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EXISTENCE OF ENDPOINTS OF MULTI-VALUED ALMOST $\varphi-$CYCLIC WEAKLY CONTRACTIVE MAPS

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Abstract. We introduce multi-valued almost $\varphi-$cyclic weakly contractive maps and prove the existence and uniqueness of endpoints in complete metric spaces when such map has the approximate cyclic endpoint property. Our results generalize the earlier results that are existing in the literature. Examples are provided in support of our results and for the justification of the hypotheses.

Keywords: Hausdorff metric, endpoint; fixed point; cyclic representation; multi-valued $\varphi-$cyclic weakly contractive map; approximate cyclic endpoint property.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

In 1969, Nadler[8] extended a well known Banach contraction principle to multi-valued mappings. Since then, fixed point theory draws many authors' attention towards the study of fixed points for multi-valued mappings in various metric spaces.

In 2003, Kirk, Srinivasan and Veeramani[5] introduced the concept of cyclic mappings and extended Banach contraction principle in the case of cyclic contraction mappings. Now a days,

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the study of the existence and uniqueness of endpoints for a multi-valued mapping in metric spaces has gained a lot of importance. For more literature, we refer [2, 3, 4, 6, 7, 10, 11, 12] and the related references therein.

Let (X,d) be a metric space, $P_{cl,bd}(X)$ be the set of all closed and bounded subsets of X and we consider the Hausdorff metric $\mathscr{H}$ on $P_{cl,bd}(X)$ induced by d, that is,
$\mathscr{H}(A,B)=\max\{\sup\limits_{u\in A} dist(u,B), \sup\limits_{v\in B} dist(A,v)\}$ where $dist(x,A)=\inf\{d(x,a):a\in A\}$. Also, we denote $\delta(x,A)=\sup\{d(x,a):a\in A\}$.

Let $T:X\to P_{cl,bd}(X)$ be a multi-valued map. A point $x\in X$ is called a fixed point (endpoint) of T if $x\in Tx$ ($Tx=\{x\}$) and we denote the set of all fixed points of T by $Fix(T)$ and the set of all endpoints of T by $End(T)$.

Definition 1.1. [7] *Let X be a metric space, $T:X\to P_{cl,bd}(X)$ be a multi-valued mapping, $f:X\to X$ be a self-map and $\{X_i\}_{i=1}^m$ be a nonempty class of nonempty subsets of X.*
Let $X_n=X_i$ if $n\equiv i$ (mod m). Then

(i) $X=\bigcup\limits_{i=1}^m X_i$ *is called a cyclic representation on X with respect to T if $Tx_i\subseteq X_{i+1}$, $x_i\in X_i$ for $i\in\{1,2,...m\}$ with $X_{m+1}=X_1$.*

(ii) $X=\bigcup\limits_{i=1}^m X_i$ *is called a cyclic representation on X with respect to f if $f(x_i)\in X_{i+1}$, $x_i\in X_i$ for $i\in\{1,2,...m\}$ with $X_{m+1}=X_1$.*

(iii) T has the approximate cyclic endpoint property if there exists a sequence $\{x_n\}_{n=1}^\infty$ such that $x_n\in X_n$ for every n and $\lim\limits_{n\to\infty}\mathscr{H}(\{x_n\},Tx_n)=0$.

(iv) We say that the fixed point problem is well posed for T if T has a unique endpoint and for any sequence $\{x_n\}$ such that $x_n\in X_n$ and $\lim\limits_{n\to\infty}\mathscr{H}(\{x_n\},Tx_n)=0$ implies that $\lim\limits_{n\to\infty}x_n=x$ for some $x\in X$.

Remark 1.1. *If $X=\bigcup\limits_{i=1}^m X_i$ is a cyclic representation on X with respect to T (or f) then $X=\bigcap\limits_{i=1}^m X_i$ contains all fixed points of T (or f).*

Throught this paper, we denote $[0,\infty)$ by $\mathbb{R}^+$. We write

$$\Phi=\{\varphi:\mathbb{R}^+\to\mathbb{R}^+/\ (i)\ \varphi(t)=0 \text{ if and only if } t=0,\ (ii)\ \varphi(t)<t \text{ for } t>0 \text{ and } (iii)\ \varphi(t_n)\to 0 \text{ implies that } t_n\to 0\}.$$

Theorem 1.1. ([7], Theorem 3.1) *Let (X,d) be a complete metric space, $\{X_i\}_{i=1}^m$ be a nonempty class of nonempty closed subsets of X, $T : X \to P_{cl,bd}(X)$ be a multi-valued map and $X = \bigcup_{i=1}^{m} X_i$ be a cyclic representation on X with respect to T. Let T be a cyclic weak $\varphi-$ contraction map for some $\varphi \in \Phi$, i.e., T satisfies*

$$\mathscr{H}(Tx,Ty) \leq d(x,y) - \varphi(d(x,y)) \tag{1}$$

for $x \in X_i$ and $y \in X_{i+1}$, $i = 1,2,...,m$ with $X_{m+1} = X_1$. Then T has a unique endpoint if and only if T has the approximate cyclic endpoint property, that is, the fixed point problem is well posed for T. Moreover, $Fix(T) = End(T)$.

We call the map T of Theorem 1.1 that satisfies the inequality (1) as a *multi-valued $\varphi-$cyclic weakly contractive map*. This name is suitable for such maps. For more details, we refer Alber and Guerre-Delabriere[1] and Rhoades[9].

Theorem 1.2. ([7], Theorem 3.4) *Let (X,d) be a complete metric space, $\{X_i\}_{i=1}^m$ be a nonempty class of nonempty closed subsets of X, $f : X \to X$ be a selfmap and $X = \bigcup_{i=1}^{m} X_i$ be a cyclic representation on X with respect to f. Let f be a cyclic $\varphi-$contraction for some $\varphi \in \Phi$, i.e.,*

$$d(fx,fy) \leq d(x,y) - \varphi(d(x,y)) \tag{2}$$

for $x \in X_i$, $y \in X_{i+1}$, $i = 1,2,...,m$ with $X_{m+1} = X_1$. Then f has a unique fixed point.

Here we observe that the φ that is applied to prove Theorem 1.1 and Theorem 1.2, the condition (ii) of $\varphi \in \Phi$ is not used any where in the proofs. Hence, we replace Φ by Φ_1 where $\Phi_1 = \{\varphi : \mathbb{R}^+ \to \mathbb{R}^+ / (i)\ \varphi(t) = 0$ if and only if $t = 0,$ and

$$(ii)\ \varphi(t_n) \to 0 \text{ implies that } t_n \to 0\}.$$

In fact Φ_1 is larger than Φ, for example, the mapping $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $\varphi(t) = t + \sin t$, $t \geq 0$, is in Φ_1 but not in Φ.

In Section 2, we define multi-valued 'almost $\varphi-$cyclic weakly contractive maps for $\varphi \in \Phi_1$' and show that the class of all such maps is larger than the 'class of multi-valued $\varphi-$cyclic weakly contractive maps for $\varphi \in \Phi_1$'. In Section 3, we prove the existence of endpoints of multi-valued almost $\varphi-$cyclic weakly contractive map (Theorem 3.1). Also, we show the importance of considering $L>0$ in the almost $\varphi-$cyclic weakly contractive map in Theorem 3.1 (Example3.1). Our result generalize Theorem 1.1.

2. Preliminaries

In the following, we define multi-valued almost $\varphi-$cyclic weakly contractive maps for $\varphi \in \Phi_1$.

Definition 2.1. *Let X be a metric space, $T : X \to P_{cl,bd}(X)$ be a multi-valued mapping, $\{X_i\}_{i=1}^m$ be a nonempty class of nonempty subsets of X and $X = \bigcup\limits_{i=1}^{m} X_i$ be a cyclic representation on X with respect to T. If there exist $L \geq 0$ and $\varphi \in \Phi_1$ such that*

$$\mathscr{H}(Tx,Ty) \leq M(x,y) - \varphi(d(x,y)) + L\delta(x,Tx) \tag{3}$$

where $M(x,y) = \max\{d(x,y), \delta(x,Tx), \delta(y,Ty), \frac{\delta(x,Ty)+\delta(y,Tx)}{2}\}$, $x \in X_i$, $y \in X_{i+1}$ for $i = 1,2,...,m$ with $X_{m+1} = X_1$ then we say that T is a multi-valued almost $\varphi-$cyclic weakly contractive map.

Here we note that a map T that satisfies (1) is continuous, whereas a map T that satisfies (3) need not be continuous. Also, we note that a map T that satisfies (1) implies that it satisfies (3) but its converse is not true due to the following example.

Example 2.1. *We consider the usual metric on $X = [0,1]$ so that X is a complete metric space and $X_1 = [0, \frac{1}{2}]$, $X_2 = [\frac{1}{4}, \frac{3}{4}]$, $X_3 = [\frac{3}{8}, 1]$ are closed subsets of X.*
We define $T : X \to P_{cl,bd}(X)$ by

$$Tx = \begin{cases} [\frac{3}{8}+x, \frac{1}{2}-x] & \text{if } \ 0 \leq x \leq \frac{1}{16} \\ [\frac{1}{2}-x, \frac{3}{8}+x] & \text{if } \ \frac{1}{16} \leq x \leq \frac{1}{8} \\ [\frac{1}{2}-\frac{x}{2}, \frac{3}{8}+\frac{x}{2}] & \text{if } \ \frac{1}{8} < x \leq \frac{1}{4} \\ [\frac{1}{2}-\frac{x}{4}, \frac{3}{8}+\frac{x}{4}] & \text{if } \ \frac{1}{4} < x \leq \frac{1}{2} \\ [\frac{1}{2}-\frac{x}{8}, \frac{3}{8}+\frac{x}{8}] & \text{if } \ \frac{1}{2} < x \leq 1. \end{cases}$$

Then $X = \bigcup\limits_{i=1}^{3} X_i$ is a cyclic representation on X with respect to T.

We define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *by* $\varphi(t) = \frac{4t}{5}$ *so that* $\varphi \in \Phi_1$.
Then T *is a multi-valued almost* $\varphi-$*cyclic weakly contractive map with* $L = 1$*, i.e.,* $\mathscr{H}(Tx,Ty) \leq M(x,y) - \varphi(d(x,y)) + \delta(x,Tx)$ *where* $x \in X_i$ *and* $y \in X_{i+1}$ *for* $i = 1,2,3$ *with* $X_4 = X_1$*. But,* T *does not satisfy the inequality* (1)*, for example,*
we choose $x = \frac{27}{112}$ *and* $y = \frac{53}{104}$ *so that* $T(x) = [\frac{85}{224}, \frac{111}{224}]$, $T(y) = [\frac{363}{832}, \frac{365}{832}]$ *and*
$\mathscr{H}(Tx,Ty) = \frac{331}{5824} \nleq \frac{391}{7280} = d(x,y) - \varphi(d(x,y))$.

Here we note that T *is not continuous at* $\frac{1}{2} \in \bigcap_{i=1}^{3} X_i = [\frac{3}{8}, \frac{1}{2}]$*, for example, we consider the sequence* $p_n = \frac{n+3}{2n+2}$ *so that* $p_n \to \frac{1}{2}$, $Tp_n = [\frac{1}{2} - \frac{p_n}{8}, \frac{3}{8} + \frac{p_n}{8}] = [\frac{7n+5}{16n+16}, \frac{7n+9}{16n+16}]$, $T\frac{1}{2} = [\frac{3}{8}, \frac{1}{2}]$ *and* $\mathscr{H}(Tp_n, T\frac{1}{2}) = \max\{0, \frac{n-1}{16n+16}\} = \frac{n-1}{16n+16} \nrightarrow 0$.

The following lemma is useful in proving our main results.

Lemma 2.1. *Let* (X,d) *be a metric space and* $\{x_n\}_{n=1}^{\infty}$ *be a sequence in* X *such that* $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$*. If* $\{x_n\}_{n=1}^{\infty}$ *is not a Cauchy sequence in* X *then there exist* $\varepsilon > 0$ *and a subsequence* $\{x_{n(k)}\}_{k=1}^{\infty}$ *of* $\{x_n\}_{n=1}^{\infty}$ *such that* $\lim_{k\to\infty} d(x_{n(k)},x_{n(k+1)}) = \varepsilon$*. Moreover, there exists a positive integer* N_0 *such that* $n(k+1) - n(k) \geq 2$ *for all* k *with* $n(k) \geq N_0$.

Proof. Since $\{x_n\}_{n=1}^{\infty}$ is not Cauchy, there exist $\varepsilon > 0$ and a strictly increasing sequence $\{n(k)\}_{k=1}^{\infty}$ of positive integers such that $n(k+1)$ is the smallest positive integer greater than $n(k)$ such that $d(x_{n(k)},x_{n(k+1)}) \geq \varepsilon$ for $k = 1,2,3,\ldots$.
We consider

$$\begin{aligned}\varepsilon \leq d(x_{n(k+1)},x_{n(k)}) &\leq d(x_{n(k+1)},x_{n(k+1)-1}) + d(x_{n(k+1)-1},x_{n(k)}) \\ &\leq d(x_{n(k+1)},x_{n(k+1)-1}) + \varepsilon \text{ for } k = 1,2,\ldots .\end{aligned}$$

Since $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$, we have $\lim_{k\to\infty} d(x_{n(k)},x_{n(k+1)}) = \varepsilon$. Also, there is a positive integer N_0 such that $d(x_n,x_{n+1}) < \varepsilon$ for $n \geq N_0$. If $n(k+1) = n(k)+1$ for some k with $n(k) \geq N_0$ then $d(x_{n(k)},x_{n(k+1)}) = d(x_{n(k)},x_{n(k)+1}) < \varepsilon$, a contradiction.
Therefore $n(k+1) - n(k) \geq 2$ for all k with $n(k) \geq N_0$. □

3. ENDPOINTS OF MULTI-VALUED ALMOST $\varphi-$CYCLIC WEAKLY CONTRACTIVE MAPS

Proposition 3.1. *Let (X,d) be a complete metric space, $\{X_i\}_{i=1}^m$ be a nonempty class of nonempty closed subsets of X and $T : X \to P_{cl,bd}(X)$ be a map. Assume that $X = \bigcup\limits_{i=1}^{m} X_i$ is a cyclic representation on X with respect to T and T is a multi-valued almost $\varphi-$cyclic weakly contractive map. Let $X_n = X_i$ whenever $n \equiv i \pmod m$ and $\{x_n\}_{n=1}^\infty$ be an arbitrary sequence in X such that $x_n \in X_n$ for every n and $\lim\limits_{n\to\infty} \mathscr{H}(\{x_n\}, Tx_n) = 0$. Then $\{x_n\}_{n=1}^\infty$ converges to a point of $\bigcap\limits_{i=1}^{m} X_i$.*

Proof. By taking $x = x_n$ and $y = x_{n+1}$ in the inequality (3), we have

$$\varphi(d(x_n,x_{n+1})) \leq M(x_n,x_{n+1}) + L\delta(x_n,Tx_n) - \mathscr{H}(Tx_n,Tx_{n+1}), \tag{4}$$

where
$$\begin{aligned} M(x_n,x_{n+1}) &= \max\{d(x_n,x_{n+1}), \delta(x_n,Tx_n), \delta(x_{n+1},Tx_{n+1}), \tfrac{\delta(x_n,Tx_{n+1})+\delta(x_{n+1},Tx_n)}{2}\} \\ &= \max\{\mathscr{H}(\{x_n\},\{x_{n+1}\}), \mathscr{H}(\{x_n\},Tx_n), \mathscr{H}(\{x_{n+1}\},Tx_{n+1}), \\ &\qquad \tfrac{\mathscr{H}(\{x_n\},Tx_{n+1})+\mathscr{H}(\{x_{n+1}\},Tx_n)}{2}\} \\ &\leq \mathscr{H}(\{x_n\},Tx_n) + \mathscr{H}(Tx_n,Tx_{n+1}) + \mathscr{H}(\{x_{n+1}\},Tx_{n+1}). \end{aligned}$$

From (4), we have

$$\begin{aligned} \varphi(d(x_n,x_{n+1})) \leq{}& \mathscr{H}(\{x_n\},Tx_n) + \mathscr{H}(Tx_n,Tx_{n+1}) + \mathscr{H}(\{x_{n+1}\},Tx_{n+1}) \\ &+ L\mathscr{H}(\{x_n\},Tx_n) - \mathscr{H}(Tx_n,Tx_{n+1}). \end{aligned}$$

On letting $n \to \infty$, it follows that $\lim\limits_{n\to\infty} \varphi(d(x_n,x_{n+1})) = 0$ so that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{5}$$

Now we prove that $\{x_n\}_{n=1}^\infty$ is Cauchy. Otherwise, by Lemma 2.1, there exist $\varepsilon > 0$ and a subsequence $\{x_{n(k)}\}_{k=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ such that

$$\lim_{k\to\infty} d(x_{n(k)},x_{n(k+1)}) = \varepsilon. \tag{6}$$

Also, there exists $N_0 \in \mathbb{Z}^+$ such that $n(k+1) - n(k) \geq 2$ for all k with $n(k) \geq N_0$. Therefore for each k with $n(k) \geq N_0$, there is an integer $l(k) \in \{1,2,...,m\}$ such that $n(k+1) - l(k) \equiv (n(k)+1) \pmod m$ so that $x_{n(k+1)-l(k)} \in X_{n(k)+1}$.

By the triangle inequality, it is easy to see that

$|d(x_{n(k)},x_{n(k+1)}) - d(x_{n(k)},x_{n(k+1)-l(k)})| \leq d(x_{n(k+1)},x_{n(k+1)-1}) + d(x_{n(k+1)-1},x_{n(k+1)-2}) + ... +$

$$d(x_{n(k+1)-l(k)+1}, x_{n(k+1)-l(k)}) \text{ for all } k.$$

On letting $k \to \infty$ from (5) and (6), it follows that

$$\lim_{k\to\infty} d(x_{n(k)}, x_{n(k+1)-l(k)}) = \varepsilon. \tag{7}$$

By taking $x = x_{n(k)}$ and $y = x_{n(k+1)-l(k)}$ in the inequality (3), we have

$\varphi(d(x_{n(k)}, x_{n(k+1)-l(k)})) \leq M(x_{n(k)}, x_{n(k+1)-l(k)}) + L\delta(x_{n(k)}, Tx_{n(k)}) - \mathscr{H}(Tx_{n(k)}, Tx_{n(k+1)-l(k)})$,

where $M(x_{n(k)}, x_{n(k+1)-l(k)}) = \max\{d(x_{n(k)}, x_{n(k+1)-l(k)}), \delta(x_{n(k)}, Tx_{n(k)}),$

$$\delta(x_{n(k+1)-l(k)}, Tx_{n(k+1)-l(k)}), \frac{\delta(x_{n(k)}, Tx_{n(k+1)-l(k)}) + \delta(x_{n(k+1)-l(k)}, Tx_{n(k)})}{2}\}.$$

It is easy to see that

$$M(x_{n(k)}, x_{n(k+1)-l(k)}) \leq \mathscr{H}(\{x_{n(k)}\}, Tx_{n(k)}) + \mathscr{H}(Tx_{n(k)}, Tx_{n(k+1)-l(k)}) + \mathscr{H}(\{x_{n(k+1)-l(k)}\}, Tx_{n(k+1)-l(k)}).$$

Therefore

$$\varphi(d(x_{n(k)}, x_{n(k+1)-l(k)}) \leq (L+1)\mathscr{H}(\{x_{n(k)}\}, Tx_{n(k)}) + \mathscr{H}(\{x_{n(k+1)-l(k)}\}, Tx_{n(k+1)-l(k)}).$$

On letting $k \to \infty$, we have $\lim_{k\to\infty} \varphi(d(x_{n(k)}, x_{n(k+1)-l(k)})) = 0$.
Since $\varphi \in \Phi_1$, we have $\lim_{k\to\infty} d(x_{n(k)}, x_{n(k+1)-l(k)}) = 0$,
a contradiction due to (7).

Therefore $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X.
Since (X,d) is a complete metric space, there exists $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

For each $1 \leq i \leq m$, we consider the subsequence $\{x_{n_i}\}$ of $\{x_n\}$ where $n_i \equiv i \pmod m$.
Since $x_{n_i} \in X_i$ and X_i is closed in X, it follows that $x \in X_i$ for every i and so $x \in \bigcap_{i=1}^{m} X_i$. □

Theorem 3.1. *In addition to the hypotheses of Proposition 3.1, if T is continuous at every point of $\bigcap_{i=1}^{m} X_i$ then T has a unique endpoint if and only if T has the approximate cyclic endpoint property.*

Proof. We assume that T has an endpoint $x \in X$ so that $x \in X_n$ for every n by Remark 1.1.
If we take $x_n = x$ for $n = 1,2,3...$ then $\lim_{n\to\infty} \mathscr{H}(\{x_n\}, Tx_n) = 0$, that is, T has the approximate cyclic endpoint property.

Conversely, we assume that T has the approximate cyclic endpoint property, i.e., there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $x_n \in X_n$ and $\lim_{n\to\infty} \mathscr{H}(\{x_n\}, Tx_n) = 0$.

Therefore by Proposition 3.1, there exists $x \in \bigcap_{i=1}^{m} X_i$ such that $\lim_{n\to\infty} x_n = x$.
By the triangle inequality, we have
$\mathscr{H}(\{x\},Tx) \leq \mathscr{H}(\{x\},\{x_n\}) + \mathscr{H}(\{x_n\},Tx_n) + \mathscr{H}(Tx_n,Tx)$ for every n and by using the continuity of T at x it follows that $\mathscr{H}(\{x\},Tx) = 0$ so that $x \in End(T)$.

Now we prove that T has a unique endpoint.
Let $x,y \in End(T)$ so that $x,y \in \bigcap_{i=1}^{m} X_i$ and

$$\begin{aligned} d(x,y) = \mathscr{H}(Tx,Ty) &\leq M(x,y) - \varphi(d(x,y) + L\delta(x,Tx) \\ &= \max\{d(x,y),0,0,d(x,y)\} - \varphi(d(x,y)). \end{aligned}$$

Therefore, $\varphi(d(x,y) = 0$ so that $x = y$. □

Remark 3.1. *Theorem 1.1 follows as a corollary to Theorem 3.1, since the inequality (1) implies the inequality (3).*

When $L = 0$ in the inequality (3), we have the following.

Corollary 3.1. *Let $\{X_i\}_{i=1}^m$ be a nonempty class of closed subsets of a complete metric space (X,d), $T : X \to P_{cl,bd}(X)$ be a multi-valued map and $X = \bigcup_{i=1}^{m} X_i$ be a cyclic representation X with respect to T. Suppose that there exists $\varphi \in \Phi_1$ such that*

$$\mathscr{H}(Tx,Ty) \leq M(x,y) - \varphi(d(x,y)) \tag{8}$$

where $M(x,y) = \max\{d(x,y),\delta(x,Tx),\delta(y,Ty),\frac{\delta(x,Ty)+\delta(y,Tx)}{2}\}$, $x \in X_i$, $y \in X_{i+1}$ for $i = 1,2,...,m$ with $X_{m+1} = X_1$. Let $X_n = X_i$ whenever $n \equiv i \pmod m$ and T is continuous on $\bigcap_{i=1}^{m} X_i$.
Then T has a unique endpoint if and only if T has the approximate cyclic endpoint property.

Corollary 3.2. *Let (X,d) be a complete metric space, $\{X_i\}_{i=1}^m$ be a nonempty class of closed subsets of X, $T : X \to P_{cl,bd}(X)$ be a multi-valued map and $X = \bigcup_{i=1}^{m} X_i$ be a cyclic representation on X with respect to T. Suppose that T satisfies*

$$\mathscr{H}(Tx,Ty) \leq M_1(x,y) - \varphi(d(x,y)) + L\delta(x,Tx) \tag{9}$$

where $L \geq 0$, $\varphi \in \Phi_1$, $M_1(x,y) = \max\{d(x,y),dist(x,Tx),dist(y,Ty),\frac{dist(x,Ty)+dist(y,Tx)}{2}\}$ where $x \in X_i$, $y \in X_{i+1}$ for $i = 1,2,...,m$ with $X_{m+1} = X_1$. Let $X_n = X_i$ whenever $n \equiv i \pmod m$ and

T is continuous at every point of $\bigcap_{i=1}^{m} X_i$. Then T has a unique endpoint if and only if T has the approximate cyclic endpoint property. Also, $Fix(T) = End(T)$.

Proof. The conclusion follows from Theorem 3.1, since the inequality (9) implies the inequality (3). We choose $x \in End(T)$. Now for any $y \in Fix(T)$, $d(x,y) \leq \mathscr{H}(Tx,Ty) \leq \max\{d(x,y),0,0,d(x,y)\} - \varphi(d(x,y))$ so that $\varphi(d(x,y)) \leq 0$ and hence $y = x \in End(T)$. Hence $Fix(T) = End(T)$. □

The following is an example in support of Theorem 3.1.

Example 3.1. *Let $X = [0,1]$ with usual metric on the real line. We define $T : X \to P_{cl,bd}(X)$ by $Tx = [a,b]$ where $a = \min\{\frac{1}{2}, 1-x\}$ and $b = \max\{\frac{1}{2}, 1-x\}$.*
Let $X_1 = [0,\frac{1}{2}]$ and $X_2 = [\frac{1}{2},1]$ so that $X_1 \cup X_2$ is a cyclic representation on X with respect to T.
We define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\varphi(t) = \begin{cases} t & \text{if } t \in [0,\frac{1}{2}), \\ \frac{t}{1+t} & \text{if } t \in [\frac{1}{2},\infty) \end{cases}$ so that $\varphi \in \Phi_1$.

Now we prove that T satisfies the inequality (3) with $L = \frac{1}{2}$.
Let $x \in X_1$ and $y \in X_2$ so that $Tx = [\frac{1}{2}, 1-x]$ and $Ty = [1-y, \frac{1}{2}]$.
We assume that $x + y \leq 1$.
For any $u \in Tx$, $dist(u,Ty) = u - \frac{1}{2}$ so that $\sup_{u \in Tx} dist(u,Ty) = \frac{1-2x}{2}$ and for any $v \in Ty$, $dist(Tx,v) = \frac{1}{2} - v$ so that $\sup_{v \in Ty} dist(Tx,v) = \frac{2y-1}{2}$.
Therefore, $\mathscr{H}(Tx,Ty) = \max\{\frac{1-2x}{2}, \frac{2y-1}{2}\} = \frac{1-2x}{2}$,
$M(x,y) = \max\{y-x, 1-2x, 2y-1, \frac{\frac{1}{2}-x+\max\{y-\frac{1}{2}, 1-x-y\}}{2}\} = 1-2x.$
It is easy to see that
$\mathscr{H}(Tx,Ty) = \frac{1-2x}{2} \leq 1-2x-(y-x) + \frac{1-2x}{2} \leq M(x,y) - \varphi(d(x,y)) + \frac{\delta(x,Tx)}{2}$.
Similarly, it is easy to verify the inequality (3) for the case $x + y > 1$.
Hence T is a multi-valued almost $\varphi-$cyclic weakly contractive map.

We choose $x_n = \begin{cases} \frac{n}{2n+4} & \text{if } n \equiv 1 \pmod 2 \\ \frac{n+4}{2n+4} & \text{if } n \equiv 0 \pmod 2. \end{cases}$
Then $x_n \in X_n$ for $n = 1,2,3...$ and $\lim_{n\to\infty} \mathscr{H}(\{x_n\}, Tx_n) = \frac{4}{2n+4} = 0$.
Hence T has the approximate cyclic endpoint property.

We observe that for any $x \in X$, $\mathscr{H}(Tx, T\frac{1}{2}) = |x - \frac{1}{2}|$ *so that* T *is continuous on* $X_1 \cap X_2 = \{\frac{1}{2}\}$. *Hence all the hypotheses of Theorem 3.1 hold and* $End(T) = \{\frac{1}{2}\}$.

Here we observe that $\mathscr{H}(T\frac{1}{4}, T\frac{3}{4}) = \frac{1}{4} \not\leq \frac{1}{6} = d(\frac{1}{4}, \frac{3}{4}) - \varphi(d(\frac{1}{4}, \frac{3}{4}))$. *Hence* T *fails to satisfy the inequality (1) so that Theorem 1.1 is not applicable.*

In the following, we give an example to show that a multi-valued almost $\varphi-$ cyclic weakly contractive map has no endpoints but may have fixed points if we relax the approximate cyclic endpoint property in Theorem 3.1.

Example 3.2. *We consider usual metric on* $X = [0,1]$ *and* $X_1 = [0, \frac{1}{2}]$, $X_2 = [\frac{1}{4}, \frac{3}{4}]$, $X_3 = [\frac{3}{8}, 1]$.

We define $T : X \to P_{cl,bd}(X)$ *by*

$$Tx = \begin{cases} [\frac{3}{8} + x, \frac{1}{2} - x] & \text{if } \; 0 \leq x \leq \frac{1}{16} \\ [\frac{1}{2} - x, \frac{3}{8} + x] & \text{if } \; \frac{1}{16} \leq x \leq \frac{1}{8} \\ [\frac{1}{2} - \frac{x}{2}, \frac{3}{8} + \frac{x}{2}] & \text{if } \; \frac{1}{8} < x \leq \frac{1}{4} \\ [\frac{1}{2} - \frac{x}{4}, \frac{3}{8} + \frac{x}{4}] & \text{if } \; \frac{1}{4} < x \leq \frac{1}{2} \\ [\frac{3}{8}, \frac{1}{2}] & \text{if } \; \frac{1}{2} \leq x \leq 1. \end{cases}$$

Then $X = \bigcup\limits_{i=1}^{3} X_i$ *is a cyclic representation on* X *with respect to* T.

We define $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *by* $\varphi(t) = \frac{4t}{5}$ *so that* $\varphi \in \Phi_1$.

Then T *satisfies (3) with* $L = 1$.

Since $\mathscr{H}(Tx, Ty) = \frac{|x-y|}{4}$ *for any* $x, y \in [\frac{1}{4}, \frac{1}{2}]$ *and* $\mathscr{H}(Tx, T\frac{1}{2}) = 0$ *for any* $x \in [\frac{1}{2}, 1]$, *it follows that* T *is continuous on* $\bigcap\limits_{i=1}^{3} X_i = [\frac{3}{8}, \frac{1}{2}]$.

If T *has the approximate cyclic endpoint property, that is, there exists a sequence* $\{x_n\}_{n=1}^{\infty}$ *such that* $\lim\limits_{n\to\infty} \mathscr{H}(\{x_n\}, Tx_n) = 0$ *where* $x_n \in X_n$ *for* $n = 1, 2, 3, \ldots$ *with* $X_n = X_i$ *whenever* $n \equiv i \pmod{m}$ *then* $\lim\limits_{n\to\infty} \mathscr{H}(\{x_{3n}\}, Tx_{3n}) = 0$.

If $\{x_{3n}\} \subseteq [\frac{3}{8}, \frac{7}{16}]$ *then* $Tx_{3n} = [\frac{1}{2} - \frac{x_{3n}}{4}, \frac{3}{8} + \frac{x_{3n}}{4}]$ *and* $\lim\limits_{n\to\infty} \mathscr{H}(\{x_{3n}\}, Tx_{3n}) = \lim\limits_{n\to\infty} \frac{3 - 6x_{3n}}{8} \neq 0$.

If $\{x_{3n}\} \subseteq [\frac{7}{16}, \frac{1}{2}]$ *then* $\lim\limits_{n\to\infty} \mathscr{H}(\{x_{3n}\}, Tx_{3n}) = \lim\limits_{n\to\infty} \frac{5x_{3n} - 2}{4} \neq 0$.

If $\{x_{3n}\} \subseteq [\frac{1}{2}, 1]$ *then* $Tx_{3n} = [\frac{3}{8}, \frac{1}{2}]$ *and* $\lim\limits_{n\to\infty} \mathscr{H}(\{x_{3n}\}, Tx_{3n}) = \lim\limits_{n\to\infty} x_{3n} - \frac{3}{8} \neq 0$,

a contradiction.

Thus T fails to satisfy the approximate cyclic endpoint property, and observe that T has no endpoints but $Fix(T) = [\frac{2}{5}, \frac{1}{2}]$.

Remark 3.2. *Remark 3.1, Example 3.1 and Example 3.2 show that Theorem 3.1 is a generalization of Theorem 1.1.*

Theorem 3.2. *Let (X,d) be a complete metric space, $\{X_i\}_{i=1}^m$ be a nonempty class of nonempty closed subsets of X, $f : X \to X$ be a selfmap and $X = \bigcup_{i=1}^{m} X_i$ be a cyclic representation on X with respect to f. Suppose that there exists $\varphi \in \Phi_1$ such that*

$$d(fx, fy) \leq M(x,y) - \varphi(d(x,y)) \tag{10}$$

where $M(x,y) = \max\{d(x,y), d(x,fx), d(y,fy), \frac{d(x,fy)+d(y,fx)}{2}\}$, $x \in X_i$, $y \in X_{i+1}$ for $i = 1,2,...,m$ with $X_{m+1} = X_1$. Further, we assume that f is continuous at every point of $\bigcap_{i=1}^{m} X_i$. Then f has a unique fixed point.

Proof. Let $X_n = X_i$ whenever $n \equiv i$ (mod m). Let $x_1 \in X_1$ be arbitrary and $x_{n+1} = f(x_n)$ so that $x_n \in X_n$ for $n = 1,2,\dots$.

We define $T : X \to P_{cl,bd}(X)$ by $Tx = \{fx\}$ for $x \in X$.

It is easy to see that T satisfies the inequality (8) and T is continuous on $\bigcap_{i=1}^{m} X_i$.

Now we show that T has the approximate cyclic endpoint property.

We assume that $fx_n \neq x_n$ for any n.

We consider

$$\begin{aligned} d(x_n,x_{n+1}) &= d(fx_{n-1}, fx_n) \\ &\leq \max\{d(x_{n-1},x_n), d(x_{n-1}, fx_{n-1}), d(x_n, fx_n), \\ &\qquad \tfrac{d(x_{n-1},fx_n)+d(x_n,fx_{n-1})}{2}\} - \varphi(d(x_{n-1},x_n)) \\ &\leq \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} - \varphi(d(x_{n-1},x_n)). \end{aligned}$$

If $d(x_{n-1},x_n) \leq d(x_n,x_{n+1})$ for some n then $\varphi(d(x_{n-1},x_n)) = 0$ so that $x_{n-1} = f(x_{n-1})$, a contradiction.

Therefore

$$d(x_n,x_{n+1}) < d(x_{n-1},x_n) - \varphi(d(x_{n-1},x_n)) \tag{11}$$

so that $\{d(x_{n-1},x_n)\}$ is a decresing sequence of nonnegative real numbers and hence $\lim\limits_{n\to\infty} d(x_{n-1},x_n)$ exists, and it is r (say), $r \geq 0$.
From (11) it follows that $\lim\limits_{n\to\infty} d(x_{n-1},x_n) = 0$ so that $\lim\limits_{n\to\infty} \mathscr{H}(\{x_{n-1}\}, Tx_{n-1}) = 0$.
Hence, by Corollary 3.1, T has a unique endpoint x.

Hence the conclusion of the theorem follows. $\square$

Remark 3.3. *Theorem 1.2 follows as a corollary to Theorem 3.2, since the inequality (2) implies the inequality (10).*

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Ya. I. Alber and S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert spaces, in: I. Gohberg, Yu. Lyubich, (Eds), New Results in Operator Theory and Its Appl., Oper. Theory Adv. Appl., 98 (1997), 7-22.

[2] A. Amini- Harandi, Endpoints of set-valued contractions in metric spaces, Nonlinear Anal., 72(1) (2010), 132-134.

[3] M. Fakhar, Endpoints of set-valued asymptotic contractions in metric spaces, Appl. Math. Lett., 24(4) (2011), 428-431.

[4] N. Hussain, A. Amini-Harandi and Y. J. Cho, Approximate endpoints for set-valued contractions in metric spaces, Fixed point Theory Appl., 2010 (2010), Article ID: 614867, 13pages.

[5] W. A. Kirk, P. S. Srinivasan and P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory., 4(1) (2003), 79-89 .

[6] S. Moradi and F. Khojasteh, Endpoints of multi-valued generalized weak contraction mappings, Nonlinear Anal., 74(6) (2011), 2170-2174.

[7] S. Moradi, Endpoints of multi-valued cyclic contraction mappings, Int. J. Nonlinear Anal. Appl., 9(1) (2018), 203-210.

[8] S. B. Nadler, Multi-valued contraction mappings, Pac. J. Math., 30(2) (1969), 475-488.

[9] B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal., 47(4) (2001), 2683-2693.

[10] D. Wardowski, Endpoints and fixed points of a set-valued contractions in cone metric spaces, Nonlinear Anal., 71 (2009), 512-516.

[11] K. Włodarczyk, D. Klim and R. Plebaniak, Existence and uniqueness of endpoints of closed set-valued asymptotic contractions in metric spaces, J. Math. Anal. Appl., 328(1) (2007), 46-57.

[12] K. Włodarczyk and R. Plebaniak, Endpoint theory for set-valued nonlinear asymptotic contractions with respect to generalized pseudodistances in uniform spaces, J. Math. Anal. Appl., 339(1) (2008), 344-358.