Available online at http://jfpt.scik.org
J. Fixed Point Theory, 2019, 2019:3
ISSN: 2052-5338

TRICYCLIC RELATIVELY NONEXPANSIVE MAPPINGS IN KOHLENBACH HYPERBOLIC SPACES

TAOUFIK SABAR, ABDELHAFID BASSOU*, MOHAMED AAMRI

Laboratory of Algebra, Analysis and Applications(L3A), Department of Mathematics and Computer Science,

Faculty of Sciences Ben M'Sik, Hassan II University of Casablanca, P.B 7955 Sidi Othmane, Casablanca,

Morocco

Abstract. In this paper, we study the structure of minimal sets of tricyclic relatively nonexpansive mappings in the setting of Kohlenbach hyperbolic spaces. This way, we obtain a best proximity point theorem for such mappings.

Keywords: best proximity point; tricyclic relatively nonexpansive; hyperbolic spaces.

2010 AMS Subject Classification: 47H09, 47H10, 54H25.

1. Introduction

Ever since its appearance, the Banach contraction principle have known a good many extensions and reformulations. Specifically, in $[1]$, it was shown that, if A and B are two nonempty closed subsets of a complete metric space and $T : A \cup B \longrightarrow A \cup B$ is cyclic mapping, that is, $T(A) \subseteq B$ and $T(B) \subseteq A$, for which there exists a $k \in (0,1)$ such that $d(Tx,Ty) \leq kd(x,y)$ for all $x \in A$ and $y \in B$, then $A \cap B \neq \varnothing$ and T has a unique fixed point in $A \cap B$. The case where

*Corresponding author

E-mail address: hbassou@gmail.com

Received April 1, 2018

$A \cap B = \varnothing$ was studied in $[2-4]$. More precisely, in $[2]$, the mapping T was assumed to be a cyclic contraction, i.e.

Definition 1.1. [2] Let A and B be nonempty subsets of a metric space (X,d), we denote by $dist\,(A,B)$ the distance between the subsets A and B, a mapping $T : A \cup B \longrightarrow A \cup B$ is said to be cyclic contraction if T is cyclic and

$$d\,(Tx,Ty) \leq kd\,(x,y) + (1-k)\,dist\,(A,B)\,,$$

for some $k \in (0,1)$ and for all $x \in A$ and $y \in B$.

A best proximity point for a cyclic mapping $T : A \cup B \longrightarrow A \cup B$ is a point $x \in A \cup B$ such that $d\,(x,Tx) = dist\,(A,B)$. Existence, uniqueness and convergence of iterates to such a point were obtained in several frameworks. First, the problem was investigated in [2] for uniformly Banach spaces. A few years later, the problem was studied from a different and more general view. For this intention, Suzuki and al, have introduced the notion of the property UC, which is a kind of a geometric notion for subsets of a metric space.

Definition 1.2. [4] Let A and B be nonempty subsets of a metric space (X,d), we say that (A,B) satisfy the UC property provided that: If (x_n) and (z_n) are sequences in A and (y_n) is a sequence in B such that

$$\lim_n d\,(x_n,y_n) = \lim_n d\,(z_n,y_n) = dist\,(A,B)\,,$$

then $\lim_n d\,(x_n,z_n) = 0$.

Lately, in [5], the current authors introduced the class of tricyclic contractions and best proximity point thereof, they also obtained an existence result of such a point for a reflexive Banach space. In this paper, we consider a hyperbolic space (which will be defined in the next section), and intend to prove the existence of best proximity point for tricyclic contractions, as well as for tricyclic relatively nonexpansive mappings.

2. Preliminaries

Definition 2.1. [6] A metric space (X,d) is a hyperbolic space if there exists a mapping $W:$ $X\times X\times[0,1]\longrightarrow X$ such that:

for all $x,y,z,w\in X$ and $\lambda,\lambda'\in[0,1]$

(i) $d(z,W(x,y,\lambda))\leq\lambda d(z,x)+(1-\lambda)d(z,y)$,

(ii) $d(W(x,y,\lambda),W(x,y,\lambda'))=|\lambda-\lambda'|d(x,y)$,

(iii) $W(x,y,\lambda)=W(y,x,1-\lambda)$,

(iv) $d(W(x,z,\lambda),W(y,w,\lambda))\leq\lambda d(x,y)+(1-\lambda)d(z,w)$.

If (i) is fulfilled alone, then (X,d,W) is a convex metric space in the sense of Takahashi [7]. One shall encounter various definitions of hyperbolic spaces, for instance, (i) and (iii) together are equivalent to (X,d,W) being a space of hyperbolic type in the sense of [8], the condition (iv) is used in [9] to define the class of hyperbolic spaces.

Definition 2.2. [7] A convex metric space is said to have the property (C) if every bounded decreasing net of nonempty, closed and convex subsets of X has a nonempty intersection.

For instance, every weakly compact convex subset of a Banach space has the property (C). Note that convex metric spaces with the property (C) generalize the notion of reflexivity from Banach to metric spaces. Let (X,d_X,W_X) and (X,d_Y,W_Y) be a two convex metric spaces, the mapping

$$d_p:(X\times Y)\times(X\times Y)\longrightarrow[0,\infty)$$

defined by

$$d_p((x_1,y_1),(x_2,y_2))=\begin{cases}(d_X(x_1,x_2)^p+d_Y(x_1,x_2)^p)^{\frac{1}{p}},1\leq p<\infty;\\ \max\{d_X(x_1,x_2),d_Y(x_1,x_2)\}\ , \qquad p=\infty.\end{cases}$$

is a distance on the Cartesian product $X\times Y$.

Lemma 2.3. [10] *The mapping* $W_{X\times Y}:(X\times Y)\times(X\times Y)\times I\longrightarrow[0,\infty)$ *defined as follows*

$$W_{X\times Y}((x_1,y_1),(x_2,y_2),\lambda)=(W_X(x_1,x_2,\lambda),W_Y(y_1,y_2,\lambda)).$$

is a convex stucture on the product metric space $(X\times Y,d_1)$.

Definition 2.4. A subset E of the convex product metric space $(X\times Y,d_1,W_{X\times Y})$ is convex if $W_{X\times Y}((x_1,y_1),(x_2,y_2),\lambda)\in E$ for all $(x_1,y_1),(x_2,y_2)\in E$ and $\lambda\in I$.

Let A,B and C be three nonempty, subsets of a convex metric space. By saying that (A,B,C) verifies a certain property we mean that it is verified by A,B and C at once. For instance, $(K_1,K_2,K_3)\subset(A,B,C)$ is equivalent to $K_1\subset A, K_2\subset B$ and $K_3\subset C$. We shall adopt the following notations:

$$D : X\times X\times X\longrightarrow[0,+\infty),$$

$$(x,y,z)\longmapsto D(x,y,z)=d(x,y)+d(y,z)+d(z,x),$$

$$\delta(A,B,C)=\inf\{D(x,y,z): x\in A,\ y\in B \text{ and } z\in C\},$$

$$\Delta(A,B,C)=\sup\{D(x,y,z): x\in A,\ y\in B \text{ and } z\in C\},$$

$$\Delta_{(x,y)}(C)=\sup\{D(x,y,z): z\in C\}$$

for all $x\in A$ and $y\in B$.

The triad $(x,y,z)\in A\times B\times C$ is said to be *proximal* in (A,B,C) if $D(x,y,z)=\delta(A,B,C)$, we set

$$\begin{aligned} A_0 &:= \left\{x\in A : D\left(x,y',z''\right)=\delta(A,B,C) \text{ for some } y'\in B \text{ and } z''\in C\right\},\\ B_0 &:= \left\{y\in B : D\left(x'',y,z'\right)=\delta(A,B,C) \text{ for some } x''\in A \text{ and } z'\in C\right\},\\ C_0 &:= \left\{z\in C : D\left(x',y'',z\right)=\delta(A,B,C) \text{ for some } x'\in A \text{ and } y''\in B\right\}. \end{aligned}$$

Clearly, $\delta(A_0,B_0,C_0)=\delta(A,B,C)$.

Definition 2.5. [5] Let A,B and C be nonempty subsets of a metric space (X,d), a mapping $T : A\cup B\cup C\longrightarrow A\cup B\cup C$ is said to be tricyclic contraction if

(1) T is a tricyclic map, i.e., $T(A)\subseteq B$, $T(B)\subseteq C$ and $T(C)\subseteq A$.
(2) $D(Tx,Ty,Tz)\leq kD(x,y,z)+(1-k)\delta(A,B,C)$, for some $k\in(0,1)$ and for all $(x,y,z)\in A\times B\times C$.

If $k=1$, then T is said to be tricyclic relatively nonexpansive.

Definition 2.6. [5] Let A,B and C be nonempty subsets of a metric space (X,d), let $T : A\cup B\cup C\longrightarrow A\cup B\cup C$ be a tricyclic mapping. A point $x\in A\cup B\cup C$ is said to be a best proximity point for T if

$$D\left(x,Tx,T^2x\right)=\delta(A,B,C).$$

Theorem 2.7. [5] *Let A,B and C be nonempty, closed, bounded and convex subsets of a reflexive Banach space X, let $T : A\cup B\cup C \longrightarrow A\cup B\cup C$ be a tricyclic contraction map. Then T has a best proximity point.*

The purpose of this work is to extend the previous existence result to Kohlenbach hyperbolic spaces and investigate the structure of the minimal sets of cyclic relatively nonexpansive mappings as well.

3. Main results

By analogous argument to the proof of Theorem 3.7 [5], we can prove the following result.

Theorem 3.1. *Let A,B and C be nonempty, closed, bounded and convex subsets of a hyperbolic space X, let $T : A\cup B\cup C \longrightarrow A\cup B\cup C$ be a tricyclic contraction map. If X has the (C) property then T has a best proximity point.*

The next definition is the tricyclic version of the definition in [11].

Definition 3.2. Let (A,B,C) be a nonempty triad of subsets of a metric space (X,d), (A,B,C) is said to be proximal compactness provided that every net $((x_i),(y_i),(z_i)) \subset A\times B\times C$ such that $D(x_i,y_i,z_i) \longrightarrow \delta(A,B,C)$ has a convergent subnet in $A\times B\times C$.

If (A,B,C) is compact in a metric space (X,d), then it is proximal compactness.

Henceforth, (A,B,C) is a nonempty, bounded, closed and convex triad of a hyperbolic space (X,d,W) such that A_0 is nonempty, X has the (C) property and (A,B,C) is a proximal compactness.

Lemma 3.3. *Let $T : A\cup B\cup C \longrightarrow A\cup B\cup C$ be a cyclic relatively nonexpansive mapping. Then there exists $(K_1,K_2,K_3) \subset (A,B,C)$ which is minimal with respect to being nonempty, closed, convex and $T-$invariant such that $\delta(K_1,K_2,K_3) = \delta(A,B,C)$.*

Proof. Let Σ be the set of all nonempty, bounded, closed and convex triads $(E,F,G) \subset (A,B,C)$ such that T is tricyclic on $E\cup F\cup G$ and $D(x,y,z) = \delta(A,B,C)$ for some $(x,y,z) \in E\times F\times G$. Since A_0 is nonempty, $(A,B,C) \in \Sigma$. Let $(E_i,F_i,G_i)_{i\in I}$ be an increasing chain of Σ.
Put $E = \bigcap_{i\in I} E_i$, $F = \bigcap_{i\in I} F_i$ and $G = \bigcap_{i\in I} G_i$, since the (C) property is provided, (E,F,G) is nonempty,

closed and convex. Besides

$$T(E) = T\left(\underset{i\in I}{\cap} E_i\right) \subseteq \underset{i\in I}{\cap} T(E_i) \subseteq \underset{i\in I}{\cap} F_i = F.$$

T is then tricyclic on $E \cup F \cup G$. Let $(x_i, y_i, z_i) \in E_i \times F_i \times G_i$ be such that $D(x_i, y_i, z_i) = \delta(A,B,C)$, since (A,B,C) is proximal compactness, (x_i, y_i, z_i) has a subsequence, say $(x_{i'}, y_{i'}, z_{i'})$, such that $x_{i'} \longrightarrow x \in E$, $y_{i'} \longrightarrow y \in F$ and $z_{i'} \longrightarrow z \in G$, hence, $D(x,y,z) = \delta(A,B,C)$. Therefore, every increasing chain in Σ is bounded above. The maximal element (K_1, K_2, K_3) is obtained by using Zorn's lemma and it's a minimal with respect to set inclusion.

Proposition 3.4. *Let $T : A \cup B \cup C \longrightarrow A \cup B \cup C$ be a cyclic relatively nonexpansive mapping, suppose $(K_1, K_2, K_3) \subseteq (A,B,C)$ is minimal, closed convex triad which is $T-$invariant such that $\delta(K_1, K_2, K_3) = \delta(A,B,C)$. Then each triad $(p,q,r) \in (K_1, K_2, K_3)$ with $D(p,q,r) = \delta(A,B,C)$ contains a diametral pair, i.e.* $\max\left\{\Delta_{(p,q)}(K_3), \Delta_{(p,r)}(K_2), \Delta_{(q,r)}(K_1)\right\} = \Delta(K_1, K_2, K_3)$.

Proof. We have

$$\left(\overline{co}(T(K_3)), \overline{co}(T(K_1)), \overline{co}(T(K_2))\right) \subseteq (K_1, K_2, K_3)$$

Hence

$$T\left(\overline{co}(T(K_3))\right) \subseteq T(K_1) \subseteq \overline{co}(T(K_1)).$$

Thus T is tricyclic on $\overline{co}(T(K_1)) \cup \overline{co}(T(K_2)) \cup \overline{co}(T(K_3))$. Let $(x_0, y_0, z_0) \in K_1 \times K_2 \times K_3$ be such that $D(x_0, y_0, z_0) = \delta(A,B,C)$. By the relatively nonexpansiveness of T, we have

$$\begin{aligned} \delta(A,B,C) &\leq \delta\left(\overline{co}(T(K_3)), \overline{co}(T(K_1)), \overline{co}(T(K_2))\right) \\ &\leq D(Tz_0, Tx_0, Ty_0) \leq D(x_0, y_0, z_0) = \delta(A,B,C). \end{aligned}$$

The minimality of (K_1, K_2, K_3) implies that

$$\overline{co}(T(K_1)) = K_2, \quad \overline{co}(T(K_2)) = K_3 \text{ and } \overline{co}(T(K_3)) = K_1.$$

Suppose $(p,q,r) \in (K_1, K_2, K_3)$ such that $D(p,q,r) = \delta(A,B,C)$ and assume that there is no diametral pair, i.e.

$$\max\left\{\Delta_{(p,q)}(K_3), \Delta_{(p,r)}(K_2), \Delta_{(q,r)}(K_1)\right\} < \Delta(K_1, K_2, K_3).$$

Put $\alpha := \max\left\{\Delta_{(p,q)}(K_3), \Delta_{(p,r)}(K_2), \Delta_{(q,r)}(K_1)\right\}$ and define

$$\begin{aligned}
E_\alpha(K_3) &= \left\{(x,y) \in K_1 \times K_2 : \Delta_{(x,y)}(K_3) \leq \alpha\right\}, \\
E_\alpha(K_1) &= \left\{\left(y',z\right) \in K_2 \times K_3 : \Delta_{(y',z')}(K_1) \leq \alpha\right\} \\
E_\alpha(K_2) &= \left\{\left(z',x'\right) \in K_3 \times K_1 : \Delta_{(z,x')}(K_2) \leq \alpha\right\},.
\end{aligned}$$

Note that $(E_\alpha(K_3), E_\alpha(K_1), E_\alpha(K_2))$ is nonemety (it contains $(p,q),(q,r),(r,p)$), closed ($E_\alpha(K_3) = \bigcap_{z\in K_3} \Psi_z^{-1}([0,\alpha])$ where $\Psi_z : K_1 \times K_2 \longrightarrow \mathbb{R}_+$, $(x,y) \longmapsto D(x,y,z)$), and convex, in this order, let $(x_1,y_1),(x_2,y_2) \in E_\alpha(K_3)$, $z \in K_3$ and $\lambda \in [0,1]$, we have

$$\begin{aligned}
D(W(x_1,x_2,\lambda), W(y_1,y_2,\lambda), z) &= d(W(x_1,x_2,\lambda), z) + d(W(y_1,y_2,\lambda), z) \\
&\quad + d(W(x_1,x_2,\lambda), W(y_1,y_2,\lambda)). \\
&\leq \lambda d(x_1,z) + (1-\lambda) d(x_2,z) + \lambda d(y_1,z) \\
&\quad + (1-\lambda) d(y_2,z) + .\lambda d(x_1,y_1) + (1-\lambda) d(x_2,y_2) \\
&= \lambda D(x_1,y_1,z) + (1-\lambda) d(x_2,y_2,z) \\
&\leq \max\{D(x_1,y_1,z), D(x_2,y_2,z)\} \leq \alpha.
\end{aligned}$$

Which means $(W(x_1,x_2,\lambda), W(y_1,y_2,\lambda)) = W_{X\times X}((x_1,y_1),(x_2,y_2),\lambda) \in E_\alpha(K_3)$, for all $(x_1,y_1),(x_2,y_2) \in E_\alpha(K_3)$ and $\lambda \in [0,1]$.

Moreover, since $((p,q),(q,r),(r,p)) \in (E_\alpha(K_3), E_\alpha(K_1), E_\alpha(K_2))$, then

$$\delta(E_\alpha(K_3), E_\alpha(K_1), E_\alpha(K_2)) = \delta(A \times B, B \times C, C \times A).$$

Indeed, let $(x,y) \in A \times B$, $(y',z) \in B \times C$ and $(z',x') \in C \times A$, we have

$$\begin{aligned}
D_1\left((x,y),\left(y',z\right),\left(z',x'\right)\right) &= d_1\left((x,y),\left(y',z\right)\right) + d_1\left(\left(y',z\right),\left(z',x'\right)\right) + d_1\left(\left(z',x'\right),(x,y)\right) \\
&= D\left(x,y',z'\right) + D\left(y,z,x'\right) \\
&\geq 2D(p,q,r) = D_1((p,q),(q,r),(r,p)).
\end{aligned}$$

Thus, $D_1((p,q),(q,r),(r,p)) = \delta(A\times B, B\times C, C\times A)$, on the other hand

$$\begin{aligned}\delta(A\times B,B\times C,C\times A) &\leq \delta(E_\alpha(K_3),E_\alpha(K_1),E_\alpha(K_2))\\ &\leq D_1((p,q),(q,r),(r,p))\\ &= \delta(A\times B,B\times C,C\times A).\end{aligned}$$

Define

$$\widetilde{T}\ :(A\times B)\cup(B\times C)\cup(C\times A)\longrightarrow(A\times B)\cup(B\times C)\cup(C\times A),\quad (x,y)\longmapsto\widetilde{T}(x,y)=(Tx,Ty)$$

Since T is tricyclic on $A\cup B\cup C$, $\widetilde{T}$ is tricyclic on $(A\times B)\cup(B\times C)\cup(C\times A)$. Furthermore $(K_1\times K_2,\ K_2\times K_3,\ K_3\times K_1)$ is minimal in

$$\widetilde{\Sigma}=\left\{\begin{array}{c}(E\times F,F\times G,G\times E)\subseteq(A\times B,B\times C,C\times A)/\\ (E\times F),(F\times G)\text{ and }(G\times E)\text{ are non-empty, bounded, closed}\\ \text{and convex with }\widetilde{T}\text{ is tricyclic on }(E\times F)\cup(F\times G)\cup(G\times E)\text{ and}\\ \delta\ (E\times F,F\times G,G\times E)=\delta\ (A\times B,B\times C,C\times A).\end{array}\right\}.$$

Now we prove that $\widetilde{T}$ is tricyclic on $E_\alpha(K_3)\cup E_\alpha(K_1)\cup E_\alpha(K_2)$.

Take note that, for all $(x,y)\in K_1\times K_2$,

$$(x,y)\in E_\alpha(K_3)\Longleftrightarrow K_3\subseteq B(x,y,\alpha).$$

Let $u=(u_1,u_2)\in E_\alpha(K_3)$.

We then must verify that $\widetilde{T}(u)=(Tu_1,Tu_2)\in E_\alpha(K_1)$, that is, $K_1\subseteq B(Tu_1,Tu_2,\alpha)$. For all $w\in K_3$, we have $D(Tw,Tu_1,Tu_2)\leq D(w,u_1,u_2)\leq\alpha$, then $T(K_3)\subseteq B(Tu_1,Tu_2,\alpha)$, therefore $\overline{co}(T(K_3))=K_1\subseteq B(Tu_1,Tu_2,\alpha)$. Hence $\widetilde{T}$ is tricyclic on $E_\alpha(K_3)\cup E_\alpha(K_1)\cup E_\alpha(K_2)$.

The minimality of $(K_1\times K_2,K_2\times K_3,K_3\times K_1)$, implies that

$$E_\alpha(K_3)=K_1\times K_2,\ E_\alpha(K_1)=K_2\times K_3\text{ and }E_\alpha(K_2)=K_3\times K_1,$$

so, for all $(x,y)\in K_1\times K_2$, $\Delta_{(x,y)}(K_3)\leq\alpha$. Consequently

$$\Delta(K_1,K_2,K_3)=\sup_{(x,y)\in K_1\times K_2}\Delta_{(x,y)}(K_3)\leq\alpha.$$

Which is contradictory with our assumpltion.

The next definition is the tricyclic version of the one introduced in [12].

Definition 3.5. Let (A,B,C) be a nonempty triad of subsets of a metric space (X,d). Let $T: A\cup B\cup C \longrightarrow A\cup B\cup C$ be a tricyclic mapping. Then a sequence (x_n) in $A\cup B\cup C$ is said to be an approximate best proximity point sequence for T if

$$\lim_{n\longrightarrow\infty} D\left(x_n, Tx_n, T^2x_n\right) = \delta\left(A,B,C\right).$$

The next lemma assures the existence of such a sequence for tricyclic relatively nonexpansive mappings.

Lemma 3.6. *Let $T: A\cup B\cup C \longrightarrow A\cup B\cup C$ be a tricyclic relatively nonexpansive mapping. Then there exists an an approximate best proximity point sequence for T in A.*

Proof. The lemma 3.3 guarantees the existence of a triad $(K_1,K_2,K_3)\subseteq(A,B,C)$ that is minimal with respect to being nonempty, closed, convex and $T-$invariant such that $\delta\left(K_1,K_2,K_3\right)=\delta\left(A,B,C\right)$ and there exists $(x',y',z')\in(K_1,K_2,K_3)$ such that

$$D\left(x',y',z'\right)=\delta\left(K_1,K_2,K_3\right)=\delta\left(A,B,C\right).$$

For every $\lambda\in(0,1)$, we define the mapping $T_\lambda: A\cup B\cup C\longrightarrow A\cup B\cup C$ as follows

$$T_\lambda x = \begin{cases} W\left(Tx,y',\lambda\right);\ x\in A, \\ W\left(Tx,z',\lambda\right);\ x\in B, \\ W\left(Tx,x',\lambda\right);\ x\in C. \end{cases}$$

Clearly T_λ is tricyclic on $A\cup B\cup C$. Besides, T_λ is a tricyclic contraction, let $(x,y,z)\in A\times B\times C$, we have

$$\begin{aligned} D\left(T_\lambda x, T_\lambda y, T_\lambda z\right) &= d\left(W\left(Tx,y',\lambda\right), W\left(Ty,z',\lambda\right)\right) + d\left(W\left(Ty,z',\lambda\right), W\left(Tz,x',\lambda\right)\right) \\ &\quad + d\left(W\left(Tz,x',\lambda\right), W\left(Tx,y',\lambda\right)\right) \\ &= \lambda D\left(Tx,Ty,Tz\right) + (1-\lambda) D\left(x',y',z'\right) \\ &\leq \lambda D\left(x,y,z\right) + (1-\lambda)\,\delta\left(A,B,C\right). \end{aligned}$$

(Theorem 3.1 implies that T_λ has a best proximity point), say $p_\lambda \in A$, for each $\lambda \in (0,1)$. Then

$$\begin{aligned}
\delta(A,B,C) &\leq D\left(p_\lambda, Tp_\lambda, T^2 p_\lambda\right) \\
&\leq D\left(p_\lambda, T_\lambda p_\lambda, T^2 p_\lambda\right) + 2d\left(T_\lambda p_\lambda, Tp_\lambda\right) \\
&\leq D\left(p_\lambda, T_\lambda p_\lambda, T_\lambda^2 p_\lambda\right) + 2d\left(T_\lambda^2 p_\lambda, T^2 p_\lambda\right) + 2d\left(T_\lambda p_\lambda, Tp_\lambda\right) \\
&\leq \delta(A,B,C) + 2d\left(T_\lambda^2 p_\lambda, T^2 p_\lambda\right) + 2d\left(T_\lambda p_\lambda, Tp_\lambda\right)
\end{aligned}$$

Since

$$d\left(T_\lambda p_\lambda, Tp_\lambda\right) = d\left(W\left(Tp_\lambda, y', \lambda\right), Tp_\lambda\right) = (1-\lambda)\, d\left(y', Tp_\lambda\right) \leq (1-\lambda)\operatorname{diam}(B).$$

And

$$\begin{aligned}
d\left(T_\lambda^2 p_\lambda, T^2 p_\lambda\right) &= d\left(T_\lambda(T_\lambda p_\lambda), T^2 p_\lambda\right) \\
&= d\left(T_\lambda(W\left(Tp_\lambda, y', \lambda\right)), T^2 p_\lambda\right) \\
&= d\left(W\left(T\left(W\left(Tp_\lambda, y', \lambda\right)\right), z', \lambda\right), T^2 p_\lambda\right) \\
&\leq \lambda d\left(T\left(W\left(Tp_\lambda, y', \lambda\right)\right), T^2 p_\lambda\right) + (1-\lambda)\, d\left(z', T^2 p_\lambda\right) \\
&\leq \lambda d\left(W\left(Tp_\lambda, y', \lambda\right), Tp_\lambda\right) + (1-\lambda)\, d\left(z', T^2 p_\lambda\right) \\
&\leq \lambda(1-\lambda)\, d\left(y', Tp_\lambda\right) + (1-\lambda)\, d\left(z', T^2 p_\lambda\right) \\
&\leq (1-\lambda)\left[\lambda \operatorname{diam}(B) + \operatorname{diam}(C)\right].
\end{aligned}$$

Thus

$$\delta(A,B,C) \leq D\left(p_\lambda, Tp_\lambda, T^2 p_\lambda\right) \leq \delta(A,B,C) + 2(1-\lambda)\left[(\lambda+1)\operatorname{diam}(B) + \operatorname{diam}(C)\right]$$

When $\lambda \longrightarrow 1^-$ in the previous relation, we have

$$D\left(p_\lambda, T_\lambda p_\lambda, T_\lambda^2 p_\lambda\right) \longrightarrow \delta(A,B,C).$$

Consequently there exists a sequence (x_n) in A such that

$$\lim_{n\longrightarrow\infty} D\left(x_n, Tx_n, T^2 x_n\right) = \delta(A,B,C).$$

Definition 3.7. Let A, B and C be nonempty subsets of (X,d). (A,B,C) is said to have the UC property if:

If (x_n) and (t_n) are sequences in A, (y_n) is in B and (z_n) is in C such that $D(x_n,y_n,z_n) \longrightarrow \delta(A,B,C)$ and $D(t_n,y_n,z_n) \longrightarrow \delta(A,B,C)$, then $d(x_n,t_n) \longrightarrow 0$.

Example. *Suppose* $\delta(A,B,C) = dist(A,B) + dist(B,C) + dist(C,A)$.
If (x_n) *and* (t_n) *are sequences in* A, (y_n) *is in* B *and* (z_n) *is in* C *such that* $D(x_n,y_n,z_n) \longrightarrow \delta(A,B,C)$ *and* $D(t_n,y_n,z_n) \longrightarrow \delta(A,B,C)$, *then* $d(x_n,y_n) \longrightarrow dist(A,B)$, $d(x_n,z_n) \longrightarrow dist(A,C)$, $d(t_n,y_n) \longrightarrow dist(A,B)$ *and* $d(t_n,z_n) \longrightarrow dist(A,C)$. *Thus,* (A,B,C) *has the UC property if either* (A,B) *or* (A,C) *does.*

Theorem 3.8. *Suppose* (A,B,C) *satisfies the UC property. Let* $T : A\cup B\cup C \longrightarrow A\cup B\cup C$ *be a tricyclic relatively nonexpansive mapping. Let* $(K_1,K_2,K_3) \subseteq (A,B,C)$ *be a minimal, closed convex triad which is* $T-$*invariant such that* $\delta(K_1,K_2,K_3) = \delta(A,B,C)$, *and a sequence* (x_n) *in* K_1 *such that* $\lim_{n\longrightarrow\infty} D\left(x_n,Tx_n,T^2x_n\right) = \delta(A,B,C)$.
Then for all $(p,q,r) \in K_1\times K_2\times K_3$ *with* $D(p,q,r) = \delta(A,B,C)$ *we have*

$$\max\left\{\limsup_{n\longrightarrow\infty} D(x_n,q,r),\ \limsup_{n\longrightarrow\infty} D(p,Tx_n,r),\ \limsup_{n\longrightarrow\infty} D\left(p,q,T^2x_n\right)\right\} = \Delta(A,B,C).$$

Furthermore, T *have a best proximity point.*

Proof. The proposition (3.4) guarentees that each point $(p,q,r) \in K_1\times K_2\times K_3$ with $D(p,q,r) = \delta(A,B,C)$ contains a diametral pair, i.e.

$$\max\left\{\Delta_{(p,q)}(K_3),\Delta_{(p,r)}(K_2),\Delta_{(q,r)}(K_1)\right\} = \Delta(K_1,K_2,K_3).$$

Let (x_n) be a sequence in K_1 such that $\lim_{n\longrightarrow\infty} D\left(x_n,Tx_n,T^2x_n\right) = \delta(A,B,C)$. Assume that there exists $(u,v,w) \in K_1\times K_2\times K_3$ and $h < \Delta(A,B,C)$ with $D(u,v,w) = \delta(A,B,C)$ and

$$\max\left\{\limsup_{n\longrightarrow\infty} D(u,Tx_n,w),\ \limsup_{n\longrightarrow\infty} D\left(u,v,T^2x_n\right),\ \limsup_{n\longrightarrow\infty} D(x_n,v,w)\right\} \le h.$$

Put

$$\begin{aligned} C_3 &= \left\{(x,y) \in K_1\times K_2 :\ \limsup_{n\longrightarrow\infty} D\left(x,y,T^2x_n\right) \le h\right\},\\ C_1 &= \left\{\left(y',z\right) \in K_2\times K_3 :\ \limsup_{n\longrightarrow\infty} D\left(x_n,y',z\right) \le h\right\},\\ C_2 &= \left\{\left(z',x'\right) \in K_3\times K_1 :\ \limsup_{n\longrightarrow\infty} D\left(x',Tx_n,z'\right) \le h\right\}. \end{aligned}$$

Clearly (C_3, C_1, C_2) is nonempty, bounded and closed. Let us prove now that it is also convex. Let $(x_1, y_1), (x_2, y_2) \in C_3$ and $\lambda \in [0,1]$. Then

$$\begin{aligned}\limsup_{n\longrightarrow\infty} D\left(W(x_1,x_2,\lambda), W(y_1,y_2,\lambda), T^2x_n\right) &\leq \limsup_{n\longrightarrow\infty}\left\{\lambda D\left(x_1,y_1,T^2x_n\right)+(1-\lambda)D\left(x_2,y_2,T^2x_n\right)\right\}\\ &\leq h.\end{aligned}$$

Since $((u,v),(v,w),(w,u)) \in C_3 \times C_1 \times C_2$, we have

$$\delta(C_1,C_2,C_3) = \delta(B\times C,\ A\times C,\ A\times B).$$

Next, we show that $\widetilde{T}$ is tricyclic on $C_3 \cup C_1 \cup C_2$. Let $(x,y) \in C_3$, then $\widetilde{T}(x,y) \in K_2 \times K_3$ and $\limsup_{n\longrightarrow\infty} D\left(x,y,T^2x_n\right) \leq h$. We have

$$\limsup_{n\longrightarrow\infty} D(x_n, Tx, Ty) \leq \limsup_{n\longrightarrow\infty} 2d\left(x_n, T^3x_n\right) + D\left(T^3x_n, Tx, Ty\right).$$

By the relatively nonexpansiveness of T, we have

$$D\left(T^3x_n, T^2x_n, Tx_n\right) \leq D\left(T^2x_n, Tx_n, x_n\right) \longrightarrow \delta(A,B,C).$$

The UC property implies that

$$d\left(x_n, T^3x_n\right) \longrightarrow 0.$$

Hence

$$\limsup_{n\longrightarrow\infty} D(x_n, Tx, Ty) \leq \limsup_{n\longrightarrow\infty} D\left(T^2x_n, x, y\right) \leq h.$$

Thus, $\widetilde{T}$ is tricyclic on $C_3 \cup C_1 \cup C_2$. By the minimality of $(K_1\times K_2,\ K_2\times K_3,\ K_3\times K_1)$, we conclude that $C_3 = K_1 \times K_2$, $C_1 = K_2 \times K_3$ and $C_2 = K_3 \times K_1$. Since $\lim_{n\longrightarrow\infty} D\left(x_n, Tx_n, T^2x_n\right) = \delta(A,B,C)$ and (A,B,C) is a proximal compactness, one may assume that

$$x_n \longrightarrow p \in K_1,\ Tx_n \longrightarrow q \in K_2 \text{ and } T^2x_n \longrightarrow r \in K_3,$$

Hence, $D(p,q,r) = \delta(A,B,C)$ and

$$\begin{aligned} D(p,q,z) &= \limsup_{n\longrightarrow\infty} D(p, Tx_n, z) \leq h,\\ D(x,q,r) &= \limsup_{n\longrightarrow\infty} D\left(x,q,T^2x_n\right) \leq h,\\ D(p,y,r) &= \limsup_{n\longrightarrow\infty} D(x_n, y, r) \leq h.\end{aligned}$$

For all $(x,y,z) \in K_1 \times K_2 \times K_3$, therefore

$$\max\left\{\Delta_{(p,q)}(K_3), \Delta_{(q,r)}(K_1), \Delta_{(r,q)}(K_1)\right\} \leq h < \Delta(A,B,C)$$

which is a contradition by the fact that (p,q,r) contains a diametral pair. Therefore

$$\max\left\{\limsup_{n\longrightarrow\infty} D(x_n,q,r),\ \limsup_{n\longrightarrow\infty} D(p,Tx_n,r),\ \limsup_{n\longrightarrow\infty} D\left(p,q,T^2x_n\right)\right\} = \Delta(K_1,K_2,K_3).$$

On the other hand,

$$\begin{aligned}\limsup_{n\longrightarrow\infty} D(x_n,q,r) &= \limsup_{n\longrightarrow\infty} D(p,Tx_n,r) = \limsup_{n\longrightarrow\infty} D\left(p,q,T^2x_n\right) \\ &= D(p,q,r) = \delta(A,B,C).\end{aligned}$$

Consequently

$$\Delta(K_1,K_2,K_3) = \delta(A,B,C).$$

And that finishes the proof.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] W.A.Kirk, P.S.Srinivasan, P.Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, Fixed Point Theory, 4 (2003), 79–89.

[2] A.A. Eldred, P. Veeramani, Existence and convergence of best proximity points, J. Math. Anal. Appl. 323 (2) (2006) 1001–1006.

[3] A.A. Eldred, W.A. Kirk, P. Veeramani, Proximal normal structure and relatively nonexpansive mappings, Studia Math. 171 (3) (2005) 283–293.

[4] T. Suzuki, M. Kikkawa, C. Vetro, The existence of the best proximity points in metric spaces with the property UC, Nonlinear Anal., Theory Method Appl. 71 (7–8) (2009) 2918–2926.

[5] T. Sabar, M.Aamri, A.Bassou, Best proximity point of tricyclic contractions, Adv. Fixed point theory, 7 (2017), No. 4, 512-523.

[6] Kohlenbach, U, Some logical metatheorems with application in functional analysis, Trans. Amer. Math. Soc. 357 (2005), 89-128.

[7] Takahashi, W: A convexity in metric space and nonexpansive mappings. Kodai Math. Semin. Rep. 22 (1970), 142-149.

[8] Goebel, K., Kirk, W.A., Iteration processes for nonexpansive mappings. In: Singh, S.P.,Thomeier, S., Watson, B.,eds., Topological Methods in Nonlinear Functional Analysis. Contemporary Mathematics 21, AMS, pp. 115-123 (1983). MR 85a:47059.

[9] Reich, S., Shafrir, I., Nonexpansive iterations in hyperbolic spaces. Nonlinear Anal., Theory Methods Appl. 15 (1990), 537-558.

[10] A.A. Abdelhakim, A convexity of functions on convex metric spaces of Takahashi and applications. J. Egypt. Math. Soc. 24 (2016), 348-354.

[11] Gabeleh, M: Minimal sets of noncyclic relatively nonexpansive mappings in convex metric spaces. Fixed Point Theory, 16 (2015), 313-322.

[12] Gabeleh and Shahzad:Some new results on cyclic relatively nonexpansive mappings in convex metric spaces. J. Inequal. Appl. 2014 (2014), Article ID 350.