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# A COMMON FIXED POINT THEOREM IN $b$-METRIC SPACES VIA SIMULATION FUNCTION

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**Abstract.** In this paper, we introduce generalized $Z$-contraction pair of maps with respect to a simulation function $\zeta$ in $b$-metric spaces and study the existence of common fixed points of such mappings in complete $b$-metric spaces. We extend it to a sequence of selfmaps. We deduce some corollaries from our main result and provide examples in support of our results.

**Keywords:** generalized $Z$-contraction pair of selfmaps; simulation function; b-metric space; fixed point; common fixed point.

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## 1. Introduction and preliminaries

In the development of non-linear analysis fixed point theory plays a prominent role in many aspects. The basic idea of b-metric was initiated from the works of Bourbaki [2] and Bakhtin [3].

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In 1993, Stefan Czerwik [7] introduced the concept of $b$-metric space (or) metric type space as a generalization of metric space and proved the Banach contraction mapping principle in this setting. Since then, many authors have been keeping their interest in finding the existence of fixed points of single valued self maps and set valued mappings in $b$-metric spaces, we refer [1, 4, 9, 10, 12, 15].

**Definition 1.1.** [7] Let X be a non-empty set. A function $d : X \times X \to [0,\infty)$ is said to be a $b$-metric if the following conditions are satisfied:

(i) $0 \leq d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$ for all $x,y \in X$, and

(iii) there exists $s \geq 1$ such that $d(x,z) \leq s\big[d(x,y) + d(y,z)\big]$ for all $x,y,z \in X$. In this case, the pair $(X,d)$ is called a $b$-metric space with coefficient $s$. Here, we observe that every metric space is a $b$-metric space, with $s = 1$

**Definition 1.2.** [5] Let $(X,d)$ be a $b$-metric space and $\{x_n\}$ a sequence in $X$.

(i) A sequence $\{x_n\}$ in X is called $b$-convergent if there exists $x \in X$ such that $d(x_n,x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in $X$ is called $b$-Cauchy if $d(x_n,x_m) \to 0$ as $n,m \to \infty$.

(iii) A $b$-metric space $(X,d)$ is said to be a complete $b$-metric space if every $b$-Cauchy sequence in X is $b$-convergent.

(iv) A set $B \subset X$ is said to be $b$-closed if for any sequence $\{x_n\}$ in $B$ such that $\{x_n\}$ is $b$-convergent to $z \in X$ then $z \in B$.

In general , a $b$-metric is not necessarily continuous.

**Example 1.1.** [8] Let $X = \mathbb{N} \cup \{\infty\}$. We define a mapping $d : X \times X \to \mathbb{R}^+$ as follows:

$$d(m,n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of m,n is even and the other is even or } \infty \\ 5 & \text{if one of m,n is odd and the other is odd or } \infty \\ 2 & \text{otherwise.} \end{cases}$$

Then $(X,d)$ is a $b$-metric space with coefficient $s = \frac{5}{2}$.

**Definition 1.3.** [5] Let $(X,d_X)$ and $(Y,d_Y)$ be two b-metric spaces. A function $f : X \to Y$

is $b$-continuous at a point $x \in X$, if it is $b$-sequentially continuous at $x$ i.e., whenever $\{x_n\}$ is $b$-convergent to $x$, $\{fx_n\}$ is $b$-convergent to $fx$.

The following lemmas are useful in proving our main results.

**Lemma 1.1.** [6] *Suppose $(X,d)$ is a metric space. Let $\{x_n\}$ be a sequence in $X$ such that $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k},x_{n_k}) \geq \varepsilon$. For each $k > 0$, corresponding to $m_k$, we can choose $n_k$ to be the smallest positive integer such that* $d(x_{m_k},x_{n_k}) \geq \varepsilon, d(x_{m_k},x_{n_k-1}) < \varepsilon$ and

(i) $\lim_{k\to\infty} d(x_{n_k-1},x_{m_k+1}) = \varepsilon$ (ii) $\lim_{k\to\infty} d(x_{n_k},x_{m_k}) = \varepsilon$

(iii) $\lim_{k\to\infty} d(x_{m_k-1},x_{n_k}) = \varepsilon$ and (iv) $\lim_{k\to\infty} d(x_{n_k},x_{m_k+1}) = \varepsilon$.

**Lemma 1.2.** [14] *Suppose $(X,d)$ is a $b$-metric space with coefficient $s \geq 1$ and $\{x_n\}$ be a sequence in $X$ such that $d(x_n,x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $n_k > m_k \geq k$ such that $d(x_{m_k},x_{n_k}) \geq \varepsilon$, $d(x_{m_k},x_{n_k-1}) < \varepsilon$ and*

(i) $\varepsilon \leq \liminf_{k\to\infty} d(x_{m_k},x_{n_k}) \leq \limsup_{k\to\infty} d(x_{m_k},x_{n_k}) \leq s\varepsilon$

(ii) $\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(x_{m_k+1},x_{n_k}) \leq \limsup_{k\to\infty} d(x_{m_k+1},x_{n_k}) \leq s^2\varepsilon$

(iii) $\frac{\varepsilon}{s} \leq \liminf_{k\to\infty} d(x_{m_k},x_{n_k+1}) \leq \limsup_{k\to\infty} d(x_{m_k},x_{n_k+1}) \leq s^2\varepsilon$

(iv) $\frac{\varepsilon}{s^2} \leq \liminf_{k\to\infty} d(x_{m_k+1},x_{n_k+1}) \leq \limsup_{k\to\infty} d(x_{m_k+1},x_{n_k+1}) \leq s^3\varepsilon$.

In 2015, Khojasteh, Shukla and Radenovic [11] introduced simulation functions and defined $Z$-contraction with respect to a simulation function.

**Definition 1.4.** [11] A simulation function is a mapping $\zeta : [0,\infty) \times [0,\infty) \to (-\infty,\infty)$ satisfying the following conditions:

$(\zeta_1)$ $\zeta(0,0) = 0$;

$(\zeta_2)$ $\zeta(t,s) < s - t$ for all $s,t > 0$;

$(\zeta_3)$ if $\{t_n\},\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0,\infty)$ then $\limsup_{n\to\infty} \zeta(t_n,s_n) < 0$.

**Remark 1.1.** Let $\zeta$ be a simulation function. If $\{t_n\},\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0,\infty)$, then $\limsup_{n\to\infty} \zeta(kt_n,s_n) < 0$ for any $k > 1$.

The following are examples of simulation functions.

**Example 1.2.** Let $\zeta : [0,\infty)\times[0,\infty) \to (-\infty,\infty)$ be defined by

(i) $\zeta(t,s) = \lambda s - t$ for all $t,s \in [0,\infty)$, where $\lambda \in [0,1)$,

(ii) $\zeta(s,t) = \frac{s}{1+s} - t$ for all $t,s \in [0,\infty)$,

(iii) $\zeta(t,s) = s - kt$ for all $t,s \in [0,\infty)$, where $k > 1$,

(iv) $\zeta(s,t) = \frac{s}{1+s} - te^t$ for all $t,s \in [0,\infty)$,

(v) $\zeta(t,s) = \frac{1}{s+1} - (t+1)$ for all $t,s \in [0,\infty)$.

**Definition 1.5.** [11] Let $(X,d)$ be a metric space and $f : X \to X$ be a selfmap of X. We say that $f$ is a $Z$-contraction with respect to $\zeta$, if there exists a simulation function $\zeta$ such that

$$\zeta(d(fx,fy),d(x,y)) \geq 0 \text{ for all } x,y \in X. \tag{1.1}$$

**Theorem 1.1.** [11] *Let $(X,d)$ be a complete metric space and $f : X \to X$ be a $Z$-contraction with respect to a certain simulation function $\zeta$, then for every $x_0 \in X$, the Picard sequence $\{f^n x_0\}$ converges in $X$ and $\lim_{n\to\infty} f^n x_0 = u$ (say) in $X$ and $u$ is the unique fixed point of $f$ in $X$.*

Recently, Olgun, Bicer and Alyildiz [13] proved the following result in complete metric spaces.

**Theorem 1.2.** [13] *Let $(X,d)$ be a complete metric space and $f : X \to X$ be a selfmap on $X$. If there exists simulation function $\zeta$ such that*

$$\zeta(d(fx,fy),M(x,y)) \geq 0 \textit{ for all } x,y \in X, \tag{1.2}$$

*where $M(x,y) = max\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\}$,*
*then for every $x_0 \in X$, the Picard sequence $\{f^n x_0\}$ converges in $X$ and $\lim_{n\to\infty} f^n x_0 = u$ (say) in $X$ and $u$ is the unique fixed point of $f$ in $X$.*

Motivated by the works of Olgun, Bicer and Alyildiz [13], we now introduce a generalized $Z$-contraction pair of maps with respect to $\zeta$ in the following.

**Definition 1.6.** Let $(X,d)$ be a $b$-metric space with coefficient $s \geq 1$. Let $f,\ g : X \to X$ be two selfmappings. If there exists a simlation function $\zeta$ such that

$$\zeta(s^4 d(fx,gy), M(x,y)) \geq 0, \tag{1.3}$$

where $M(x,y) = \max\ \{d(x,y),\ d(x,fx),\ d(y,gy),\ \frac{1}{2s}[d(x,gy)+d(y,fx)]\}$
for all $x,y \in X$, then we say that $(f,g)$ is a generalized $Z$-contraction pair of maps.

If $g = f$ and $s = 1$, then we say that $f$ is a generalized $Z$-contraction map of a metric space $X$.

**Example 1.3.** Let $X = [0,1]$ and let $d : X \times X \to [0,\infty)$ be defined by
$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ (x+y)^2 & \text{if } x \neq y. \end{cases}$
Clearly $(X,d)$ is a $b$-metric space with coefficient $s = 2$.
We define $f,g : X \to X$ by

$$fx = \frac{x}{8} \text{ for all } x \in [0,1] \text{ and } gx = \begin{cases} 0 & \text{if } x \in [0,1]/\{\frac{1}{2}\} \\ \frac{1}{16} & \text{if } x = \frac{1}{2}. \end{cases}$$

We now define $\zeta : [0,\infty) \times [0,\infty) \to [0,\infty)$ by $\zeta(t,s) = \frac{4t}{5+t} - s$.
We have the following possible cases.
<u>*Case*</u> (i): $x = \frac{1}{2}, y = \frac{1}{2}$.
In this case, $fx = \frac{1}{16}, gy = \frac{1}{16}, s^4 d(fx,gy) = 0$ and $M(x,y) = (\frac{9}{16})^2$.
Now, we consider

$$\zeta(s^4 d(fx,gy), M(x,y)) = \frac{4M(x,y)}{5+M(x,y)} - s^4 d(fx,gy) = \frac{4(\frac{9}{16})^2}{5+(\frac{9}{16})^2} - 16(0) > 0.$$

<u>*Case*</u> (ii): $x \neq \frac{1}{2}, y = \frac{1}{2}$.
In this case, $fx = \frac{x}{8}, gy = \frac{1}{16}, s^4 d(fx,gy) = 16(\frac{x}{8} + \frac{1}{16})^2 = \frac{1}{4}(x+\frac{1}{2})^2$ and
$M(x,y) \geq d(x,y) = (x+\frac{1}{2})^2$.
Now, we consider

$$\zeta(s^4 d(fx,gy), M(x,y)) = \frac{4M(x,y)}{5+M(x,y)} - s^4 d(fx,gy) \geq \frac{4d(x,y)}{5+d(x,y)} - \frac{1}{4} d(x,y) \geq 0$$

Case (iii): $x \neq \frac{1}{2}, y \neq \frac{1}{2}$.

In this case, $fx = \frac{x}{8}, gy = 0, s^4 d(fx,gy) = 16(\frac{x}{8})^2$ and

$M(x,y) \geq d(x,fx) = (\frac{9x}{8})^2$.

Now, we consider

$$\zeta(s^4 d(fx,gy), M(x,y)) = \frac{4M(x,y)}{5+M(x,y)} - s^4 d(fx,gy) \geq \frac{4d(x,fx)}{5+d(x,fx)} - \frac{16}{81} d(x,fx) \geq 0.$$

Case (iv): $x = \frac{1}{2}, y \neq \frac{1}{2}$.

In this case, $fx = \frac{1}{16}, gy = 0, s^4 d(fx,gy) = \frac{1}{16})$ and

$M(x,y) \geq d(x,y) = (\frac{1}{2}+y)^2$.

Now, we consider

$$\begin{aligned}\zeta(s^4 d(fx,gy), M(x,y)) &= \frac{4M(x,y)}{5+M(x,y)} - s^4 d(fx,gy)\\ &\geq \frac{4d(x,y)}{5+d(x,y)} - \frac{1}{16}\\ &= \frac{4(\frac{1}{2}+y)^2}{5+(\frac{1}{2}+y)^2} - \frac{1}{16} \geq 0.\end{aligned}$$

Hence in all the cases, the inequality (1.3) holds. Thus $(f,g)$ is a generalized $Z$-contraction pair of maps.

In Section 2, we prove the existence of common fixed points of a pair of selfmaps $(f,g)$, where $(f,g)$ is a generalized $Z$-contraction pair of maps with respect to a simulation function $\zeta$ in complete $b$-metric spaces. We extend it to a sequence of self maps. We deduce some corollaries from our main result and provide examples in support of our results in Section 3.

## 2. Main results

**Proposition 2.1.** *Let $(X,d)$ be a $b$-metric space with coefficient $s \geq 1$ and $f,g : X \to X$ be two selfmaps. Assume that $f,g$ is a generalized $Z$-contraction pair of maps. Then $u$ is a fixed point $f$ if and only if $u$ is a fixed point of $g$. Moreover, in that case $u$ is unique.*

**Proof.** Let $u$ be a fixed point of $f$. i.e., $fu = u$.

Suppose $gu \neq u$.

We consider

$$\zeta(s^4 d(fu, gfu), M(u, fu)) = \zeta(s^4 d(u, gu), M(u, u)) \geq 0, \tag{2.1}$$

where

$$\begin{aligned} M(u,u) = & \max\{d(u,u),\ d(u,fu),\ d(u,gu),\ \frac{1}{2s}[d(u,gu)+d(u,fu)]\} \\ =& \max\{0,0,d(u,gu),\frac{1}{2s}d(u,gu)\} \\ =& d(u,gu). \end{aligned}$$

Now using the value of $M(u,u)$ in (2.1), we get

$$0 \leq \zeta(s^4 d(u,gu), M(u,u)) = \zeta(s^4 d(u,gu), d(u,gu)) < d(u,gu) - s^4 d(u,gu) \leq 0,$$

a contradiction.

Hence $gu = u$, so that $u$ is a common fixed point of $f$ and $g$.

Similarly, it is easy to see that if $u$ is a fixed point of $g$ then $u$ is a fixed point of $f$ also.

Suppose $u$ and $v$ are two common fixed point of $f$ and $g$ with $u \neq v$. From the inequality (1.3), we have

$$\zeta(s^4 d(u,v), M(u,v)) = \zeta(s^4 d(fu, gv), M(u,v)) \geq 0, \tag{2.2}$$

where

$$\begin{aligned} M(u,v) =& \max\{d(u,v),\ d(u,fu),\ d(v,gv),\ \frac{1}{2s}[d(u,gv)+d(v,fu)]\} \\ =& \max\{d(u,v),0,0,\frac{1}{s}d(u,v)\} \\ =& d(u,v). \end{aligned}$$

Now using the value of $M(u,v)$ in (2.1), we get

$$0 \leq \zeta(s^4 d(fu,gv), M(u,v)) = \zeta(s^4 d(u,v), d(u,v)) < d(u,v) - s^4 d(u,v) \leq 0,$$

a contradiction.

Hence $u = v$. Therefore the proposition follows.

**Theorem 2.1.** *Let $(X,d)$ be a complete b-metric space with coefficient $s \geq 1$, and $f,g : X \to X$ be two selfmaps. Assume that $f,g$ is a generalized Z-contraction pair of maps. Then $f,g$ have a unique common fixed point in $X$, provided either $f$ or $g$ is b-continuous.*

**Proof.** Let $x_0 \in X$ be arbitrary. Since $fX \subseteq X$ and $gX \subseteq X$, there exists $x_1, x_2 \in X$ such that $fx_0 = x_1$ and $gx_1 = x_2$. Similarly there exist $x_3, x_4 \in X$ such that $fx_2 = x_3$ and $gx_3 = x_4$.
In general, we construct a sequence $\{x_n\}$ by $fx_{2n} = x_{2n+1},\ gx_{2n+1} = x_{2n+2}$ for $n = 0,1,2,\dots$.

Suppose that $x_{2n} = x_{2n+1}$ for some $n$, then $x_{2n} = fx_{2n}$ so that $x_{2n}$ is a fixed point of $f$.
Hence by Proposition 2.1, we have $x_{2n}$ is a fixed point of $g$ also so that $x_{2n}$ is a common fixed point of $f$ and $g$.
Similarly, if $x_{2n+1} = x_{2n+2}$ for some $n$. Then $x_{2n+1}$ is a common fixed point of $f$ and $g$.
Hence in both the cases, $f$ and $g$ have a common fixed point.
Hence with out loss of generality, we assume that $x_n \neq x_{n+1}$ for all $n$.
Now, we consider

$$\zeta(s^4 d(x_{2n+1}, x_{2n+2}), M(x_{2n}, x_{2n+1})) = \zeta(s^4 d(fx_{2n},\ gx_{2n+1}), M(x_{2n}, x_{2n+1})) \geq 0, \tag{2.3}$$

where

$$\begin{aligned} M(x_{2n}, x_{2n+1}) &= \max\{d(x_{2n}, x_{2n+1}),\ d(x_{2n}, fx_{2n}),\ d(x_{2n+1}, gx_{2n+1}), \\ &\qquad \frac{1}{2s}[d(x_{2n}, gx_{2n+1}) + d(x_{2n+1}, fx_{2n})]\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \\ &\qquad \frac{1}{2s}[d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})]\} \\ &= \max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\}. \end{aligned}$$

Suppose that $d(x_{2n}, x_{2n+1}) < d(x_{2n+1}, x_{2n+2})$ for some $n \in \mathbb{N}$.
Hence, we have $M(x_{2n}, x_{2n+1}) = d(x_{2n+1}, x_{2n+2})$.
Now using the value of $M(x_{2n}, x_{2n+1})$ and from (2.3), we have

$$\begin{aligned} 0 \leq \zeta(s^4 d(x_{2n+1}, x_{2n+2}), M(x_{2n+1}, x_{2n+1})) &= \zeta(s^4 d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n+2})) \\ &< d(x_{2n+1}, x_{2n+2}) - s^4 d(x_{2n+1}, x_{n+2}) \leq 0, \end{aligned}$$

a contradiction.

Therefore $d(x_{2n}, x_{2n+1}) \geq d(x_{2n+1}, x_{2n+2})$.

Similarly, we can prove that $d(x_{2n+1},x_{2n+2}) \geq d(x_{2n+2},x_{2n+3})$.

Hence $d(x_{n+1},x_{n+2}) \leq d(x_n,x_{n+1})$ for all $n$.

Therefore $\{d(x_n,x_{n+1})\}$ is a decreasing sequence and bounded below by zero. Thus there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(x_n,x_{n+1}) = r$.

Suppose that $r > 0$. Now, using condition $(\zeta_3)$ with $t_n = d(x_{2n+1},x_{2n+2})$ and $s_n = d(x_{2n},x_{2n+1})$, we have $0 \leq \limsup\limits_{n\to\infty} \zeta(s^4 d(x_{2n+1},x_{2n+2}), M(x_{2n},x_{2n+1})) < 0$, a contradiction.

Therefore $r = 0$. i.e.,

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{2.4}$$

Now we prove that $\{x_n\}$ is a $b$-Cauchy sequence. For this it is sufficient to show that the subsequence $\{x_{2n}\}$ is a $b$-Cauchy sequence in $X$. Suppose that $\{x_{2n}\}$ is not a $b$-Cauchy sequence. Then there exist an $\varepsilon > 0$ and sequences positive integers $\{2m_k\}$ and $\{2n_k\}$ with $2n_k > 2m_k > k$ such that

$$d(x_{2m_k},x_{2n_k}) \geq \varepsilon \text{ and } d(x_{2m_k},x_{2n_k-2}) < \varepsilon. \tag{2.5}$$

Now, we consider

$$\zeta(s^4 d(x_{2n_k+1},x_{2m_k}), M(x_{2n_k},x_{2m_k-1})) = \zeta(s^4 d(fx_{2n_k},\ gx_{2m_k-1}), M(x_{2n_k},x_{2m_k-1})) \geq 0, \tag{2.6}$$

where

$$\begin{aligned} M(x_{2n_k},x_{2m_k-1}) = \max\{& d(x_{2n_k},x_{2m_k-1}),\ d(x_{2n_k},fx_{2n_k}), d(x_{2m_k-1},gx_{2m_k-1}), \\ & \frac{1}{2s}[d(x_{2n_k},gx_{2m_k-1}) + d(fx_{2n_k},x_{2m_k-1})]\} \\ = \max\{& d(x_{2n_k},x_{2m_k-1}), d(x_{2n_k},x_{2n_k+1}), d(x_{2m_k},x_{2m_k-1}), \\ & \frac{1}{2s}[d(x_{2n_k},x_{2m_k}) + d(x_{2n_k+1},x_{2m_k-1})]\}. \end{aligned}$$

Now, we consider the following two cases

Case (i) : $s = 1$.

In this case $(X,d)$ is a metric space. Then by Lemma 1.1 there exist $\varepsilon > 0$ and sequence of positive integers $\{2n_k\}$ and $\{2m_k\}$ such that $2n_k > 2m_k \geq k$ with $d(x_{2m_k},x_{2n_k}) \geq \varepsilon$ and $d(x_{2m_k},x_{2n_k-2}) < \varepsilon$ satisfying (i)- (iv) of Lemma 1.1.

Hence we have

$$M(x_{2n_k},x_{2m_k-1}) = \max\{d(x_{2n_k},x_{2m_k-1}),d(x_{2n_k},x_{2n_k+1}),d(x_{2m_k-1},x_{2m_k}), \frac{d(x_{2n_k},x_{2m_k})+d(x_{2m_k-1},x_{2n_k+1})}{2}\}. \tag{2.7}$$

On taking limit as $k\to\infty$ we have $\lim_{k\to\infty} M(x_{2n_k},x_{2m_k-1}) = \varepsilon$.

Using condition $(\zeta_3)$ with $t_n = d(x_{2n_k+1},x_{2m_k})$ and $s_n = M(x_{2n_k},x_{2m_k-1})$, we have

$0 \le \limsup_{k\to\infty} \zeta(d(x_{2n_k+1},x_{2m_k}),M(x_{2n_k},x_{2m_k-1})) < 0$,

a contradiction.

Case (ii) : $s > 1$.

In this case, by Lemma 1.2 there exist $\varepsilon > 0$ and sequences of positive integers $\{2n_k\}$ and $\{2m_k\}$ such that $2n_k > 2m_k \ge k$ with $d(x_{2m_k},x_{2n_k}) \ge \varepsilon$ and $d(x_{2m_k},x_{2n_k-2}) < \varepsilon$ satisfying (i) - (iv) of Lemma 1.2.

Again taking limit as $k\to\infty$ in the equation (2.7) and using conditions (i) - (iv) of Lemma 1.2, we have $\lim_{k\to\infty} M(x_{2n_k},x_{2m_k-1}) = \max\{s^2\varepsilon,0,0,\frac{s\varepsilon+s^3\varepsilon}{2s}\} = s^2\varepsilon$.

Hence from (1.3), we have

$0 \le \zeta(s^4 d(fx_{2n_k},gx_{2m_k-1}),M(x_{2n_k},x_{2m_k-1}))$.

Now we have

$$\begin{aligned} 0 &\le \limsup_{k\to\infty} \zeta(s^4 d(fx_{2n_k},gx_{2m_k-1}),M(x_{2n_k},x_{2m_k-1})) \\ &\le \limsup_{k\to\infty}[M(x_{2n_k},x_{2m_k-1}) - s^4 d(x_{2n_k+1},x_{2m_k})] \\ &\le \limsup_{k\to\infty} M(x_{2n_k},x_{2m_k-1}) - s^4 \liminf_{k\to\infty} d(x_{2n_k+1},x_{2m_k}) \le s^2\varepsilon - s^4(\frac{\varepsilon}{s}) < 0, \end{aligned}$$

a contradiction.

Therefore by case (i) and case (ii), we have $\{x_{2n}\}$ is a b-Cauchy sequence in $(X,d)$.

Hence $\{x_n\}$ is a $b$-Cauchy sequence in $(X,d)$. Since $X$ is a complete $b$-metric space, we have $\{x_n\}$ is $b$-convergent to some point $x$(say) in $X$.

Therefore $x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} fx_{2n}$ and $x = \lim_{n\to\infty} x_{2n+2} = \lim_{n\to\infty} gx_{2n+1}$ so that $\lim_{n\to\infty} fx_{2n} = x = \lim_{n\to\infty} gx_{2n+1}$.

We, assume that $f$ is $b$-continuous.

Since $x_{2n}\to x$ as $n\to\infty$, we have $fx_{2n}\to fx$ as $n\to\infty$.

Hence

$$0 \leq d(x, fx) \leq s(d(x, fx_{2n}) + d(fx_{2n}, fx)) \to 0 \text{ as } n \to \infty$$

so that $d(x, fx) = 0$.

Hence $x$ is a fixed point of $f$.

Now by proposition 2.1, we have $x$ is a unique common fixed point of $f$ and $g$.

Similarly, we can prove that $x$ is a unique common fixed point of $f$ and $g$ whenever $g$ is $b$-continuous.

Hence the theorem follows.

**Theorem 2.2.** *Let $(X, d)$ be a complete b-metric space with coefficient $s \geq 1$. Let $\{f_n\}$ be a sequence of selfmaps defined on a b-metric space $(X, d)$. Assume that for each $i \neq 1$, $(f_1, f_i)$ is a generalized Z-contraction pair of maps. Then the sequence $\{f_i\}$ has a unique common fixed point in X, provided that at least one of the maps $f_i$ is b-continuous.*

**Proof** Fix $i \neq 1$. Since $(f_1, f_i)$ is a generalized $Z$-contraction pair of maps, by Theorem 2.1, we have $(f_1, f_i)$ is a unique common fixed point in $X$.

Hence the conclusion of theorem follows.

## 3. Corollaries and examples

**Corollary 3.1.** *Let $(X, d)$ be a complete b-metric space with coefficient $s \geq 1$ and $f : X \to X$ be a selfmap on $X$. If there exists simulation function $\zeta$ such that*

$$\zeta(s^4 d(fx, fy), M(x, y)) \geq 0 \textit{ for all } x, y \in X, \tag{3.1}$$

*where $M(x,y) = max\{d(x,y), d(x, fx), d(y, fy), \frac{d(x,fy)+d(y,fx)}{2}\}$,*
*then $f$ has a unique fixed point in X, provided $f$ is b-continuous.*

**Proof.** Follows by choosing $g = f$ in Theorem 2.1.

**Remark 3.1.** Corollary 3.1 extends Theorem 1.2 to $b$-metric spaces.

**Corollary 3.2.** *Let $(X, d)$ be a complete b-metric space with coefficient $s \geq 1$. Let $f, g : X \to X$ be two selfmaps on $X$. Assume that there exist two continuous functions $\psi, \varphi : [0, \infty) \to [0, \infty)$*

*with* $\varphi(t) < t \leq \psi(t)$ *for all* $t > 0$ *and* $\varphi(t) = \psi(t) = 0$ *if and only if* $t = 0$ *such that*

$$\psi(s^4 d(fx, gy)) \leq \varphi(M(x,y)) \tag{3.2}$$

*where* $M(x,y) = max\{d(x,y), d(x,fx), d(y,gy), \frac{d(x,gy)+d(y,fx)}{2}\}$,
*for all* $x, y \in X$.
*Then* $f$ *and* $g$ *have a common fixed point in X, provided either* $f$ *or* $g$ *is b-continuous.*

**Proof.** Follows from Theorem 2.1 by choosing $\zeta(s,t) = \varphi(t) - \psi(s)$ for all $t, s \in [0, \infty)$.

**Remark 3.2.** If $g = f$ and $s = 1$ in Theorem 2.1 then Theorem 1.2 follows as a corollary.

The following is an example in support of Theorem 2.1.

**Example 3.1.** Let $X = [0, \infty)$ and let $d : X \times X \to [0, \infty)$ be defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 4 & \text{if } x, y \in [0,1) \\ 5 + \frac{1}{x+y} & \text{if } x, y \in [1, \infty) \\ \frac{66}{25} & \text{otherwise.} \end{cases}$$

Then clearly $d$ is a complete $b$-metric space with coefficient $s = \frac{25}{24}$.
Here we observe that when $x = \frac{3}{2}, z = 2 \in [1, \infty)$ and $y \in [0,1)$, we have
$d(x,z) = 5 + \frac{1}{x+z} = 5 + \frac{2}{7} = \frac{37}{7}$ and $d(x,y) + d(y,z) = \frac{66}{25} + \frac{66}{25} = \frac{132}{25}$ so that

$$d(x,z) \nleq d(x,y) + d(y,z).$$

Hence, the given $d$ is a $b$-metric with $s = \frac{25}{24} (> 1)$ but not a metric.
Now we define $f, g : X \to X$ by

$$fx = \begin{cases} \frac{x}{4} + 2 & \text{if } x \in [0,1) \\ 3x - 2 & \text{if } x \in [1, \infty) \end{cases} \quad \text{and} \quad gx = \begin{cases} x & \text{if } x \in [0,1) \\ \frac{1}{x} & \text{if } x \in [1, \infty). \end{cases}$$

Then clearly $f$ and $g$ are $b$-continuous functions.

Now we define $\zeta : [0, \infty) \times [0, \infty) \to [0, \infty)$ by $\zeta(s,t) = \frac{4}{5}t - s$.

We have the following possible cases.

<u>*Case*</u> (i): $x, y \in [0,1)$.
In this case, $fx = \frac{x}{4} + 2 \in [1, \infty), gy = y \in [0,1), d(x,y) = 4$,
$d(x,fx) = \frac{66}{25}, d(y,gy) = 4$ and $\frac{d(x,gy)+d(y,fx)}{2s} = \frac{4+\frac{66}{25}}{2\times\frac{25}{24}} = \frac{1992}{625}$.

Therefore $M(x,y) = 4$ and $s^4 d(fx,gy) = (\frac{25}{24})^4 \times \frac{66}{25} = \frac{1031250}{331776}$.

Now,we consider

$$\zeta(s^4 d(fx,gy), M(x,y)) = \frac{4}{5}M(x,y) - s^4 d(fx,gy) = \frac{4}{5} \times 4 - \frac{1031250}{331776} > 0.$$

*<u>Case</u>* (ii): $x, y \in [1,\infty)$.

In this case, $fx = 3x-2 \in [1,\infty), gy = \frac{1}{y} \in [0,1), d(x,y) = 5 + \frac{1}{x+y} \geq 5$,
$d(x,fx) = 5 + \frac{1}{4x-2} \geq 5, d(y,gy) = \frac{66}{25}$, and $\frac{d(x,gy)+d(y,fx)}{2s} = \frac{\frac{66}{25}+5+\frac{1}{x+y}}{2\times\frac{25}{24}} \geq \frac{39}{10}$.
Therefore $M(x,y) \geq d(x,y) \geq 5$ and $s^4 d(fx,gy) = (\frac{25}{24})^4 \times \frac{66}{25} = \frac{1031250}{331776}$.

Now, we consider

$$\begin{aligned}\zeta(s^4 d(fx,gy), M(x,y)) &= \frac{4}{5}M(x,y) - s^4 d(fx,gy)\\ &\geq \frac{4}{5}d(x,y) - s^4 d(fx,gy)\\ &\geq \frac{4}{5} \times 5 - \frac{1031250}{331776} > 0.\end{aligned}$$

*<u>Case</u>* (iii): $x \in [0,1), y \in [1,\infty)$.

In this case, $fx = \frac{x}{4} + 2 \in [1,\infty), gy = \frac{1}{y} \in [0,1), d(x,y) = \frac{66}{25}$,
$d(x,fx) = \frac{66}{25}, d(y,gy) = \frac{66}{25}$,and $\frac{d(x,gy)+d(y,fx)}{2s} = \frac{4+5+\frac{1}{y+\frac{x}{4}+2}}{2\times\frac{25}{24}} = \frac{12}{25} \times (9 + \frac{4}{8+x+4y}) \geq 4$.
Therefore $M(x,y) \geq \frac{d(x,gy)+d(y,fx)}{2s} \geq 4$ and $s^4 d(fx,gy) = (\frac{25}{24})^4 \times \frac{66}{25} = \frac{1031250}{331776}$.

Now we consider

$$\begin{aligned}\zeta(s^4 d(fx,gy), M(x,y)) &= \frac{4}{5}M(x,y) - s^4 d(fx,gy)\\ &\geq \frac{4}{5}\frac{d(x,gy)+d(y,fx)}{2s} - s^4 d(fx,gy)\\ &\geq \frac{4}{5} \times 4 - \frac{11031250}{331776} > 0.\end{aligned}$$

*<u>Case</u>* (iv): $x \in [1,\infty), y \in [0,1)$.

In this case, $fx = 3x-2 \in [1,\infty), gy = y \in [0,1), d(x,y) = \frac{66}{25}$,
$d(x,fx) = 5 + \frac{1}{4x-2} \geq 5, d(y,gy) = 4$and $\frac{d(x,gy)+d(y,fx)}{2s} = \frac{(\frac{66}{25}+\frac{66}{25})}{2\times\frac{25}{24}} = \frac{1584}{625}$.
Therefore $M(x,y) \geq d(x,fx) \geq 5$ and $s^4 d(fx,gy) = (\frac{25}{24})^4 \times \frac{66}{25} = \frac{1031250}{331776}$.

We now con sider

$$\zeta(s^4 d(fx,gy),M(x,y)) = \frac{4}{5}M(x,y) - s^4 d(fx,gy) \geq \frac{4}{5}d(x,fx) - s^4 d(fx,gy)$$
$$\geq \frac{4}{5}\times 5 - \frac{1031250}{331776} > 0.$$

Hence $(f,g)$ is a generalized $Z$-contraction pair of maps and satisfy all the hypotheses of Theorem 2.1 and $x=1$ is the unique common fixed point of $f$ and $g$.

**Example 3.2.** Let $X$, $d$, $\zeta$ as in Example 3.1 and define $f_1$, $f_i$ for $i \geq 2$ by

$$f_1(x) = \begin{cases} x & \text{if } x \in [0,1) \\ \frac{1}{x} & \text{if } x \in [1,\infty). \end{cases} \quad \text{and} \quad f_i(x) = \begin{cases} \frac{x}{4}+i & \text{if } x \in [0,1) \\ 3x-2 & \text{if } x \in [1,\infty). \end{cases}$$

Then $(f_1, f_i)$ for $i \geq 2$ is a generalized $Z$-contraction pair of maps and the sequence $\{f_i\}$ has a unique common fixed point 1.

## Conflict of Interests

The authors declare that there is no conflict of interests.

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