothesis (H_1)-(H_2) and (i)-(iv) above are made. Let $z_\infty \in \operatorname{dom}\varphi(0,\cdot)$ be given and suppose that*

- (H_3) *$(z_n)_n \subset H$ converges strongly to z_∞ with $\sup_n \varphi(0,z_n) < +\infty$;*
- (H_4) *$(\mu^n)_n \subset S_\Sigma$ converges to $\mu^\infty \in S_\Sigma$, with respect to $w(L^\infty_{\mathscr{M}(U)}(I), L^1_{\mathscr{C}(U)}(I))$-topology on S_Σ.*

Then, the sequence $(x_{z_n,\mu^n}(\cdot))_n$ of absolutely continuous solutions of the problem

$$(P_{z_n,\mu^n}) \qquad \begin{cases} -\dot{x}_{z_n,\mu^n}(t) \in \partial\varphi(t, x_{z_n,\mu^n}(t)) + \int_{\Gamma(t)} g(t, x_{z_n,\mu^n}(t), u)\mu^n_t(du) \\ \quad \text{a.e. } t \in I, \\ x_{z_n,\mu^n}(0) = z_n, \end{cases}$$

converges uniformly on I to the unique absolutely continuous solution of $(P_{z_\infty,\mu^\infty})$.

Proof. We show that any subsequence of $(x_{z_n,\mu^n}(\cdot))_n$ converges uniformly on I to $x_{z_\infty,\mu^\infty}(\cdot)$. Recall that, for each n, $x_{z_n,\mu^n}(\cdot)$ denotes the unique absolutely continuous solution of

$$\text{(6)} \qquad \begin{cases} -\dot{x}_{z_n,\mu^n}(t) \in \partial\varphi(t, x_{z_n,\mu^n}(t)) + h_{\mu^n}(t, x_{z_n,\mu^n}(t)) \quad \text{a.e. } t \in I \\ x_{z_n,\mu^n}(0) = z_n \in \operatorname{dom}\varphi(0,\cdot), \end{cases}$$

and $x_{z_\infty,\mu^\infty}(\cdot)$ is the unique absolutely continuous solution of

$$\text{(7)} \qquad \begin{cases} -\dot{x}_{z_\infty,\mu^\infty}(t) \in \partial\varphi(t, x_{z_\infty,\mu^\infty}(t)) + h_{\mu^\infty}(t, x_{z_\infty,\mu^\infty}(t)) \quad \text{a.e. } t \in I \\ x_{z_\infty,\mu^\infty}(0) = z_\infty \in \operatorname{dom}\varphi(0,\cdot). \end{cases}$$

In view of Theorem 3.3, for any $n \in \mathbb{N} \cup \{+\infty\}$, we may write

$$\text{(8)} \qquad \|x_{z_n,\mu^n}(\cdot)\|_\infty \le \|z_n\| + [\xi_n(T)]^{\frac{1}{2}}$$

where

$$\xi_n(T) := b_n(T) + 2\sigma T \int_0^T b_n(\tau)\beta^2(\tau)\exp(2\sigma\int_\tau^T \theta\beta^2(\theta)\, d\theta)\, d\tau,$$

with for any $t \in I$,

$$\begin{aligned} b_n(t) &:= t[\alpha_n + 2\sigma(1+\|z_n\|)^2 \int_0^t \beta^2(\tau)\,d\tau], \\ \alpha_n &:= (k^2(0)+3(\rho+1)^2)\int_0^T \dot{a}^2(t)\,dt + 2[T+\varphi(0,z_n)]. \end{aligned}$$

From (a) and (III) of Theorem 3.3, one has for any n,

$$\int_0^T \|\dot{x}_{z_n,\mu^n}(t)\|^2\,dt \leq \alpha_n + \sigma(1+\|z_n\| + [\xi_n(T)]^{\frac{1}{2}})^2 \int_0^T \beta^2(t)\,dt. \tag{9}$$

Define now,

$$M := \sup_{n\in\mathbb{N}\cup\{+\infty\}} \varphi(0,z_n).$$

Thanks to (H_3), the element M lies in $\mathbb{R}_+$. Put

$$\eta_T := \sup_{n\in\mathbb{N}\cup\{+\infty\}} (\|z_n\| + [\xi_n(T)]^{\frac{1}{2}}),$$

which is finite by assumption (H_3) and

$$w_T := \alpha_T + \sigma(1+\eta_T)^2 \int_0^T \beta^2(t)\,dt,$$

where

$$\alpha_T := (k^2(0)+3(\rho+1)^2)\int_0^T \dot{a}^2(t)\,dt + 2[T+M].$$

It results from (8) and (H_3) that

$$\sup_{n\in\mathbb{N}\cup\{+\infty\}} \|x_{z_n,\mu^n}(\cdot)\|_\infty \leq \eta_T < +\infty,$$

and then, for any $t \in I$, via (III) from Theorem 3.3, one gets

$$\sup_{n\in\mathbb{N}\cup\{+\infty\}} \|h_{\mu^n}(t, x_{z_n,\mu^n}(t))\| \leq (1+\eta_T)\beta(t) \quad \text{a.e. } t \in I. \tag{10}$$

Furthermore, in view of (9) one obtains, for any n,

$$\sup_{n\in\mathbb{N}\cup\{+\infty\}} \int_0^T \|\dot{x}_{z_n,\mu^n}(t)\|^2\,dt \leq w_T < +\infty. \tag{11}$$

Let's prove that $(x_{z_n,\mu^n}(\cdot))$ converges uniformly in $\mathscr{C}_H(I)$ to $x_{z_\infty,\mu^\infty}(\cdot)$ by proving that any subsequence of $(x_{z_n,\mu^n}(\cdot))$ has a subsequence converging uniformly in $\mathscr{C}_H(I)$ to $x_{z_\infty,\mu^\infty}(\cdot)$.

Fix any subsequence of $(x_{z_n,\mu^n}(\cdot))$, that we do not relabel. Next, in the light of (11), extracting another subsequence, we may suppose that the corresponding subsequence $(\dot{x}_{z_n,\mu^n}(\cdot))$ converges weakly in $L^2_H(I)$ to some map $a(\cdot)$. Hence, for any $t \in I$,

$$\int_0^t \dot{x}_{z_n,\mu^n}(s)\,ds \longrightarrow \int_0^t a(s)\,ds \quad \text{weakly in } H.$$

Define a map $x(\cdot) \in \mathscr{C}_H(I)$ by

$$x(t) := z_\infty + \int_0^t a(s)ds,$$

it results, for all $t \in I$

$$x_{z_n,\mu^n}(t) \longrightarrow x(t) \quad \text{weakly in } H.$$

We prove the pointwise convergence of the subsequence $(x_{z_n,\mu^n}(\cdot))$ to $x_{z_\infty,\mu^\infty}(\cdot)$. From the definitions of $x_{z_n,\mu^n}(\cdot)$ and $x_{z_\infty,\mu^\infty}(\cdot)$, we have, for almost $t \in I$ and all n

$$\begin{aligned}
&-\dot{x}_{z_n,\mu^n}(t) \in \partial\varphi(t, x_{z_n,\mu^n}(t)) + h_{\mu^n}(t, x_{z_n,\mu^n}(t))\\
&-\dot{x}_{z_\infty,\mu^\infty}(t) \in \partial\varphi(t, x_{z_\infty,\mu^\infty}(t)) + h_{\mu^\infty}(t, x_{z_\infty,\mu^\infty}(t))\\
&x_{z_n,\mu^n}(0),\ x_{z_\infty,\mu^\infty}(0) \in \operatorname{dom}\varphi(0,\cdot).
\end{aligned}$$

The monotonicity property of $\partial\varphi(t,\cdot)$ ensures that

$$\langle -\dot{x}_{z_n,\mu^n}(t) + \dot{x}_{z_\infty,\mu^\infty}(t) - h_{\mu^n}(t, x_{z_n,\mu^n}(t)) + h_{\mu^\infty}(t, x_{z_\infty,\mu^\infty}(t)), x_{z_n,\mu^n}(t) - x_{z_\infty,\mu^\infty}(t)\rangle \geq 0,$$

for all n and almost $t \in I$. Hence,

$$\begin{aligned}
\langle \dot{x}_{z_n,\mu^n}(t) - \dot{x}_{z_\infty,\mu^\infty}(t), x_{z_n,\mu^n}(t) - x_{z_\infty,\mu^\infty}(t)\rangle \quad &\leq \\
&\langle -h_{\mu^n}(t, x_{z_n,\mu^n}(t)) + h_{\mu^\infty}(t, x_{z_\infty,\mu^\infty}(t)), x_{z_n,\mu^n}(t) - x_{z_\infty,\mu^\infty}(t)\rangle,
\end{aligned}$$

that is,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}(\|x_{z_n,\mu^n}(t) - x_{z_\infty,\mu^\infty}(t)\|^2) \quad &\leq \\
&\langle h_{\mu^n}(t, x_{z_n,\mu^n}(t)) - h_{\mu^\infty}(t, x_{z_\infty,\mu^\infty}(t)), x_{z_\infty,\mu^\infty}(t) - x_{z_n,\mu^n}(t)\rangle.
\end{aligned}$$

Remark that,

$$\langle h_{\mu^n}(t,x_{z_n,\mu^n}(t))-h_{\mu^\infty}(t,x_{z_\infty,\mu^\infty}(t)),x_{z_\infty,\mu^\infty}(t)-x_{z_n,\mu^n}(t)\rangle$$
$$=\langle h_{\mu^n}(t,x_{z_n,\mu^n}(t))-h_{\mu^n}(t,x_{z_\infty,\mu^\infty}(t)),x_{z_\infty,\mu^\infty}(t)-x_{z_n,\mu^n}(t)\rangle$$
$$+\langle h_{\mu^n}(t,x_{z_\infty,\mu^\infty}(t))-h_{\mu^\infty}(t,x_{z_\infty,\mu^\infty}(t)),x_{z_\infty,\mu^\infty}(t)-x_{z_n,\mu^n}(t)\rangle.$$

Due to (11) and since $x_{z_n,\mu^n}(\cdot)$, $x_{z_\infty,\mu^\infty}(\cdot)$ are continuous from I to$H$, there exists $\eta>0$ such that, for all n and for almost all $t\in I$,

$$x_{z_n,\mu^n}(t),x_{z_\infty,\mu^\infty}(t)\in B[0,\eta].$$

Thanks to the assumptions on g, there exists a non-negative function $\gamma_\eta(\cdot)\in L^2_{\mathbb{R}}(I)$ such that, for any n and for any $t\in I$, $h_{\mu^n}(t,\cdot)$ is $\gamma_\eta(t)$-Lipschitz on $B[0,\eta]$. Then, one has

$$\frac{d}{dt}(\|x_{z_n,\mu^n}(t)-x_{z_\infty,\mu^\infty}(t)\|^2)\le 2\gamma_\eta(t)\|x_{z_n,\mu^n}(t)-x_{z_\infty,\mu^\infty}(t)\|^2+2\varsigma_n(t), \tag{12}$$

for all n and for almost all $t\in I$, where

$$\varsigma_n(t)=\langle h_{\mu^n}(t,x_{z_\infty,\mu^\infty}(t))-h_{\mu^\infty}(t,x_{z_\infty,\mu^\infty}(t)),x_{z_\infty,\mu^\infty}(t)-x_{z_n,\mu^n}(t)\rangle. \tag{13}$$

Taking the assumption (III) of Theorem 3.3, and the definitions of h_{μ^n} and h_{μ^∞} into account, for any n and for all $t\in I$, one gets

$$|\varsigma_n(t)|\le 4\eta(1+\eta)\beta(t). \tag{14}$$

So, the sequence $(\varsigma_n(\cdot))$ is bounded in $L^1_{\mathbb{R}}(I)$. Next, we prove that
$\lim_n\int_0^s\varsigma_n(t)\,dt=0$ for any $s\in I$. Using the definitions of h_{μ^n} and h_{μ^∞},

$$\begin{aligned}\int_0^s\varsigma_n(t)dt &= \int_0^s\int_{\Gamma(t)}\langle g(t,x_{z_\infty,\mu^\infty}(t),u),x_{z_\infty,\mu^\infty}(t)-x_{z_n,\mu^n}(t)\rangle\mu^n_t(du)dt\\ &\quad-\int_0^s\int_{\Gamma(t)}\langle g(t,x_{z_\infty,\mu^\infty}(t),u),x_{z_\infty,\mu^\infty}(t)-x_{z_n,\mu^n}(t)\rangle\mu^\infty_t(du)dt.\end{aligned}$$

Consider for $(t,v,u)\in I\times H\times U$,

$$\phi(t,v,u):=\langle g(t,x_{z_\infty,\mu^\infty}(t),u),x_{z_\infty,\mu^\infty}(t)-v\rangle\mathbf{1}_{[0,s]}(t).$$

Then, for any $t \in I$, the function $\phi(t,\cdot,\cdot)$ is sequentially continuous on $H^w \times U$. Set $x_{z_\infty,\mu^\infty}(\cdot) := x(\cdot)$, for all $(t,n,u) \in I \times (\mathbb{N} \cup \{\infty\}) \times U$, one gets

$$|\phi(t, x_{z_n,\mu^n}(t), u)| \leq 2\eta(1+\eta)\beta(t). \tag{15}$$

This equality holds true

$$\begin{aligned}\int_0^s \varsigma_n(t)\, dt &= \textstyle\int_0^T \int_U \phi(t, x_{z_n,\mu^n}(t), u)\mu_t^n(du)\, dt - \\ &\quad \textstyle\int_0^T \int_U \phi(t, x_{z_n,\mu^n}(t), u)\mu_t^\infty(du)\, dt,\end{aligned}$$

because for each n and for each $t \in I$, μ_t^n and μ_t^∞ are probability measures satisfying $\mu_t^n(\Gamma(t)) = \mu_t^\infty(\Gamma(t)) = 1$.

Define the Young measures $\theta^n, \rho^n, \theta \in \mathscr{Y}(I, \lambda, H \times U)$ by $\theta_t^n := \delta_{x_{z_n,\mu^n(t)}} \otimes \mu_t^n$, $\rho_t^n := \delta_{x_{z_n,\mu^n(t)}} \otimes \mu_t^\infty$, and $\theta_t := \delta_{x(t)} \otimes \mu_t^\infty$, we may write the last equality as follows

$$\int_0^s \varsigma_n(t)\, dt = \int_{I\times H\times U} \phi\, d\theta^n - \int_{I\times H\times U} \phi\, d\rho^n.$$

In view of Lemma 3.1, one gets

$$\lim_{n\to\infty} \int_0^s \varsigma_n(t)\, dt = \int_{I\times H\times U} \phi\, d\theta - \int_{I\times H\times U} \phi\, d\theta = 0.$$

Then, we deduce that the subsequence $(x_{z_n,\mu^n}(\cdot))$ converges pointwise to $x_{z_\infty,\mu^\infty}(\cdot)$ by applying Lemma 2.2 to the inequality (12).

It remains to prove that the convergence is uniform. Due to (12) and (13), we have, for almost all $t \in I$,

$$\frac{d}{dt}(\|x_{z_n,\mu^n}(t) - x_{z_\infty,\mu^\infty}(t)\|^2) \leq 2\gamma_\eta(t)\|x_{z_n,\mu^n}(t) - x_{z_\infty,\mu^\infty}(t)\|^2 + 2|\varsigma_n(t)|,$$

and

$$\begin{aligned}|\varsigma_n(t)| &\leq (\|h_{\mu^n}(t, x_{z_\infty,\mu^\infty}(t))\| + \|h_{\mu^\infty}(t, x_{z_\infty,\mu^\infty}(t))\|)\|x_{z_\infty,\mu^\infty}(t) - x_{z_n,\mu^n}(t)\| \\ &\leq 2(1+\eta)\beta(t)\|x_{z_\infty,\mu^\infty}(t) - x_{z_n,\mu^n}(t)\|.\end{aligned}$$

It results that $\varsigma_n(t) \to 0$ for almost all $t \in I$, because of the pointwise convergence of the subsequence $(x_{z_n,\mu^n}(\cdot))$ to $x_{z_\infty,\mu^\infty}(\cdot)$ on I. Combining with (14), entail, via the dominated convergence theorem,

$$\lim_{n\to\infty}\int_0^T |\varsigma_n(t)|\, dt = 0.$$

Thanks to Lemma 2.1, the subsequence $(x_{z_n,\mu^n}(\cdot))$ converges uniformly to $x_{z_\infty,\mu^\infty}(\cdot)$ in $\mathscr{C}_H(I)$. Thus, $x_{z_\infty,\mu^\infty}(\cdot)$ is the unique uniform cluster point on I of any subsequence $(x_{z_n,\mu^n}(\cdot))_n$. This implies that the whole sequence $(x_{z_n,\mu^n}(\cdot))_n$ converges uniformly on I to the unique absolutely continuous solution of $(P_{z_\infty,\mu^\infty})$, namely $x_{z_\infty,\mu^\infty}(\cdot)$.

Corollary 3.1. *Under hypothesis (H_1)-(H_2) and (i)-(iv), let $x_0 \in \operatorname{dom}\varphi(0,\cdot)$ be a fixed initial datum. Then, the map*

$$G : S_\Sigma \to \mathscr{C}_H(I),\ \mu \mapsto G(\mu) := x_\mu(\cdot),$$

where $x_\mu(\cdot)$ is the unique absolutely continuous solution of $(P_{\mathscr{R}})$, is continuous with respect to $w(L^\infty_{\mathscr{M}(U)}(I), L^1_{\mathscr{C}(U)}(I))$-topology on S_Σ and the uniform convergence topology on $\mathscr{C}_H(I)$.

Proof. Under our hypothesis on U and $\Gamma(\cdot)$, S_Σ is compact for the $w(L^\infty_{\mathscr{M}(U)}(I), L^1_{\mathscr{C}(U)}(I))$-topology and the induced topology on S_Σ is metrizable (because $L^1_{\mathscr{C}(U)}(I)$ is separable and S_Σ is included in the unit ball of $L^\infty_{\mathscr{M}(U)}(I)$). Thus, from the sequential continuity of the map $G(\cdot)$, for the topologies under consideration results its continuity. The latter is a straightforward consequence of Theorem 3.4.

We close this section by recalling the following relaxed problem (see Theorem 5.9 [28])

Theorem 3.5. *Under assumptions (H_1)-(H_2) and (i)-(iv), suppose that, the cost functional $J : I \times H \times U \longrightarrow \mathbb{R}$ is an integrand such that for each $t \in I$, $J(t,\cdot,\cdot)$ is continuous on $H \times U$, J is also bounded from below. Assume further that for any bounded sequence $(x_n(\cdot))$ in $(\mathscr{C}_H(I), \|\cdot\|_\infty)$ and for any sequence $(\zeta_n(\cdot))$ in S_Γ, the sequence $(J(\cdot, x_n(\cdot), \zeta_n(\cdot)))$ is uniformly integrable in $L^1_{\mathbb{R}}(I)$. Denote by $x_\mu(\cdot)$ (resp. $x_\zeta(\cdot)$) the unique absolutely continuous solution of $(P_{\mathscr{R}})$ (resp. $(P_{\mathscr{O}})$), with for any $\zeta(\cdot) \in S_\Gamma$*

$$(P_{\mathscr{O}}) \qquad \begin{cases} -\dot{x}(t) \in \partial\varphi(t, x(t)) + g(t, x(t), \zeta(t)) & \text{a.e. } t \in I \\ x(0) = x_0 \in \operatorname{dom}\varphi(0,\cdot). \end{cases}$$

Then, the control problem $(P.R)$ *has an optimal solution. Furthermore, one has*

$$\min(P.R) = \inf(P.O),$$

where

$$\inf_{\zeta(\cdot)\in S_\Gamma} \int_I J(t, x_\zeta(t), \zeta(t))\, dt \tag{P.O}$$

and

$$\inf_{\mu\in S_\Sigma} \int_I \int_U J(t, x_\mu(t), u)\mu_t(du)\, dt. \tag{P.R}$$

The latter is called the relaxed problem.

4. Application to control and viscosity

4.1. Some semi-continuity properties of the value function associated with $(P_{y,\mu})$

We study here, the value function of control problems submitted to perturbed differential inclusions similar to $(P_{\mathscr{R}})$, the controls are Young measures. The continuous dependence result is crucial in the statement of the variational properties of the value functions in the control problems under consideration and the development below.

We refer to [25], in the particular case of ordinary differential equations (that is $\varphi \equiv 0$), for a study of the sensitivity and the convexity up to a square property of the corresponding value functions.

Proposition 4.1. *Suppose that hypothesis* (H_1)-(H_2) *and* (i)-(iv) *above are made. Let* $\psi : H \to \mathbb{R}\cup\{+\infty\}$ *be a proper lower semi-continuous function which is bounded below. Let* $t \in I$ *and* $z \in \operatorname{dom}\varphi(0,\cdot)$. *Moreover, suppose that* $(\psi_n)_n$ *is an increasing sequence of bounded below continuous functions on* H, *that converges pointwisely to* ψ.

Then, for any sequences $(t_n)_{n\geq 1} \subset I$ *converging to* t, *and* $(z_n) \subset \operatorname{dom}\varphi(0,\cdot)$ *converging to* z, *one has*

$$\liminf_{n\to+\infty} W_{\psi_n}(z_n, t_n) \geq W_\psi(z, t),$$

where $W_\alpha(y,s) = \inf_{\mu \in S_\Sigma} \alpha(x_{y,\mu}(s))$ *for any function* $\alpha : H \to \mathbb{R} \cup \{+\infty\}$ *that is bounded below and all* $(y,s) \in H \times I$, $x_{y,\mu}(\cdot)$ *is the unique absolutely continuous solution of* $(P_{y,\mu})$.

Proof. Fix any integer $n \geq 1$. Thanks to Corollary 3.1 and the continuity of $\psi_n(\cdot)$ on H, the function $\mu \mapsto \psi_n(x_{z_n,\mu}(t_n))$ is continuous on S_Σ endowed with the $w(L^\infty_{\mathscr{M}(U)}(I), L^1_{\mathscr{C}(U)}(I))$-topology. Since S_Σ is a compact space, there exists some relaxed control $\mu^n \in S_\Sigma$ such that

$$\inf_{\mu \in S_\Sigma} \psi_n(x_{z_n,\mu}(t_n)) = W_{\psi_n}(z_n,t_n) = \psi_n(x_{z_n,\mu^n}(t_n)), \tag{16}$$

with $x_{z_n,\mu^n}(\cdot)$ is the absolutely continuous solution of (P_{z_n,μ^n}).
The sequential compactness of S_Σ yields that there is a subsequence of $(\mu^n)_{n\geq 1}$ (still denoted (μ^n)) which converges to some measure $\mu \in S_\Sigma$, with respect to $w(L^\infty_{\mathscr{M}(U)}(I), L^1_{\mathscr{C}(U)}(I))$-topology.
The assumptions of Theorem 3.4 are satisfied. Hence, the latter ensures that the sequence $(x_{z_n,\mu^n}(\cdot))_n$ converges uniformly on I to $x_{z,\mu}(\cdot)$, the absolutely continuous solution of $(P_{z,\mu})$.

Fix any integer $k \geq 1$. In view of (16) and the increasing behavior of $(\psi_n)_n$ for any $n \geq k$, one gets

$$W_{\psi_n}(z_n,t_n) = \psi_n(x_{z_n,\mu^n}(t_n)) \geq \psi_k(x_{z_n,\mu^n}(t_n)). \tag{17}$$

Thanks to the uniform convergence on I of $(x_{z_n,\mu^n}(\cdot))_n$ to $x_{z,\mu}(\cdot)$ and the continuity at t of $x_{z,\mu}(\cdot)$, one has

$$\begin{aligned} &\limsup_{n\to+\infty} \|x_{z_n,\mu^n}(t_n) - x_{z,\mu}(t)\| \\ &\leq \limsup_{n\to+\infty} (\|x_{z_n,\mu^n}(\cdot) - x_{z,\mu}(\cdot)\|_\infty + \|x_{z,\mu}(t_n) - x_{z,\mu}(t)\|) = 0. \end{aligned}$$

Then, making use of the continuity of the function $\psi_k(\cdot)$ on H entails that

$$\lim_{n\to+\infty} \psi_k(x_{z_n,\mu^n}(t_n)) = \psi_k(x_{z,\mu}(t)).$$

Next, passing to the limit inferior on n in (17), one has for any $k \in \mathbb{N}$

$$\liminf_{n\to+\infty} W_{\psi_n}(z_n,t_n) \geq \psi_k(x_{z,\mu}(t)).$$

Hence,

$$\liminf_{n\to+\infty} W_{\psi_n}(z_n,t_n) \geq \sup_{k\in\mathbb{N}} \psi_k(x_{z,\mu}(t)) = \psi(x_{z,\mu}(t)),$$

the last equality follows from the increasing pointwise convergence of $(\psi_k)_k$ to ψ. Consequently, since $\mu \in S_\Sigma$, it results that

$$\liminf_{n\to+\infty} W_{\psi_n}(z_n,t_n) \geq \inf_{\mu\in S_\Sigma} \psi(x_{z,\mu}(t)) = W_\psi(z,t).$$

Proposition 4.2. *Suppose that hypothesis (H_1)-(H_2) and (i)-(iv) above are made. Consider an L^1-bounded normal integrand $\phi : I\times H\times U \to \mathbb{R}$. Let $t\in I$ and $z\in \operatorname{dom}\varphi(0,\cdot)$. Moreover, let $(\phi_n)_n$ be an increasing sequence of L^1-bounded Carathéodory integrands defined on $I\times H\times U$ such that $\phi = \sup_{n\in\mathbb{N}} \phi_n$.*
Let $(z_n)_{n\in\mathbb{N}} \subset \operatorname{dom}\varphi(0,\cdot)$ be a sequence converging to z in H, and define the value function

$$W_\phi(y) := \inf_{\mu\in S_\Sigma} \int_0^T \int_U \phi(t,x_{y,\mu}(t),u)\mu_t(du)\,dt,$$

where the map $x_{y,\mu}(\cdot)$ is the unique absolutely continuous solution of $(P_{y,\mu})$.
Then,

$$\liminf_{n\to+\infty} W_{\phi_n}(z_n) \geq W_\phi(z)$$

where, for each $n\geq 1$,

$$\begin{aligned} W_{\phi_n}(z_n) &:= \inf_{\mu\in S_\Sigma} \int_0^T \int_U \phi_n(t,x_{z_n,\mu}(t),u)\mu_t(du)\,dt, \\ &:= \inf_{\zeta\in S_\Gamma} \int_0^T \phi_n(t,x_{z_n,\zeta}(t),\zeta(t))\,dt. \end{aligned}$$

Proof. Remark first that an approximating sequence such as $(\phi_n)_n$ always exists (we refer the reader to [13]).

The last equality of the proposition is an immediate consequence of the relaxation theorem (Theorem 3.5). The L^1-boundedness of the integrands ϕ_n guaranties the uniform integrability. Moreover, in view of Corollary 3.1 and the fiber product tool of Theorem 3.2, for any fixed $n\in\mathbb{N}$, the map

$$\mu \in S_\Sigma \mapsto \int_0^T \int_U \phi_n(t,x_{z_n,\mu}(t),u)\mu_t(du)\,dt,$$

is continuous with respect to the $w(L^\infty_{\mathscr{M}(U)}(I), L^1_{\mathscr{C}(U)}(I))$-topology on the compact space S_Σ. Hence, there exists a relaxed control $\mu^n \in S_\Sigma$ that satisfies

$$W_{\phi_n}(z_n) = \int_0^T \int_U \phi_n(t, x_{z_n,\mu^n}(t), u)\mu_t^n(du)\, dt.$$

Since S_Σ is sequentially compact, there exist an element $\mu \in S_\Sigma$ and a subsequence still denoted by $(\mu^n)_n$ which converges to μ with respect to $w(L^\infty_{\mathscr{M}(U)}(I), L^1_{\mathscr{C}(U)}(I))$-topology.
Thanks to Theorem 3.4, the corresponding sequence $(x_{z_n,\mu^n}(\cdot))_n$ converges uniformly to $x_{z,\mu}(\cdot)$, the solution of $(P_{z,\mu})$.
Next, fix any $k \in \mathbb{N}$. For any $n \geq k$, the monotonicity of $(\phi_n)_n$ leads to,

$$W_{\phi_n}(z_n) = \int_0^T \int_U \phi_n(t, x_{z_n,\mu^n}(t), u)\mu_t^n(du)\, dt$$

$$\geq \int_0^T \int_U \phi_k(t, x_{z_n,\mu^n}(t), u)\mu_t^n(du)\, dt. \tag{18}$$

Due to Theorems 3.4 and 3.2, since $\phi_k(\cdot)$ is an L^1-bounded Carathéodory integrand, one also has

$$\lim_{n\to+\infty} \int_0^T \int_U \phi_k(t, x_{z_n,\mu^n}(t), u)\mu_t^n(du)\, dt = \int_0^T \int_U \phi_k(t, x_{z,\mu}(t), u)\mu_t(du)\, dt.$$

Then, taking the inferior limit on n in (18) entails , for all $k \in \mathbb{N}$,

$$\liminf_{n\to+\infty} W_{\phi_n}(z_n) \geq \int_0^T \int_U \phi_k(t, x_{z,\mu}(t), u)\mu_t(du)\, dt.$$

We may apply the monotone convergence theorem to conclude that

$$\begin{aligned} \liminf_{n\to+\infty} W_{\phi_n}(z_n) &\geq \int_0^T \int_U (\sup_{k\in\mathbb{N}} \phi_k)(t, x_{z,\mu}(t), u)\mu_t(du)\, dt \\ &= \int_0^T \int_U \phi(t, x_{z,\mu}(t), u)\mu_t(du)\, dt, \end{aligned}$$

and thus, as $\mu \in S_\Sigma$, one gets $\liminf_{n\to+\infty} W_{\phi_n}(z_n) \geq W_\phi(z)$.

Corollary 4.1. *Suppose that hypothesis (H_1)-(H_2) and (i)-(iv) above are made. Consider an interval $I := [a,b]$, $(0 \leq a < b))$, and an L^1-bounded Carathéodory integrand $\phi : I \times H \times U \to \mathbb{R}$. Let G_ϕ be a function defined by*

$$G_\phi : \operatorname{dom}\varphi(a,\cdot) \to \mathbb{R}, \quad G_\phi(z) := \sup_{\nu\in\mathscr{R}} \int_I \int_U \phi(t, x_{z,\nu}(t), u)\nu_t(du)\, dt,$$

with $x_{z,\nu} : I \to H$ the unique absolutely continuous solution of

$$(P_{z,\nu}) \qquad \begin{cases} -\dot{x}_{z,\nu}(t) \in \partial\varphi(t, x_{z,\nu}(t)) + \int_U g(t, x_{z,\nu}(t), u)\nu_t(du) \text{ a.e. in } I \\ x_{z,\nu}(a) = z \in \operatorname{dom}\varphi(a,\cdot), \end{cases}$$

and $\mathscr{R} := \{\nu : I \to \mathscr{M}^1_+(U),\, t \mapsto \nu_t$ is $\lambda-$measurable$\}$.
Then, G_ϕ is sequentially upper semi-continuous on $\operatorname{dom}\varphi(a,\cdot)$ with respect to the topology induced by the norm of H.

Proof. Associate Σ with the constant set-valued map $\Gamma \equiv U$ on I. Then, the set $\mathscr{R}$ coincides with S_Σ, and the space $(\mathscr{R}, w(L^\infty_{\mathscr{M}(U)}(I), L^1_{\mathscr{C}(U)}(I)))$ is compact metrizable. Hence, conclude from Proposition .

4.2. Application to viscosity theory

We establish here, some new properties of the lower value function of minimization problems submitted to $(P_{y,\mu})$. In order to produce a viscosity type solution to the problem under consideration, we will state two technical results.
The following theorem is a dynamic programming inspired by those widely used in the viscosity theory for control problems subject to ordinary differential equations. The next one is a tool which allows us to provide the existence of viscosity subsolutions connected with our control problem, in the finite dimensional setting.

Theorem 4.1. (Theorem of dynamic programming)
Suppose that hypothesis (H_1)-(H_2) and (i)-(iv) above are made. Consider an L^1-bounded Carathéodory integrand $J : I \times H \times U \to \mathbb{R}$. Set $\mathscr{D} = \{(\tau, z) :\ \tau \in [0, T[,\ z \in \operatorname{dom}\varphi(\tau,\cdot)\}$, and let $\rho > 0$ be such that $\tau + \rho < T$. Let $V_J : \mathscr{D} \to \mathbb{R}$ be the value function defined by

$$V_J(\tau, z) := \sup_{\mu \in \mathscr{R}} \int_\tau^T \int_U J(t, x_{z,\mu}(t), u)\mu_t(du)\, dt,$$

with $x_{z,\mu}(\cdot)$ the unique absolutely continuous solution of the problem

$$(P_{z,\mu}(\tau)) \qquad \begin{cases} -\dot{x}_{z,\mu}(t) \in \partial\varphi(t, x_{z,\mu}(t)) + \int_U g(t, x_{z,\mu}(t), u)\mu_t(du) \\ \text{a.e. } t \in [\tau, T], \\ x_{z,\mu}(\tau) = z \in \operatorname{dom}\varphi(\tau, \cdot). \end{cases}$$

Then,

$$V_J(\tau, z) := \max_{\mu \in \mathscr{R}} \{ \int_\tau^{\tau+\rho} \int_U J(t, x_{z,\mu}(t), u)\mu_t(du)\, dt + V_J(\tau+\rho, x_{z,\mu}(\tau+\rho)) \}$$

where

$$V_J(\tau+\rho, x_{z,\mu}(\tau+\rho)) := \sup_{\nu \in \mathscr{R}} \int_{\tau+\rho}^T \int_U J(t, v_{z,\nu}(t), u)\nu_t(du)\, dt,$$

the set $\mathscr{R}$ is defined above and the map $v_{z,\nu}(\cdot)$ denotes the solution on $[\tau+\rho, T]$ of the dynamical system associated with the control ν and starting from $v_{z,\nu}(\tau+\rho) = x_{z,\mu}(\tau+\rho)$, namely $(P_{x_{z,\mu}(\tau+\rho),\nu}(\tau+\rho))$.

Proof. For any $\mu \in \mathscr{R}$, one has

$$\int_\tau^T \int_U J(t, x_{z,\mu}(t), u)\mu_t(du)\, dt =$$
$$\int_\tau^{\tau+\rho} \int_U J(t, x_{z,\mu}(t), u)\mu_t(du)\, dt + \int_{\tau+\rho}^T \int_U J(t, x_{z,\mu}(t), u)\mu_t(du)\, dt.$$

The absolutely continuous $y : [\tau+\rho, T] \to H$ is the restriction to $[\tau+\rho, T]$ of the mapping $x_{z,\mu}(\cdot)$ solution of the problem

$$\begin{cases} -\dot{y}(t) \in \partial\varphi(t, y(t)) + \int_U g(t, y(t), u)\mu_t(du) \text{ a.e. } t \in [\tau+\rho, T], \\ y(\tau+\rho) = x_{z,\mu}(\tau+\rho). \end{cases}$$

Since $\mu \in \mathscr{R}$, it results

$$V_J(\tau+\rho, x_{z,\mu}(\tau+\rho)) \geq \int_{\tau+\rho}^T \int_U J(t, x_{z,\mu}(t), u)\mu_t(du)\, dt,$$

and then

$$\int_\tau^T \int_U J(t, x_{z,\mu}(t), u)\mu_t(du)\, dt \leq$$
$$\int_\tau^{\tau+\rho} \int_U J(t, x_{z,\mu}(t), u)\mu_t(du)\, dt + V_J(\tau+\rho, x_{z,\mu}(\tau+\rho)).$$

The inequality $\leq$ holds true (take the supremum on $\mu \in \mathscr{R}$).

Now, define

$$W_J(\tau,z) := \sup_{\mu\in\mathscr{R}} \{\int_\tau^{\tau+\rho} \int_U J(t,x_{z,\mu}(t),u)\mu_t(du)\, dt + V_J(\tau+\rho, x_{z,\mu}(\tau+\rho))\}.$$

Then, there exists $\mu^1 \in \mathscr{R}$ that verifies

$$W_J(\tau,z) = \int_\tau^{\tau+\rho} \int_U J(t,x_{z,\mu^1}(t),u)\mu^1_t(du)\, dt + V_J(\tau+\rho, x_{z,\mu^1}(\tau+\rho)). \tag{19}$$

In fact, by virtue of Corollaries 3.1, 4.1, the fiber product theorem 3.2, and since J is an L^1-bounded Carathéodory integrand, the map

$$\mu \mapsto \int_\tau^{\tau+\rho} \int_U J(t,x_{z,\mu}(t),u)\mu_t(du)\, dt + V_J(\tau+\rho, x_{z,\mu}(\tau+\rho))$$

from $(\mathscr{R}, w(L^\infty_{\mathscr{M}(U)}(I), L^1_{\mathscr{C}(U)}(I)))$ into $\mathbb{R}$ is upper semi-continuous. Hence, the compactness of $\mathscr{R}$ for the latter weak-$*$ topology justifies the existence of μ^1.

Following the same arguments, there exists $\mu^2 \in \mathscr{R}$, such that

$$V_J(\tau+\rho, x_{z,\mu^1}(\tau+\rho)) = \int_{\tau+\rho}^T \int_U J(t,v_{z,\mu^2}(t),u)\mu^2_t(du)\, dt, \tag{20}$$

the map $v_{z,\mu^2}(\cdot)$ is the trajectory linked to μ^2 on $[\tau+\rho,T]$ with initial condition $v_{z,\mu^2}(\tau+\rho) = x_{z,\mu^1}(\tau+\rho)$. Define now $\tilde{\mu} : [\tau,T] \to \mathscr{M}^1_+(U)$,

$$\tilde{\mu}_t := \mathbf{1}_{[\tau,\tau+\rho[}(t)\mu^1_t + \mathbf{1}_{[\tau+\rho,T]}(t)\mu^2_t. \tag{21}$$

It's clear that $\tilde{\mu} \in \mathscr{R}$. Then, let $\tilde{x}_{z,\tilde{\mu}}(\cdot) : [\tau,T] \to H$ be the the unique absolutely continuous solution of the problem $(P_{z,\tilde{\mu}}(\tau))$. As the solution is unique on $[\tau,\tau+\rho]$ and $[\tau+\rho,T]$ respectively, one has

$$\tilde{x}_{z,\tilde{\mu}}(t) = \begin{cases} x_{z,\mu^1}(t) & \text{if } t \in [\tau,\tau+\rho], \\ v_{z,\mu^2}(t) & \text{if } t \in [\tau+\rho,T]. \end{cases}$$

In view of (19), (20) and (21), one gets

$$\begin{aligned} & W_J(\tau,z) \\ & = \int_\tau^{\tau+\rho} \int_U J(t,x_{z,\mu^1}(t),u)\mu^1_t(du)\, dt + \int_{\tau+\rho}^T \int_U J(t,v_{z,\mu^2}(t),u)\mu^2_t(du)\, dt \\ & = \int_\tau^T \int_U J(t,\tilde{x}_{z,\tilde{\mu}}(t),u)\tilde{\mu}_t(du)\, dt \\ & \leq V_J(\tau,z). \end{aligned}$$

This ends the proof.

Lemma 4.1. *Consider an upper semi-continuous function $\Lambda : I \times H \times \mathscr{M}^1_+(U) \to \mathbb{R}$ defined by*

$$\Lambda(t,z,\mu) := \Lambda_1(t,z,\mu) + \Lambda_2(t,z),$$

where $\Lambda_1 : I \times H \times \mathscr{M}^1_+(U) \to \mathbb{R}$ is a continuous mapping and $\Lambda_2 : I \times H \to \mathbb{R}$ is an upper semi-continuous integrand such that, for any bounded subset B of H, $\Lambda_2|_{I\times B}$ is bounded. Let $(t_0,x_0) \in \mathscr{D}$. Suppose also that

$$\sup_{\nu \in \mathscr{M}^1_+(U)} \Lambda(t_0,x_0,\nu) < -\eta \text{ for some } \eta > 0.$$

Then, there exists a real number $\rho > 0$ such that

$$\sup_{\mu \in \mathscr{R}} \int_{t_0}^{t_0+\rho} \Lambda(t, x_{x_0,\mu}(t), \mu_t)\, dt < -\frac{\rho\eta}{2},$$

where for any $\mu \in \mathscr{R}$, $x_{x_0,\mu}(\cdot)$ is the unique absolutely continuous solution of the problem

$$(P_{x_0,\mu}(t_0)) \qquad \begin{cases} -\dot{x}_{x_0,\mu}(t) \in \partial\varphi(t, x_{x_0,\mu}(t)) + \int_U g(t, x_{x_0,\mu}(t), u)\mu_t(du) \\ \text{a.e. } t \in [t_0, T], \\ x_{x_0,\mu}(t_0) = x_0 \in \operatorname{dom}\varphi(t_0,\cdot). \end{cases}$$

Proof. The mapping $(t,x) \mapsto \sup_{\nu \in \mathscr{M}^1_+(U)} \Lambda(t,x,\nu)$ is upper semi-continuous on $I \times H$. This is due to the fact that the space $(\mathscr{M}^1_+(U), w(\mathscr{M}(U)), \mathscr{C}(U))$ is compact metrizable and the continuity hypothesis on Λ_1 which guaranties that the mapping $(t,x) \mapsto \sup_{\nu \in \mathscr{M}^1_+(U)} \Lambda_1(t,x,\nu)$ is upper semi-continuous on $I \times H$. As a result, there exists a real number $\delta_0 > 0$ such that

$$\sup_{\nu \in \mathscr{M}^1_+(U)} \Lambda(t,x,\nu) < -\frac{\eta}{2} \tag{22}$$

whenever $0 < t - t_0 \leq \delta_0$ and $\|x - x_0\| \leq \delta_0$.
From assertion (b) of Theorem 3.3, there exists $M_0 > 0$ independent of $\mu \in \mathscr{R}$, satisfying

$$\|x_{x_0,\mu}(t) - x_{x_0,\mu}(s)\| \leq (t-s)^{\frac{1}{2}} M_0 \text{ for all } t_0 \leq s \leq t \leq T.$$

Choose now, $\rho \in]0, \delta_0[$ such that $\rho^{\frac{1}{2}} M_0 < \delta_0$. For any $\mu \in \mathscr{R}$, by (22), the function $t \mapsto \Lambda(t, x_{x_0,\mu}(t), \mu_t)$ is bounded and λ-measurable on $[t_0, t_0+\rho]$. Thus, integrating on $[t_0, t_0+\rho]$, one deduces that

$$\int_{t_0}^{t_0+\rho} \Lambda(t, x_{x_0,\mu}(t), \mu_t)\, dt \le \int_{t_0}^{t_0+\rho} \sup_{\nu \in \mathscr{M}_+^1(U)} \Lambda(t, x_{x_0,\mu}(t), \nu)\, dt \le -\frac{\rho\eta}{2},$$

the proof is then complete.

Theorem 4.2. (Existence of viscosity subsolutions) *Let $H = \mathbb{R}^d$. Suppose that $\varphi : I \times \mathbb{R}^d \to \mathbb{R}_+$ satisfies (H_2), the function $\varphi(\cdot,\cdot)$ is convex and globally Lipschitz on $I \times \mathbb{R}^d$. Let U be a compact metric space. Suppose also that*

- (H_5) *the function $g : I \times \mathbb{R}^d \times U \longrightarrow \mathbb{R}^d$ is continuous, uniformly Lipschitz continuous with respect to its second variable and bounded, that is, there exists $L > 0$ such that $\|g(t,z,u)\| \le L$ for all $(t,z,u) \in I \times H \times U$;*
- (H_6) *the cost function $J : I \times \mathbb{R}^d \times U \to \mathbb{R}$ is bounded and continuous.*

Denote by $V_J(\cdot,\cdot)$ the value function of Theorem 4.1, and by $H(\cdot,\cdot,\cdot)$ the upper Hamiltonian on $I \times \mathbb{R}^d \times \mathbb{R}^d$ given by

$$\begin{aligned} &H(t,z,\xi) \\ &= \sup_{\nu \in \mathscr{M}_+^1(U)} \{\langle \xi, -\int_U g(t,z,u)\, \nu(du)\rangle + \int_U J(t,z,u)\, \nu(du)\} + \delta^*(\xi, -\partial\varphi(t,z)), \end{aligned}$$

where the function $\delta^*(\cdot, -\partial\varphi(t,z))$ is the support function of $-\partial\varphi(t,z) \subset \mathbb{R}^d$.
Then, V_J is a viscosity subsolution of the Hamilton-Jacobi-Bellman equation

$$\frac{\partial V}{\partial t}(t,x) + H(t,x,\nabla V(t,x)) = 0,$$

that is to say: for any $\phi \in \mathscr{C}^1(I \times \mathbb{R}^d)$ such that $V_J - \phi$ reaches a local maximum at $(t_0,x_0) \in I \times \mathbb{R}^d$, one has

$$\frac{\partial \phi}{\partial t}(t_0,x_0) + H(t_0,x_0,\nabla\phi(t_0,x_0)) \ge 0.$$

Proof. We proceed by contradiction. Suppose that there exist some $\phi \in \mathscr{C}^1(I \times \mathbb{R}^d)$ and a point (t_0,x_0) for which

$$\frac{\partial \phi}{\partial t}(t_0,x_0) + H(t_0,x_0,\nabla\phi(t_0,x_0)) < -\eta \ \text{ for some } \eta > 0.$$

In view of Proposition 1.17 in [29], for each $t \in I$, the set-valued mapping $z \in \mathbb{R}^d \to \partial\varphi(t,z)$ is upper semi-continuous with convex compact values in $\mathbb{R}^d$. Since $\varphi(\cdot,\cdot)$ is globally Lipschitz, then, the range of $\partial\varphi(\cdot,\cdot)$ is a bounded set.

It results that the function

$$(t,x) \in I \times \mathbb{R}^d \mapsto \Lambda_2(t,z) := \delta^*(\nabla\phi(t,z), -\partial\varphi(t,z)),$$

is upper semi-continuous. Thanks to the continuity of $\nabla\phi(\cdot,\cdot)$ and the boundedness of the range of $\partial\varphi(\cdot,\cdot)$, for any bounded subset B of $\mathbb{R}^d$, $\Lambda_2|_{I\times B}$ is bounded.

Under our hypothesis, the function $\Lambda_1 : I \times \mathbb{R}^d \times \mathscr{M}^1_+(U) \mapsto \mathbb{R}$ defined by

$$\Lambda_1(t,z,\nu) := \int_U J(t,z,u)\,\nu(du) + \langle \nabla\phi(t,z), -\int_U g(t,z,u)\,\nu(du)\rangle + \frac{\partial\phi}{\partial t}(t,z)$$

is continuous, $\mathscr{M}^1_+(U)$ is endowed with the $w(\mathscr{M}(U),\mathscr{C}(U))$-topology. Hence, applying Lemma to $\Lambda := \Lambda_1 + \Lambda_2$ and find some $\rho > 0$ such that

$$\frac{-\rho\eta}{2} \geq \sup_{\mu\in\mathscr{R}}\{\int_{t_0}^{t_0+\rho} \frac{\partial\phi}{\partial t}(t,x_{x_0,\mu}(t))dt + \int_{t_0}^{t_0+\rho}\int_U J(t,x_{x_0,\mu}(t),u)\mu_t(du)dt$$
$$+ \int_{t_0}^{t_0+\rho} \delta^*(\nabla\phi(t,x_{x_0,\mu}(t)), -\partial\varphi(t,x_{x_0,\mu}(t)))dt$$
$$+ \int_{t_0}^{t_0+\rho}\int_U \langle\nabla\phi(t,x_{x_0,\mu}(t)), -g(t,x_{x_0,\mu}(t),u)\rangle\mu_t(du)dt\}, \tag{23}$$

where for $\mu \in \mathscr{R}$, $x_{x_0,\mu}(\cdot) : [t_0,T] \to H$ denotes the unique absolutely continuous solution of the problem $(P_{x_0,\mu}(t_0))$.

Relying on the proof of Lemma , note that decreasing ρ is damageless. Then, thanks to Theorem 4.1 of dynamic programming, we have

$$V_J(t_0,x_0) := \max_{\mu\in\mathscr{R}}\{\int_{t_0}^{t_0+\rho}\int_U J(t,x_{x_0,\mu}(t),u)\mu_t(du)\,dt +$$
$$V_J(t_0+\rho, x_{x_0,\mu}(t_0+\rho))\}. \tag{24}$$

As $V_J - \phi$ has a local maximum at (t_0,x_0), for ρ small enough and any $\mu \in \mathscr{R}$, we may write

$$V_J(t_0,x_0) - \phi(t_0,x_0) \geq V_J(t_0+\rho, x_{x_0,\mu}(t_0+\rho)) - \phi(t_0+\rho, x_{x_0,\mu}(t_0+\rho)).$$

The latter inequality along with (24) yield

$$0 \leq \sup_{\mu\in\mathscr{R}}\{\int_{t_0}^{t_0+\rho}\int_U J(t,x_{x_0,\mu}(t),u)\mu_t(du)dt + \phi(t_0+\rho, x_{x_0,\mu}(t_0+\rho)) - \phi(t_0,x_0)\}. \tag{25}$$

It's clear that making use of the $\mathscr{C}^1$ regularity of ϕ and the absolute continuity of $x_{x_0,\mu}(\cdot)$, for any $\mu \in \mathscr{R}$, we have

$$\begin{aligned}
&\phi(t_0+\rho, x_{x_0,\mu}(t_0+\rho)) - \phi(t_0,x_0)\\
&= \int_{t_0}^{t_0+\rho} \frac{\partial \phi}{\partial t}(t, x_{x_0,\mu}(t))dt + + \int_{t_0}^{t_0+\rho} \langle \nabla\phi(t, x_{x_0,\mu}(t)), \dot{x}_{x_0,\mu}(t)\rangle dt
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{t_0}^{t_0+\rho} \frac{\partial \phi}{\partial t}(t, x_{x_0,\mu}(t))dt + \int_{t_0}^{t_0+\rho} \delta^*(\nabla\phi(t, x_{x_0,\mu}(t)), -\partial\varphi(t, x_{x_0,\mu}(t)))dt\\
&\quad + \int_{t_0}^{t_0+\rho} \langle \nabla\phi(t, x_{x_0,\mu}(t)), -\int_U g(t, x_{x_0,\mu}(t), u)\mu_t(du)\rangle dt,
\end{aligned} \tag{26}$$

by the definition of the support function and the differential inclusion in $(P_{x_0,\mu}(t_0))$.

Now, upper bounding the difference $\phi(t_0+\rho, x_{x_0,\mu}(t_0+\rho)) - \phi(t_0,x_0)$ in (25), along with (26), the supremum in (23) is then non-negative, this is in contradiction with (23).

Remark 4.1. *In order to show further that V_J is a supersolution to the Hamilton-Jacobi-Bellman equation, we have to add extra conditions on φ, J and g, so that, the control problem has the form*

$$\begin{cases}
\dot{x}_{z,\mu}(t) = -\nabla\varphi(t, x_{z,\mu}(t)) + \int_U g(t, x_{z,\mu}(t), u)\mu_t(du)\\
\text{a.e. } t \in I,\\
x_{z,\mu}(0) = z \in \operatorname{dom}\varphi(0,\cdot),
\end{cases}$$

that is, $\partial\varphi(t,x) = \{\nabla\varphi(t,x)\}$, $(t,x) \in I \times \mathbb{R}^d$. Consequently, V_J is a viscosity solution.

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Conflict of Interests

The authors declare that there is no conflict of interests.

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