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A NEW GENERAL ITERATIVE METHODS FOR MULTIVALUED NONEXPANSIVE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, strong convergence theorems by general approximation methods for nonexpansive multivalued mappings and variational inequality problems are established under some suitable conditions in real Banach spaces. The methods in the paper are novel and different from those in early and recent literature. The results obtained in this paper are significant improvement on important recent results .

Keywords: general iterative method; multivalued mappings; *k*- strongly accretive operators and *L*-Lipchizian operators.

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1. Introduction

Let (X,d) be a metric space, K be a nonempty subset of X and $T : K \to 2^K$ be a multivalued mapping. An element $x \in K$ is called a fixed point of T if $x \in Tx$. For single valued mapping, this reduces to $Tx = x$. The fixed point set of T is denoted by $F(T) := \{x \in D(T) : x \in Tx\}$.

For several years, the study of fixed point theory for *multivalued nonlinear mappings* has attracted, and continues to attract, the interest of several well known mathematicians (see, for

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example, Brouwer [4], Kakutani [20], Nash [16, 17], Geanakoplos [21], Nadla [15], Downing and Kirk [11]).

Interest in the study of fixed point theory for multivalued nonlinear mappings stems, perhaps, mainly from its usefulness in real-world applications such as *Game Theory* and *Non-Smooth Differential Equations*.

Game Theory. Nash showed the existence of equilibria for non-cooperative *static* games as a direct consequence of *multivalued* Brouwer or Kakutani fixed point theorem. More precisely, under some regularity conditions, given a game, there always exists a *multi-valued mapping* whose fixed points coincide with the equilibrium points of the game. This, among other things, made Nash a recipient of Nobel Prize in Economic Sciences in 1994. However, it has been remarked that the applications of this theory to equilibrium are mostly *static*: they enhance understanding conditions under which equilibrium may be achieved but do not indicate how to construct a process starting from a non-equilibrium point and convergent to equilibrium solution. This is part of the problem that is being addressed by *iterative methods for fixed point of multi-valued mappings*. Consider a game $G=(u_n,K_n)$ with N players denoted by n, $n=1,\cdots,N$, where $K_n\subset\mathbb{R}^{m_n}$ is the set of possible strategies of the n'th player and is assumed to be nonempty, compact and convex and $u_n : K := K_1\times K_2\cdots\times K_N\to\mathbb{R}$ is the payoff (or gain function) of the player n and is assumed to be continuous. The player n can take *individual actions*, represented by a vector $\sigma_n\in K_n$. All players together can take a *collective action*, which is a combined vector $\sigma=(\sigma_1,\sigma_2,\cdots,\sigma_N)$. For each n, $\sigma\in K$ and $z_n\in K_n$, we will use the following standard notations:

$$K_{-n} := K_1\times\cdots\times K_{n-1}\times K_{n+1}\times\cdots\times K_N,$$

$$\sigma_{-n} := (\sigma_1,\cdots,\sigma_{n-1},\sigma_{n+1},\cdots,\sigma_N),$$

$$(z_n,\sigma_{-n}) := (\sigma_1,\cdots,\sigma_{n-1},z_n,\sigma_{n+1},\cdots,\sigma_N).$$

A strategy $\bar{\sigma}_n \in K_n$ permits the n'th player to maximize his gain *under the condition* that the *remaining players* have chosen their strategies σ_{-n} if and only if

$$u_n(\bar{\sigma}_n, \sigma_{-n}) = \max_{z_n \in K_n} u_n(z_n, \sigma_{-n}).$$

Now, let $T_n : K_{-n} \to 2^{K_n}$ be the multivalued map defined by

$$T_n(\sigma_{-n}) := \operatorname*{Arg\,max}_{z_n \in K_n} u_n(z_n, \sigma_{-n}) \ \forall \sigma_{-n} \in K_{-n}.$$

Definition. A collective action $\bar{\sigma} = (\bar{\sigma}_1, \cdots, \bar{\sigma}_N) \in K$ is called a *Nash equilibrium point* if, for each n, $\bar{\sigma}_n$ is the best response for the n'th player to the action $\bar{\sigma}_{-n}$ made by the remaining players. That is, for each n,

$$u_n(\bar{\sigma}) = \max_{z_n \in K_n} u_n(z_n, \bar{\sigma}_{-n}) \tag{1}$$

or equivalently,

$$\bar{\sigma}_n \in T_n(\bar{\sigma}_{-n}). \tag{2}$$

This is equivalent to $\bar{\sigma}$ is a fixed point of the multivalued map $T : K \to 2^K$ defined by

$$T(\sigma) := [T_1(\sigma_{-1}), T_2(\sigma_{-2}), \cdots, T_N(\sigma_{-N})].$$

From the point of view of social recognition, game theory is perhaps the most successful area of application of *fixed point theory of multivalued mappings*. However, it has been remarked that the applications of this theory to equilibrium are mostly static: they enhance understanding conditions under which equilibrium may be achieved but do not indicate how to construct a process starting from a non-equilibrium point and convergent to equilibrium solution. *This is part of the problem that is being addressed by iterative methods for fixed point of multivalued mappings.*

Let D be a nonempty subset of a normed space E. The set D is called *proximinal* (see, *e.g.*, [19]) if for each $x \in E$, there exists $u \in D$ such that

$$d(x,u) = \inf\{\|x-y\| : y \in D\} = d(x,D),$$

where $d(x,y) = \|x-y\|$ for all $x,y \in E$. Every nonempty, closed and convex subset of a real Hilbert space is proximinal. Let $CB(D)$, $K(D)$ and $P(D)$ denote the family of nonempty closed bounded subsets, nonempty compact subsets, and nonempty proximinal bounded subsets of D respectively. The *Hausdorff metric* on $CB(K)$ is defined by:

$$H(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\}$$

for all $A,B \in CB(K)$. A multi-valued mapping $T : D(T) \subseteq E \to CB(E)$ is called *L- Lipschitzian* if there exists $L > 0$ such that

$$H(Tx,Ty) \leq L\|x-y\| \ \forall x,y \in D(T). \tag{3}$$

When $L \in (0,1)$, we say that T is a *contraction*, and T is called *nonexpansive* if $L = 1$.

Existence theorems for fixed point of *multi-valued* contractions and nonexpansive mappings using the Hausdorff metric have been proved by several authors (see, e.g., Nadler [15], Lim [18]). Later, an interesting and rich fixed point theory for such maps and more general maps was developed which has applications in control theory, convex optimization, differential inclusion, and economics (see, Gorniewicz [21] and references cited therein).

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$. An operator $A : H \to H$ is said to be *strongly positive bounded* if there exists a constant $c > 0$ such that

$$\langle Ax,x\rangle_H \geq c\|x\|^2 \ \forall \ x \in H.$$

An operator $A : H \to H$ is called *monotone* if

$$\langle Ax-Ay, x-y\rangle_H \geq 0 \ \forall \ x,y \in H,$$

A is called *k-strongly monotone* if there exists $k \in (0,1)$ such that for each $x,y \in H$ such that

$$\langle Ax-Ay, x-y\rangle_H \geq k\|x-y\|^2.$$

Remark 1. *From the definion of A, we note that strongly positive bounded linear operator A is a* $\|A\|$*-Lipschitzian and c- strongly monotone operator.*

Let C be a nonempty closed and convex subset of real Hilbert space H, and $F : C \to H$ be a nonlinear map. Then, a variational inequality problem with respect to C and F is to find $x^* \in C$ such that

$$\langle Fx^*, p - x^* \rangle \geq 0, \ \forall p \in C.$$

Applications of variational inequality problems span as diverse disciplines as differential equations, time-optimal control, optimization, mathematical programming, mechanics, finance and so on (see, for example, [13] for more details).

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems and variational inequality problem; see, e.g., [10, 27, 29] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in F(T)} \frac{1}{2}\langle Ax, x \rangle - \langle b, x \rangle \tag{4}$$

In [27], Xu proved that the sequence $\{x_n\}$ defined by iterative method below with initial guess $x_0 \in H$ chosen arbitrary:

$$x_{n+1} = \alpha_n b + (I - \alpha_n A)Tx_n \ n \geq 0. \tag{5}$$

converges strongly to the unique solution of the minimization problem (4), where T is a nonexpansive mappings in H and A a strongly positive bounded linear operator. On other hand, Marino and Xu [5] considered an iterative method for a nonexpansive mapping. Let f be a contraction on H and $A : H \to H$ be a strongly positive bounded linear operator. The sequence $\{x_n\}$ defined by iterative method below with initial guess $x_0 \in H$ chosen arbitrary:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n A)Tx_n \ n \geq 0. \tag{6}$$

converges strongly to the fixed point of T, which is a unique solution of the following variational inequality

$$\langle Ax^* - \gamma f(x^*), x^* - p \rangle \leq 0 \ \ \forall p \in F(T).$$

Question 1: Can results of Marino and Xu [5], Xu [27] be extend from single-valued nonexpansive mappings to multivalued nonexpansive mappings?

Question 2: Can results of Marino and Xu [5], Xu [27] be extend from Hilber spaces to real Banach spaces ?

Question 3: We know that nonexpansive mapping is more general than contraction. What happen if the contraction is replaced by nonexpansive mapping ?

Question 4: We know that k- strongly accretive operators and L-Lipchizian operators is more general than strongly positive bounded operators. What happen if the strongly positive bounded linear operators is replaced by k- strongly accretive operators and L-Lipchizian operators ?

The purpose of this paper is to give affirmative answers to these questions mentioned above. Motivated by Marino and Xu [5], we construct an iterative algorithm and prove strong convergence theorems for approximating fixed points of mutivalued nonexpansive mappings which is also the solution of some variational inequality problems in real Banach spaces having weakly continuous duality maps. No compactness assumption is made. The algorithm and results presented in this paper improve and extend some recents results. Finally, our method of proof is of independent interest.

2. Preliminaries

Let E be a real Banach space with norm $\|\cdot\|$ and dual E^*. For any $x \in E$ and $x^* \in E^*$, $\langle x^*, x\rangle$ is used to refer to $x^*(x)$. Let $\varphi : [0,+\infty) \to [0,\infty)$ be a stricly increasing continuous function such that $\varphi(0) = 0$ and $\varphi(t) \to +\infty$ as $t \to \infty$. Such a function φ is called gauge. Associed to a gauge a duality map $J_\varphi : E \to 2^{E^*}$ defined by:

$$J_\varphi(x) := \{x^* \in E^* : \langle x, x^*\rangle = ||x||\varphi(||x||), ||x^*|| = \varphi(||x||)\}, x \in E. \tag{7}$$

If the gauge is defined by $\varphi(t) = t$, then the corresponding duality map is called the *normalized duality map* and is denoted by J. Hence the normalized duality map is given by

$$J(x) := \{x^* \in E^* : \langle x, x^*\rangle = ||x||^2 = ||x^*||^2 =\}, \forall x \in E.$$

Notice that

$$J_\varphi(x) = \frac{\varphi(||x||)}{||x||} J(x), x \neq 0.$$

A normed linear space E is said to be strictly convex if the following holds:

$$\|x\| = \|y\| = 1,\ x \neq y \ \Rightarrow\ \left\|\frac{x+y}{2}\right\| < 1.$$

The modulus of convexity of E is the function $\delta_E : (0,2] \to [0,1]$ defined by:

$$\delta_E(\varepsilon) := \inf\left\{1 - \frac{1}{2}\|x+y\| \ : \|x\| = \|y\| = 1,\ \|x-y\| \geq \varepsilon\right\}.$$

E is uniformly convex if and only if $\delta_E(\varepsilon) > 0$ for every $\varepsilon \in (0,2]$. For $p > 1$, E is said to be p-uniformly convex if there exists a constant $c > 0$ such that $\delta_E(\varepsilon) \geq c\varepsilon^p$ for all $\varepsilon \in (0,2]$.

Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. E is said to be *smooth* if the limit

$$\lim_{t \to 0^+} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S$. E is said to be uniformly smooth if it is smooth and the limit is attained uniformly for each $x, y \in S$.

Let E be a real normed linear space of dimension ≥ 2. The *modulus of smoothness* of E , ρ_E, is defined by:

$$\rho_E(\tau) := \sup\left\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau\right\};\quad \tau > 0.$$

It is known that a normed linear space E is *uniformly smooth* if

$$\lim_{\tau \to 0} \frac{\rho_E(\tau)}{\tau} = 0.$$

If there exists a constant $c > 0$ and a real number $q > 1$ such that $\rho_E(\tau) \leq c\tau^q$, then E is said to be *q-uniformly smooth*. Typical examples of such spaces are the L_p, ℓ_p and W_p^m spaces for $1 < p < \infty$ where,

$$L_p\ (or\ l_p)\ or\ W_p^m \text{ is } \begin{cases} 2-\text{uniformly smooth and } p-\text{uniformly convex} & \text{if } \ 2 \leq p < \infty; \\ 2-\text{uniformly convex and } p-\text{uniformly smooth} & \text{if } \ 1 < p < 2. \end{cases}$$

Let J_q denote the *generalized duality mapping* from E to 2^{E^*} defined by

$$J_q(x) := \left\{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\right\}$$

where $\langle ., . \rangle$ denotes the generalized duality pairing. J_2 is called the *normalized duality mapping* and is denoted by J.

It is known that E is smooth if only if each duality map J_φ is single-valued, that E is Frechet differentiable if and only if each duality map J_φ is norm-to-norm continuous in E, and that E is uniformly smooth if and only if each duality map J_φ is norm-to-norm uniformly continuous on bounded subsets of E.

Following Browder [6], we say that a Banach space has a weakly continuous duality map if there exists a gauge φ such that J_φ is a single-valued and is weak-to-weak* sequentially continous, i.e., if $(x_n) \subset E$, $x_n \xrightarrow{w} x$, then $J_\varphi(x_n) \xrightarrow{w^*} J_\varphi(x)$. It is know that l^p $(1 < p < \infty)$ has a weakly continuous duality map with gauge $\varphi(t) = t^{p-1}$. (see [9] fore more detais on duality maps).

Remark 2. *Note also that a duality mapping exists in each Banach space. We recall from* [1] *some of the examples of this mapping in $l_p, L_p, W^{m,p}$-spaces,* $1 < p < \infty$.

(i) $l_p: \; Jx = \|x\|_{l_p}^{2-p} y \in l_q, \; x = (x_1, x_2, \cdots, x_n, \cdots), \; y = (x_1|x_1|^{p-2}, x_2|x_2|^{p-2}, \cdots, x_n|x_n|^{p-2}, \cdots),$

(ii) $L_p: \; Ju = \|u\|_{L_p}^{2-p} |u|^{p-2} u \in L_q,$

(iii) $W^{m,p}: \; Ju = \|u\|_{W^{m,p}}^{2-p} \sum_{|\alpha \leq m|} (-1)^{|\alpha|} D^\alpha \Big(|D^\alpha u|^{p-2} D^\alpha u \Big) \in W^{-m,q},$

where $1 < q < \infty$ *is such that* $1/p + 1/q = 1$.

Finally recall that a Banach space E satisfies Opial property (see, e.g., [22]) if $\limsup_{n\to+\infty} \|x_n - x\| < \limsup_{n\to+\infty} \|x_n - y\|$ whenever $x_n \xrightarrow{w} x$, $x \neq y$. A Banach space E that has a weakly continuous duality map satisfies Opial's property.

Given a gauge φ and E be a smooth real Banach space. A map $A : E \to E$ is called *accretive* if for each $x, y \in E$

$$\big\langle Ax - Ay, J_\varphi(x - y) \big\rangle \geq 0.$$

A is called k- *strongly accretive* if there exists $k \in (0,1)$ such that for each $x, y \in E$

$$\langle Ax - Ay, J_\varphi(x-y)\rangle \geq k\varphi(||x-y||)||x-y||. \tag{8}$$

In a Hilbert space, the normalized duality map is the identity map. Hence, in Hilbert spaces, *strongly monotonicity* and *strongly accretivity* coincide.

Remark 3. *If* $\varphi(t) = t^{q-1},\ q > 1$, *inequality* (8) *becomes*

$$\langle Ax - Ay, J_q(x-y)\rangle \geq k\|x-y\|^q.$$

Definition 1. *Let E be real Banach space and $T : D(T) \subset E \to 2^E$ be a multivalued mapping. $I - T$ is said to be demiclosed at* 0 *if for any sequence $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to p and $d(x_n, Tx_n)$ converges to zero, then $p \in Tp$.*

Lemma 1 (Demi-closedness Principle, [24])**.** *Let E be a uniformly convex Banach space satisfying the Opial condition, K be a nonempty closed and convex subset of E. Let $T : K \to CB(K)$ be a multivalued nonexpansive mapping with convex-values. Then $I - T$ is demi-closed at zero.*

Lemma 2 ([18])**.** *Let E be a real Banach spaces. Then, the following inequality holds*

$$\Phi(\|x+y\|) \leq \Phi(\|x\|) + \langle y, J_\varphi(x+y)\rangle$$

for all x,y $\in E$.
In particular, for all x,y $\in E$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, J(x+y)\rangle.$$

Theorem 3. [7] *Let $q > 1$ be a fixed real number and E be a smooth Banach space. Then the following statements are equivalent:*
(i) *E is q-uniformly smooth.*
(ii) *There is a constant $d_q > 0$ such that for all $x, y \in E$*

$$\|x+y\|^q \leq \|x\|^q + q\langle y\,,\, J_q(x)\rangle + d_q\|y\|^q.$$

(iii) There is a constant $c_1 > 0$ such that

$$\langle x-y\,,J_q(x)-J_q(y)\rangle \le c_1\|x-y\|^q \;\; \forall \; x,y\in E.$$

Lemma 4 (Xu, [28]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that $a_{n+1} \le (1-\alpha_n)a_n + \sigma_n$ for all $n \ge 0$, where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\sigma_n\}$ is a sequence in R such that*
(a) $\sum_{n=0}^{\infty} \alpha_n = \infty$, (b) $\limsup_{n\to\infty} \frac{\sigma_n}{\alpha_n} \le 0$ or $\sum_{n=0}^{\infty} |\sigma_n| < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 5 (Chidume et al. [8]). *Let X be a reflexive real Banach space and let $A\,,B \in CB(X)$. Assume that B is weakly closed. Then, for every $a \in A$, there exists $b \in B$ such that*

$$\|a-b\| \le H(A,B).$$

Definition 2. *A function $f : E \to \mathbb{R}$ is said to be strongly convex if there exists $\alpha > 0$ such that for every $x,y \in E$ with $x \ne y$ and $\lambda \in (0,1)$, the following inequality holds:*

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y) - \alpha\|x-y\|^2. \tag{9}$$

Lemma 6. *Let E be normed linear space and $f : E \to \mathbb{R}$ a real-valued differentiable convex function. Assume that f is strongly convex. Then the differential map $\nabla f : E \to E^*$ is strongly monotone, i.e., there exists a positive constant k such that*

$$\langle \nabla f(x) - \nabla f(y), x-y\rangle \ge k\|x-y\|^2 \; \forall x,y \in E. \tag{10}$$

Lemma 7. [25] *Let C and D be nonempty subsets of a smooth real Banach space E with $D \subset C$ and $Q_D : C \to D$ a retraction from C into D. Then Q_D is sunny and nonexpansive if and only if*

$$\langle z - Q_D z, J(y - Q_D z)\rangle \le 0 \tag{11}$$

for all $z \in C$ and $y \in D$.

It is noted that Lemma 7 still holds if the normalized duality map is replaced by the general duality map J_φ, where φ is gauge function.

Remark 4. *If K is a nonempty closed convex subset of a Hilbert space H, then the nearest point projection P_K from H to K is the sunny nonexpansive retraction.*

Lemma 8. [26] *Let be a real Hilbert space H. Let $A : H \to H$ be a k-strongly monotone and L-Lipschitzian operator with $k > 0$, $L > 0$. Assume that $0 < \eta < \frac{2k}{L^2}$ and $\tau = \eta\Big(k - \frac{L^2\eta}{2}\Big)$. Then for each $t \in \Big(0, min\{1, \frac{1}{\tau}\}\Big)$, we have*

$$\|(I - t\eta A)x - (I - t\eta A)y\| \leq (1 - t\tau)\|x - y\| \; x, y \in H.$$

Lemma 9. *Let $q > 1$ be a fixed real number and E be a q-uniformly smooth real Banach space with constant d_q. Let $A : E \to E$ be a k-strongly accretive and L-Lipschitzian operator with $k > 0$, $L > 0$. Assume that $0 < \eta < \Big(\frac{kq}{d_q L^q}\Big)^{\frac{1}{q-1}}$ and $\tau = \eta\Big(k - \frac{d_q L^q \eta^{q-1}}{q}\Big)$. Then for each $t \in \Big(0, min\{1, \frac{1}{\tau}\}\Big)$, we have*

$$\|(I - t\eta A)x - (I - t\eta A)y\| \leq (1 - t\tau)\|x - y\| \; x, y \in E.$$

Proof. Using (ii) of Theorem 3 and properties of A, we have

$$\begin{aligned}
\|(I - t\eta A)x - (I - t\eta A)y\|^q &\leq \|x - y\|^q + q\langle t\eta Ay - t\eta Ax\,,\, J_q(x - y)\rangle + d_q\|t\eta Ax - t\eta Ay\|^q \\
&\leq \|x - y\|^q - qt\eta\,\langle Ax - Ay\,,\, J_q(x - y)\rangle + d_q(t\eta)^q\|Ax - Ay\|^q \\
&\leq \|x - y\|^q - qtk\eta\,\|x - y\|^q + d_q(Lt\eta)^q\|x - y\|^q \\
&\leq \Big(1 - qtk\eta + d_q L^q t^q \eta^q\Big)\|x - y\|^q.
\end{aligned}$$

Therefore

$$\|(I - t\eta A)x - (I - t\eta A)y\| \leq \Big(1 - qtk\eta + d_q L^q t^q \eta^q\Big)^{\frac{1}{q}}\|x - y\|. \tag{12}$$

Using definition of η, inequality (12) and inequality $(1 + x)^s \leq 1 + sx$, for $x > -1$ and $0 < s < 1$, we have

$$\begin{aligned}
\|(I - t\eta A)x - (I - t\eta A)y\| &\leq \Big(1 - tk\eta + \frac{d_q L^q t^q \eta^q}{q}\Big)\|x - y\| \\
&\leq \Big(1 - t\eta(k - \frac{d_q L^q \eta^{q-1}}{q}\Big)\|x - y\| \\
&\leq (1 - t\tau)\|x - y\|,
\end{aligned}$$

etablishing the lemma. □

Remark 5. *Lemma 9 is one generalization of Lemma 8.*

3. Main results

In what follows, we use the following iteration scheme: let E be a q-uniformly smooth real Banach space and uniformly convex real Banach space and $T : E \to CB(E)$ be a multivalued nonexpansive mapping with convex-values and f be a nonexpansive mapping on E.

Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in E$ by:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \eta A)y_n,\ n \geq 0, \tag{13}$$

$$\|y_n - y_{n-1}\| \leq H(Tx_n, Tx_{n-1})\ \forall n \geq 1, \tag{14}$$

where $y_n \in Tx_n$ and $\{\alpha_n\}$ is real sequence in $(0,1)$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Remark 6. *From y_{n-1}, the existence of y_n in* (13) *satisfying* (14) *is garanted by Lemma 5.*

We now prove the following theorem.

Theorem 10. *Let $q > 1$ be a fixed real number and E be a q-uniformly smooth and uniformly convex real Banach space having a weakly continuous duality map. Let $T : E \to CB(E)$ be a multivalued nonexpansive mapping with convex-values and f be a nonexpansive mapping on E such that $= F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. Let $A : E \to E$ be a k-strongly accretive and L-Lipschitzian operator with $k > 0,\ L > 0$. Assume that $0 < \eta < \left(\frac{kq}{d_q L^q}\right)^{\frac{1}{q-1}}$ and $0 < \gamma < \tau$, where $\tau = \eta\left(k - \frac{d_q L^q \eta^{q-1}}{q}\right)$.*

Then, the sequence $\{x_n\}$ *defined by* (13) *and* (14) *converges strongly to a fixed point* x^* *of* T, *which solves the following variational inequality*

$$\langle \eta Ax^* - \gamma f(x^*), J_\varphi(x^* - p)\rangle \leq 0 \ \forall p \in F(T). \tag{15}$$

Proof. We first show that the uniqueness of the solution of the variational inequality (15). Suppose both $x^* \in F(T)$ and $x^{**} \in F(T)$ are solutions to (15), then

$$\langle \eta Ax^* - \gamma f(x^*), J_\varphi(x^* - x^{**})\rangle \leq 0 \tag{16}$$

and

$$\langle \eta Ax^{**} - \gamma f(x^{**}), J_\varphi(x^{**} - x^*)\rangle \leq 0 \tag{17}$$

Adding up (16) and (17) yields

$$\langle \eta Ax^{**} - \eta Ax^* + \gamma f(x^*) - \gamma f(x^{**}), J_\varphi(x^{**} - x^*)\rangle \leq 0. \tag{18}$$

$$\begin{aligned} \frac{d_q L^q \eta^{q-1}}{q} > 0 \quad &\Longleftrightarrow \quad k - \frac{d_q L^q \eta^{q-1}}{q} < k \\ &\Longleftrightarrow \quad \eta\left(k - \frac{d_q L^q \eta^{q-1}}{q}\right) < k\eta \\ &\Longleftrightarrow \quad \tau < k\eta. \end{aligned}$$

It follows that

$$0 < \gamma < \tau < k\eta.$$

Noticing that

$$\langle \eta Ax^{**} - \eta Ax^* + \gamma f(x^*) - \gamma f(x^{**}), J_\varphi(x^{**} - x^*)\rangle \geq (k\eta - \gamma)\|x^* - x^{**}\|\varphi(\|x^* - x^{**}\|),$$

which implies that $x^* = x^{**}$ and the uniqueness is proved. Below we use x^* to denote the unique solution of (15).

Without loss of generality, we can assume $\alpha_n \in \Big(0, min\{1, \dfrac{1}{\tau}\}\Big)$. We prove that the sequence $\{x_n\}$ is bounded. Let $p \in F(T)$. From (13) and the fact $Tp = \{p\}$, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n \gamma f(x_n) + (I - \alpha_n \eta A) y_n - p\| \\
&\leq \alpha_n \gamma \|x_n - p\| + \alpha_n \|\eta A p - \gamma f(p)\| + (1 - \alpha_n \tau)\|y_n - p\| \\
&\leq \alpha_n \gamma \|x_n - p\| + \alpha_n \|\eta A p - \gamma f(p)\| + (1 - \alpha_n \tau) H(Tx_n, Tp) \\
&= [1 - (\tau - \gamma)\alpha_n]\|x_n - p\| + \alpha_n \|\eta A p - \gamma f(p)\| \leq \max\{\|x_n - p\|, \frac{1}{\tau - \gamma}\|\eta A p - \gamma f(p)\|\}
\end{aligned}$$

By induction, it is easy to see that

$$\|x_{n+1} - p\| \leq \max\{\|x_0 - p\|, \frac{1}{\tau - \gamma}\|\eta A p - \gamma f(p)\|\}, \ n \geq 1.$$

Hence, $\{x_n\}$, $f(x_n)$ and $\{y_n\}$ are bounded.

From (13) and (14), it follows that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n \gamma f(x_n) + (I - \alpha_n \eta A) y_n - \alpha_{n-1} \gamma f(x_{n-1}) - (I - \alpha_{n-1} \eta) A y_{n-1}\| \\
&= \|\alpha_n \big(\gamma f(x_n) - \gamma f(x_{n-1})\big) + (\alpha_n - \alpha_{n-1}) \gamma f(x_{n-1}) \\
&\quad + (I - \alpha_n \eta A)(y_n - y_{n-1}) + (\alpha_{n-1} - \alpha_n) \eta A y_{n-1}\| \\
&\leq \alpha_n \gamma \|x_n - x_{n-1}\| + (1 - \alpha_n \tau)\|y_n - y_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\gamma f(x_{n-1}) + \eta A y_{n-1}\| \\
&\leq \alpha_n \gamma \|x_n - x_{n-1}\| + (1 - \alpha_n \tau) H(Tx_n, Tx_{n-1}) + |\alpha_n - \alpha_{n-1}|\|\gamma f(x_{n-1}) + \eta A y_{n-1}\|.
\end{aligned}$$

Hence,

$$\|x_{n+1} - x_n\| \leq [1 - (\tau - \gamma)\alpha_n]\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| M_1, \tag{19}$$

where $M_1 > 0$ is such that $\sup_n\{\|\gamma f(x_{n-1})\| + \|\eta A y_{n-1}\|\} \leq M_1$. Hence, from (19) and Lemma 4, we deduce

$$\lim_{n \to +\infty} \|x_{n+1} - x_n\| = 0.$$

At the same time, we note that

$$\|x_{n+1} - y_n\| = \alpha_n \|\gamma f(x_n) - \eta A y_n\| \to 0.$$

Therefore, we have

$$\lim_{n\to+\infty}\|x_n-y_n\|=0.$$

Hence

$$d(x_n,Tx_n)=0,\ n\to\infty. \tag{20}$$

Let t_0 be a fixed real number such that $t_0\in\Big(0,min\{1\ ,\frac{1}{\tau}\}\Big)$. We observe that $Q_{F(T)}(I+(t_0\gamma f-t_0\eta A))$ is a contraction, where $Q_{F(T)}$ is the sunny nonexpansive retraction from E to $F(T)$. Indeed, for all $x,y\in E$, by Lemma 9, we have

$$\begin{aligned}
&\|Q_{F(T)}(I+(t_0\gamma f-t_0\eta A))x-Q_{F(T)}(I+(t_0\gamma f-t_0\eta A))y\|\\
\le\ &\|(I+(t_0\gamma f-t_0\eta A))x-(I+(t_0\gamma f-t_0\eta A))y\|\\
\le\ &t_0\gamma\|f(x)-f(y)\|+\|(I-t_0\eta A)x-(I-t_0\eta A)y\|\\
\le\ &(1-t_0(\tau-\gamma))\|x-y\|.
\end{aligned}$$

Banach's Contraction Mapping Principle guarantees that $Q_{F(T)}(I+(t_0\gamma f-t_0\eta A))$ has a unique fixed point, say $x_1\in E$. That is, $x_1=Q_{F(T)}(I+(t_0\gamma f-t_0\eta A))x_1$. Thus, in view of Lemma 7, it is equivalent to the following variational inequality problem

$$\langle\eta Ax_1-\gamma f(x_1),J_\varphi(x_1-p)\rangle\le 0\ \ \forall\ \ p\in F(T).$$

By the uniqueness of the solution of (15), we have $x_1=x^*$.

Next, we prove that

$$\limsup_{n\to+\infty}\langle\eta Ax^*-\gamma f(x^*),J_\varphi(x^*-x_n)\rangle\le 0.$$

Since E is reflexive and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that x_{n_k} converges weakly to a in E and

$$\limsup_{n\to+\infty}\langle\eta Ax^*-\gamma f(x^*),J_\varphi(x^*-x_n)\rangle=\lim_{k\to+\infty}\langle\eta Ax^*-\gamma f(x^*),J_\varphi(x^*-x_{n_k})\rangle.$$

From (20) and Lemma 1, we obtain $a\in F(T)$. On other hand, the assumption that the duality mappiing J_φ is weakly continuous and (15), we then have

$$\begin{aligned}
\limsup_{n\to+\infty}\langle\eta Ax^*-\gamma f(x^*),J_\varphi(x^*-x_n)\rangle &= \lim_{k\to+\infty}\langle\eta Ax^*-\gamma f(x^*),J_\varphi(x^*-x_{n_k})\rangle\\
&= \langle\eta Ax^*-\gamma f(x^*),J_\varphi(x^*-a)\rangle\le 0.
\end{aligned}$$

Finally, we show that $x_n \to x^*$. In fact, since $\Phi(t) = \int_0^t \varphi(\sigma)d\sigma$, $\forall t \geq 0$, and φ is a gauge function, then for $1 \geq k \geq 0$, $\Phi(kt) \leq k\Phi(t)$. From (13) and Lemma 2, we get that

$$\begin{aligned}
&\Phi(\|x_{n+1} - x^*\|) = \Phi(\|\alpha_n \gamma f(x_n) + (I - \alpha_n \eta A)y_n - x^*\|) \\
\leq\;& \Phi(\|\alpha_n(\gamma f(x_n) - \gamma f(x^*) + (I - \alpha_n \eta A)(y_n - x^*)\|) + \alpha_n \langle \eta A x^* - \gamma f(x^*), J_\varphi(x^* - x_{n+1})\rangle \\
\leq\;& \Phi(\alpha_n \|\gamma f(x_n) - \gamma f(x^*)\| + (1 - \alpha_n \tau)\|y_n - x^*\|) + \alpha_n \langle \eta A x^* - \gamma f(x^*), J_\varphi(x^* - x_{n+1})\rangle \\
\leq\;& \Phi(\alpha_n \gamma \|x_n - x^*\| + (1 - \alpha_n \tau)\|x_n - x^*\|) + \alpha_n \langle \eta A x^* - \gamma f(x^*), J_\varphi(x^* - x_{n+1})\rangle \\
\leq\;& \Phi([1 - (\tau - \gamma)\alpha_n]\|x_n - x^*\|) + \alpha_n \langle \eta A x^* - \gamma f(x^*), J_\varphi(x^* - x_{n+1})\rangle \\
\leq\;& [1 - (\tau - \gamma)\alpha_n]\Phi(\|x_n - x^*\|) + \alpha_n \langle \eta A x^* - \gamma f(x^*), J_\varphi(x^* - x_{n+1})\rangle
\end{aligned}$$

From Lemma 4, its follows that $x_n \to x^*$. This completes the proof. □

Corollary 11. *Let H be a real Hilbert space. Let $T : H \to CB(H)$ be a multivalued nonexpansive mapping with convex-values and f be a nonexpansive mapping on H such that $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. Let $A : H \to H$ be a k-strongly monotone and L-Lipschitzian operator with $k > 0$, $L > 0$. Assume that $0 < \eta < \dfrac{2k}{L^2}$ and $0 < \gamma < \tau$, where $\tau = \eta\left(k - \dfrac{L^2\eta}{2}\right)$. Then, the sequence $\{x_n\}$ defined by* (13) *and* (14) *converges strongly to a fixed point x^* of T, which solves the following variational inequality*

$$\langle \eta A x^* - \gamma f(x^*), x^* - p\rangle \leq 0 \;\; \forall p \in F(T). \tag{21}$$

Proof. Hilbert spaces are 2-uniformly smooth and uniforly convex, the proof follows from Theorem 10. □

Corollary 12. *Assume that $E = l_q$, $1 < q < \infty$. Let $T : E \to CB(E)$ be a multivalued nonexpansive mapping with convex-values and f be a self-contraction mapping on E. Let $A : E \to E$ be a k-strongly accretive and L-Lipschitzian operator with $k > 0$, $L > 0$. Assume that $0 < \eta < \left(\dfrac{kq}{d_q L^q}\right)^{\frac{1}{q-1}}$ and $0 < \gamma < \tau$, where $\tau = \eta\left(k - \dfrac{d_q L^q \eta^{q-1}}{q}\right)$. Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in E$ by:*

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \eta A)y_n, \; n \geq 0, \tag{22}$$

$$\|y_n - y_{n-1}\| \leq H(Tx_n, Tx_{n-1}) \; \forall n \geq 1, \tag{23}$$

where $y_n \in Tx_n$ and $\{\alpha_n\}$ is real sequence in $(0,1)$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0$;
(ii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.
Then, the sequence $\{x_n\}$ defined by (22) *and* (23) *converges strongly to a fixed point x^* of T, which solves the following variational inequality*

$$\langle \eta Ax^* - \gamma f(x^*), J_\varphi(x^* - p)\rangle \le 0 \;\; \forall p \in F(T). \tag{24}$$

Proof. Since $E = l_q$, $1 < q < \infty$ are q-uniformly smooth, uniformly convex and has a weakly continuous duality map. Using the fact that contractions are nonexpansives mappings. The proof follows from Theorem 10. □

Corollary 13. *Let H be a real Hilbert space. Let $T : H \to CB(H)$ be a multivalued nonexpansive mapping with convex-values and f be a nonexpansive mapping on H such that $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. Let $A : H \to H$ be a k-strongly positive bounded linear operator with $k > 0$. Assume that $0 < \eta < \frac{2k}{\|A\|^2}$ and $0 < \gamma < \tau$, where $\tau = \eta\left(k - \frac{\|A\|^2\eta}{2}\right)$.*
Then the sequence $\{x_n\}$ defined by (13) *and* (14) *converges strongly to a fixed point x^* of T, which solves the following variational inequality*

$$\langle \eta Ax^* - \gamma f(x^*), x^* - p\rangle \le 0 \;\; \forall p \in F(T). \tag{25}$$

Proof. By Remark 1, the proof follows from Theorem 10. □

Corollary 14. *Let H be a real Hilbert space. Let $T : H \to CB(H)$ be a multivalued nonexpansive mapping with convex-values and f be a nonexpansive mapping on H such that $F(T) \neq \emptyset$ and $Tp = \{p\}$ for all $p \in F(T)$. Let $g : H \to \mathbb{R}$ be a differentiable,strongly convex real-valued function. Suppose the differential map $\nabla g : H \to H$ is L-Lipschitz. Assume that $0 < \eta < \frac{2k}{L^2}$ and $0 < \gamma < \tau$, where $\tau = \eta\left(k - \frac{L^2\eta}{2}\right)$.*
Let $\{x_n\}$ be a sequence defined iteratively from arbitrary $x_0 \in H$ by:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n \eta \nabla g) y_n,\; n \ge 0, \tag{26}$$

$$\|y_n - y_{n-1}\| \le H(Tx_n, Tx_{n-1}) \;\forall n \ge 1, \tag{27}$$

where $y_n \in Tx_n$ and $\{\alpha_n\}$ is real sequence in $(0,1)$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Then, the sequence defined by (26) *and* (27) *converges strongly to a fixed point x^* of T, which solves the following variational inequality*

$$\langle \eta \nabla g(x^*) - \gamma f(x^*), x^* - p \rangle \leq 0 \;\; \forall p \in F. \tag{28}$$

Proof. From strong convexity and differentiablity of g, it follows that ∇g is k-strongly monotone. Since ∇g Lipschitz. Therefore the follows from Theorem 10. □

We now give example of space E and mapping T satisfying the assumptions of Theorem 10. Let $E = l_p,\ 1 < p < \infty$, and $T : E \to CB(E)$ be the mapping defined by:

$$Tx = \begin{cases} x \;\; \text{if } x \neq 0, \\ \\ y \in E : \quad 0 < \|y\|_{l_p} \leq 1 \;\text{ if } x = 0. \end{cases}$$

It is well known (see, e.g., [9]) that $E = l_p,\ 1 < p < \infty$, has weakly continuous duality map. The map T is nonexpansive with convex values. Therefore, the spaces E and the map T satisfies all the assumptions of Theorem 10.

Remark 7. *Real sequences that satisfy conditions (i) and (ii) and re given by:* $\alpha_n = \dfrac{1}{\sqrt{n}}$.

Remark 8. *For numerous applications to approximate fixed points of nonexpansive mappings, see the celebrated monograph of Berinde* [2]*. As remarked by Charles Byrne* [3]*, most of the maps that arise in image reconstruction and signal processing are nonexpansive in nature.*

Conflict of Interests

The authors declare that there is no conflict of interests.

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