Available online at http://jfpt.scik.org
J. Fixed Point Theory, 2019, 2019:7
ISSN: 2052-5338

PPF DEPENDENT COUPLED FIXED POINTS VIA $C-$CLASS FUNCTIONS

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Abstract. We define mixed monotone property and coupled fixed point under the concept of PPF dependence and prove the existence of PPF dependent coupled fixed points of a non-self Banach space valued mapping. Our results extend the results of Bhaskar and Lakshmikantham [12] and Harjani, Lopez and Sadarangani [14] under the concept of PPF dependence. To illustrate the phenomena, we provide an example in support of our main result.

Keywords: PPF dependent fixed point; Razumikhin class; mixed monotone property; PPF dependent coupled fixed point; $\alpha-$admissible mapping; $C-$class function.

2010 AMS Subject Classification: 47H10, 54H25.

1. INTRODUCTION

In 1922, Banach [5] established a fixed point theorem known as Banach contraction principle and it is a fundemental result to get the existence of fixed points in analysis. There are many generalizations of Banach contraction principle by changing either the domain space or extending a single valued mapping to multi-valued mappings. In particular, Ran and Reurings [21] extended Banach's fixed point theorem in complete metric space endowed with a partial ordering. Guo and Lakshmikantham [11] introduced mixed monotone operators through which

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Received March 16, 2019

Gnana Bhaskar and Lakshmikantham [12] established the existence of coupled fixed points of mappings satisfying mixed monotone property in partially ordered metric spaces, for more details we refer [3, 14, 16, 19].

In 1977, Bernfeld, Lakshmikantham and Reddy[7] introduced the concept of fixed point for mappings that have different domains and ranges which is called PPF (Past, Present and Future) dependent fixed point. Furthermore, they gave notion of Banach type contraction for non-self mapping and proved the existence of PPF dependent fixed points in the Razumikhin class for Banach type contraction mappings. The PPF dependent fixed point theorems are useful for providing the solutions of nonlinear functional differential and integral equations which may depend upon the past history, present data and future consideration. Several mathematicians proved the existence of PPF dependent fixed point of single-valued and multi-valued mappings, for more details we refer [4, 6, 8, 9, 10, 15, 18].

In 2012, Samet, Vetro and Vetro [22] introduced the concept of $\alpha-$admissible self mappings and proved the existence of fixed points using contractive type conditions involving an $\alpha-$admissible mapping in complete metric spaces and such study was continued in [1, 2, 13, 17, 20].

In 2014, Ciric, Alsulami, Salmi and Vetro [8] introduced the concept of triangular α_c-admissible mappings with respect to η_c non-self mappings and established the existence of PPF dependent fixed points for contraction mappings involving triangular α_c-admissible mappings with respect to η_c non-self mappings in the Razumikhin class.

Throughout this paper, we denote the real line by $\mathbb{R}$, $\mathbb{R}^+ = [0,\infty)$, and $\mathbb{N}$ is the set of all natural numbers, $\mathbb{Z}$ is the set of intergers. Let $(E, ||.||_E)$ be a Banach space and we denote it simply by E. Let $I = [a,b] \subseteq \mathbb{R}$ and $E_0 = C(I,E)$, the set of all continuous functions on I equipped with the supremum norm $||.||_{E_0}$ and we define it by $||\phi||_{E_0} = \sup\limits_{a\leq t\leq b} ||\phi(t)||_E$ for $\phi \in E_0$. We use the following proposition in proving our results.

Proposition 1.1. *If $\{a_n\}$ and $\{b_n\}$ are two real sequences, $\{b_n\}$ is bounded, then* $\liminf(a_n + b_n) \leq \liminf a_n + \limsup b_n$.

In Section 2 of this paper, we present preliminaries in which we present basic definitions, lemmas that are needed to develop the paper. Also we extend the concept of coupled fixed points from the metric space setting to the concept of PPF dependent coupled fixed points in $E_0 \times E_0$.

In Section 3, we prove the existence of PPF dependent coupled fixed points of a function $F : E_0 \times E_0 \to E$ having the mixed c-monotone property. An example is provided to illustrate the main result of this paper.

2. Preliminaries

Definition 2.1. [12] Let $(X, \preceq)$ be a partially ordered metric space and $F : X \times X \to X$ be a mapping. We say that F has the mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and is monotone non-increasing in y, that is for any $x, y \in X$,

$$\begin{cases} \text{(i)} \ x_1, x_2 \in X, \ x_1 \preceq x_2 \ \text{implies} \ F(x_1, y) \preceq F(x_2, y) \\ \text{and} \\ \text{(ii)} \ y_1, y_2 \in X, \ y_1 \preceq y_2 \ \text{implies} \ F(x, y_2) \preceq F(x, y_1). \end{cases} \tag{2.1}$$

Definition 2.2. [12] An element $(x,y) \in X \times X$ is said to be a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.

Theorem 2.3. [12] *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \to X$ be a continuous function having the mixed monotone property on X. Assume that there exists a $k \in [0,1)$ with*

$$d(F(x,y), F(u,v)) \leq \frac{k}{2}[d(x,u) + d(y,v)] \tag{2.2}$$

for any $u \preceq x$, $y \preceq v$. If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$ then there exist $x, y \in X$ such that $F(x,y) = x$ and $F(y,x) = y$.

Theorem 2.4. [12] *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Assume that X has the following properties:*

(i) *if a non-decreasing sequence $\{x_n\} \subset X$ with $x_n \to x$, then $x_n \preceq x$ for any n,*

(ii) *if a non-increasing sequence $\{y_n\} \subset X$ with $y_n \to y$, then $y \preceq y_n$ for any n.*

Let $F : X \times X \to X$ be a function having the mixed monotone property on X. Assume that there *exists a* $k \in [0,1)$ *with*

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}[d(x,u)+d(y,v)] \tag{2.3}$$

for any $u \preceq x,\ y \preceq v$. *If there exist* $x_0,y_0 \in X$ *such that* $x_0 \preceq F(x_0,y_0)$ *and* $F(y_0,x_0) \preceq y_0$ *then there exist* $x,y \in X$ *such that* $F(x,y)=x$ *and* $F(y,x)=y$.

Definition 2.5. [17] Let T be a self mapping on X and let $\alpha : X \times X \to \mathbb{R}^+$ be a function. We say that T is $\alpha-$admissible mapping if for any $x,y \in X,\ \alpha(x,y) \geq 1 \implies \alpha(Tx,Ty) \geq 1$.

Karapinar, Kumam and Salimi [17] introduced the notion of triangular $\alpha-$admissible mappings as follows.

Definition 2.6. [17] Let T be a self mapping on X and let $\alpha : X \times X \to \mathbb{R}^+$ be a function. Then T is said to be a triangular $\alpha-$admissible mapping if for any $x,y,z \in X$,

$\alpha(x,y) \geq 1 \implies \alpha(Tx,Ty) \geq 1$ and

$\alpha(x,z) \geq 1,\ \alpha(z,y) \geq 1 \implies \alpha(x,y) \geq 1.$

Definition 2.7. [19] Let $F : X \times X \to X$ and $\alpha : X^2 \times X^2 \to \mathbb{R}^+$ be two mappings. We say that F is $\alpha-$admissible if for any $x,y,u,v \in X$ we have

$\alpha((x,y),(u,v)) \geq 1 \implies \alpha((F(x,y),F(y,x)),(F(u,v),F(v,u))) \geq 1.$

Harjani, Lopez, Sadarangani[14] proved some generalizations of the results of [12].

Theorem 2.8. [14] *Let* $(X,\preceq)$ *be a partially ordered set and suppose there exists a metric* d *on* X *such that* (X,d) *is a complete metric space. Let* $F : X \times X \to X$ *be a mapping having the mixed monotone property on* X *such that*

$$\psi(d(F(x,y),F(u,v))) \leq \psi(max\{d(x,u),d(y,v)\}) - \phi(max\{d(x,u),d(y,v)\})$$

for any $u \preceq x$ *and* $y \preceq v$, *where* ϕ,ψ *are altering distance functions.*

Suppose that either

(a) F *is continuous or*

(b) X *has the following properties :*

(i) *if a non-decreasing sequnce* $\{x_n\} \subset X$ *with* $x_n \to x$, *then* $x_n \preceq x$ *for all* n,

(ii) *if a non-increasing sequence* $\{y_n\} \subset X$ *with* $y_n \to y$, *then* $y \preceq y_n$ *for all* n.

If there exist $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then there exist $x, y \in X$ such that $F(x,y) = x$ and $F(y,x) = y$.

In 2014, Ansari [1] introduced the concept of $C-$ class function as follows.

Definition 2.9. [1] A mapping $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called a $C-$class function if it is continuous and for any $s, t \in \mathbb{R}^+$, the function G satisfies the following conditions:

(i) $G(s,t) \leq s$ and

(ii) $G(s,t) = s$ implies that either $s = 0$ or $t = 0$.

The family of all $C-$class functions is denoted by ζ.

Example 2.10. [1] The following functions belong to ζ.

(i) $G(s,t) = s - t$ for all $s, t \in \mathbb{R}^+$.

(ii) $G(s,t) = ks$ for all $s, t \in \mathbb{R}^+$ where $0 < k < 1$.

(iii) $G(s,t) = \frac{s}{(1+t)^r}$ for all $s, t \in \mathbb{R}^+$ where $r \in \mathbb{R}^+$.

(iv) $G(s,t) = s\beta(s)$ for all $s, t \in \mathbb{R}^+$ where $\beta : \mathbb{R}^+ \to [0,1)$ is continuous.

(v) $G(s,t) = s - \phi(s)$ for all $s, t \in \mathbb{R}^+$ where $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and $\phi(t) = 0$ if and only if $t = 0$.

(vi) $G(s,t) = sh(s,t)$ for all $s, t \in \mathbb{R}^+$ where $h : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous such that $h(s,t) < 1$ for all $s, t \in \mathbb{R}^+$.

For a fixed $c \in I$, the Razumikhin class R_c of functions in E_0 is defined by $R_c = \left\{ \phi \in E_0 / \, ||\phi||_{E_0} = ||\phi(c)||_E \right\}$. Clearly every constant function from I to E belongs to R_c so that R_c is a non-empty subset of E_0 .

Definition 2.11. [7] Let R_c be the Razumikhin class of continuous functions in E_0. We say that

(i) the class R_c is algebraically closed with respect to the difference if $\phi - \psi \in R_c$, whenever $\phi, \psi \in R_c$.

(ii) the class R_c is topologically closed if it is closed with respect to the topology on E_0 by the norm $||.||_{E_0}$.

The Razumikhin class of functions R_c has the following properties.

Theorem 2.12. [4] *Let R_c be the Razumikhin class of functions in E_0. Then*

(i) *for any $\phi \in R_c$ and $\alpha \in \mathbb{R}$, we have $\alpha\phi \in R_c$.*

(ii) *the Razumikhin class R_c is topologically closed with respect to the norm defined on E_0.*

(iii) $\bigcap\limits_{c\in[a,b]} R_c = \{\phi \in E_0/\phi : I \to E \text{ is constant}\}$.

Definition 2.13. [7] Let $T : E_0 \to E$ be a mapping. A function $\phi \in E_0$ is said to be a PPF dependent fixed point of T if $T\phi = \phi(c)$ for some $c \in I$.

Definition 2.14. [7] Let $T : E_0 \to E$ be a mapping. Then T is called a Banach type contraction if there exists $k \in [0,1)$ such that $||T\phi - T\psi||_E \leq k||\phi - \psi||_{E_0}$ for any $\phi, \psi \in E_0$.

Theorem 2.15. [7] *Let $T : E_0 \to E$ be a Banach type contraction. Let R_c be algebraically closed with respect to the difference and topologically closed. Then T has a unique PPF dependent fixed point in R_c.*

Definition 2.16. Let $c \in I$. Let $T : E_0 \to E$ and $\alpha : E \times E \to \mathbb{R}^+$ be two functions. Then T is said to be a α_c-admissible mapping if for any $\phi, \psi \in E_0$,

$$\alpha(\phi(c), \psi(c)) \geq 1 \implies \alpha(T\phi, T\psi) \geq 1. \tag{2.4}$$

Ciric, Alsulami, Salimi and Vetro[8] introduced the concept of triangular α_c-admissible mapping with respect to η_c as follows.

Definition 2.17. [8] Let $c \in I$ and $T : E_0 \to E$. Let $\alpha, \eta : E \times E \to \mathbb{R}^+$ be two functions. Then T is said to be a triangular α_c-admissible mapping with respect to η_c if for any $\phi, \psi, \varphi \in E_0$,

$$\begin{cases} \text{(i) } \alpha(\phi(c), \psi(c)) \geq \eta(\phi(c), \psi(c)) \implies \alpha(T\phi, T\psi) \geq \eta(T\phi, T\psi) \\ \qquad\qquad\qquad\qquad \text{and} \\ \text{(ii) } \alpha(\phi(c), \psi(c)) \geq \eta(\phi(c), \psi(c)),\ \alpha(\psi(c), \varphi(c)) \geq \eta(\psi(c), \varphi(c)) \\ \qquad\qquad \implies \alpha(\phi(c), \varphi(c)) \geq \eta(\phi(c), \varphi(c)). \end{cases} \tag{2.5}$$

Note that if $\eta(x,y) = 1$ for any $x, y \in E$, then we say that T is a triangular α_c- admissible mapping and if $\alpha(x,y) = 1$ for any $x, y \in E$, then we say that T is a triangular η_c-subadmissible mapping.

In the following we define mixed $c-$monotone property, α_c-admissible mapping, PPF dependent coupled fixed point and triangular α_c-admissible mapping for a non-self mapping $F: E_0 \times E_0 \to E$.

Definition 2.18. Let $c \in I$. Let $(E, \preceq)$ be a partially ordered Banach space and $F: E_0 \times E_0 \to E$ be a function. We say that F has the mixed $c-$monotone property if for any $\phi, \psi \in E_0$,

$$\text{(i) } \phi_1, \phi_2 \in E_0,\ \phi_1(c) \preceq \phi_2(c) \implies F(\phi_1, \phi) \preceq F(\phi_2, \phi)$$

and

$$\text{(ii) } \psi_1, \psi_2 \in E_0,\ \psi_1(c) \preceq \psi_2(c) \implies F(\psi, \psi_2) \preceq F(\psi, \psi_1).$$

Definition 2.19. Let $c \in I$. Let $F: E_0 \times E_0 \to E$ and $\alpha: E^2 \times E^2 \to \mathbb{R}^+$ be two functions. Then F is said to be α_c-admissible mapping if for any $\phi_1, \phi_2, \psi_1, \psi_2 \in E_0$,

$$\alpha((\phi_1(c), \psi_1(c)), (\phi_2(c), \psi_2(c))) \geq 1 \implies$$
$$\alpha((F(\phi_1, \psi_1), F(\psi_1, \phi_1)), (F(\phi_2, \psi_2)), F(\psi_2, \phi_2))) \geq 1.$$

Definition 2.20. Let $c \in I$. An element $(\phi, \psi) \in E_0 \times E_0$ is said to be a PPF dependent coupled fixed point of the mapping $F: E_0 \times E_0 \to E$ if $F(\phi, \psi) = \phi(c)$ and $F(\psi, \phi) = \psi(c)$.

Definition 2.21. Let $c \in I$. Let $F: E_0 \times E_0 \to E$ and $\alpha: E^2 \times E^2 \to \mathbb{R}^+$ be two functions. Then F is said to be triangular α_c-admissible mapping if it satisfies the following conditions:

(i) F is α_c-admissible and

(ii) for any $\phi_1, \phi_2, \phi_3, \psi_1, \psi_2, \psi_3 \in E_0$,

$$\alpha((\phi_1(c), \psi_1(c)), (\phi_2(c), \psi_2(c))) \geq 1 \text{ and } \alpha((\phi_2(c), \psi_2(c)), (\phi_3(c), \psi_3(c))) \geq 1$$
$$\implies \alpha((\phi_1(c), \psi_1(c)), (\phi_3(c), \psi_3(c))) \geq 1.$$

Lemma 2.22. *Let $c \in I$. Let F be a triangular α_c-admissible mapping. We define two sequences $\{\phi_n\}$ and $\{\psi_n\}$ by $F(\phi_n, \psi_n) = \phi_{n+1}(c)$ and $F(\psi_n, \phi_n) = \psi_{n+1}(c)$ for $n \in \mathbb{N} \cup \{0\}$, where $\phi_0, \psi_0 \in R_c$ are such that*

(i) $\alpha((F(\phi_0, \psi_0), F(\psi_0, \phi_0)), (\phi_0(c), \psi_0(c))) \geq 1$ and

(ii) $\alpha((\psi_0(c), \phi_0(c)), (F(\psi_0, \phi_0), F(\phi_0, \psi_0))) \geq 1$.

Then for any $m, n \in \mathbb{N}$ with $m < n$, we have

$\alpha((\phi_n(c), \psi_n(c)), (\phi_m(c), \psi_m(c))) \geq 1$ *and* $\alpha((\psi_m(c), \phi_m(c)), (\psi_n(c), \phi_n(c))) \geq 1$.

Proof. Runs as that of Lemma 12 in [8]. □

3. Main results

We denote

$\Psi = \{\psi \mid \psi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, nondecreasing and $\psi(t) = 0$ if and only if $t = 0\}$.

Theorem 3.1. *Let $c \in I$. Let $(E, \preceq)$ be a partially ordered Banach space. Let $F : E_0 \times E_0 \to E$ be a continuous function having the mixed c-monotone property. Let $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be a C-class function. Assume that*

(i) *there exist $\phi, \psi \in \Psi$ such that for any $\phi_1, \phi_2, \psi_1, \psi_2 \in E_0$ with*

$\phi_1(c) \preceq \phi_2(c),\ \psi_2(c) \preceq \psi_1(c)$ *and* $\alpha((\phi_2(c), \psi_2(c)), (\phi_1(c), \psi_1(c))) \geq 1 \implies$

$$\begin{aligned}\psi(||F(\phi_2, \psi_2) - F(\phi_1, \psi_1)||_E) \leq G(\psi(max\{||\phi_2 - \phi_1||_{E_0}, ||\psi_2 - \psi_1||_{E_0}\}),\\ \phi(max\{||\phi_2 - \phi_1||_{E_0}, ||\psi_2 - \psi_1||_{E_0}\})),\end{aligned} \tag{3.1}$$

(ii) *F is a triangular α_c-admissible mapping,*

(iii) *R_c is algebraically closed with respect to the difference,*

(iv) *$F(E_0 \times E_0) \subseteq R_c(c)$, where $R_c(c) = \{\phi(c) \mid \phi \in R_c\}$,*

(v) *there exist $\phi_0, \psi_0 \in R_c$ such that*

$$\begin{cases} \phi_0(c) \preceq F(\phi_0, \psi_0),\ F(\psi_0, \phi_0) \preceq \psi_0(c), \\ \alpha((F(\phi_0, \psi_0), F(\psi_0, \phi_0)), (\phi_0(c), \psi_0(c))) \geq 1 \\ \qquad\qquad and \\ \alpha((\psi_0(c), \phi_0(c)), (F(\psi_0, \phi_0), F(\phi_0, \psi_0))) \geq 1. \end{cases} \tag{3.2}$$

Then F has a PPF dependent coupled fixed point in $R_c \times R_c$,
i.e., there exist $\phi^*, \psi^* \in R_c$ such that $F(\phi^*, \psi^*) = \phi^*(c)$ and $F(\psi^*, \phi^*) = \psi^*(c)$.
Moreover, if $(E, \preceq)$ is a totally ordred set then $\phi^* = \psi^*$.

Proof. From (v), we have $\phi_0, \psi_0 \in R_c$ such that $\phi_0(c) \preceq F(\phi_0, \psi_0),\ F(\psi_0, \phi_0) \preceq \psi_0(c)$.
From (iv), we have there exist $\phi_1, \psi_1 \in R_c$ such that $F(\phi_0, \psi_0) = \phi_1(c)$ and $F(\psi_0, \phi_0) = \psi_1(c)$.
Clearly $\phi_0(c) \preceq \phi_1(c)$ and $\psi_1(c) \preceq \psi_0(c)$.

From (3.2), we have

$$\left\{\begin{array}{c}\alpha((\phi_1(c),\psi_1(c)),(\phi_0(c),\psi_0(c)))\geq 1\\ \text{and}\\ \alpha((\psi_0(c),\phi_0(c)),(\psi_1(c),\phi_1(c)))\geq 1.\end{array}\right. \tag{3.3}$$

From (3.1), we have

$$\left\{\begin{array}{l}\psi(||F(\phi_1,\psi_1)-F(\phi_0,\psi_0)||_E)\leq G(\psi(max\{||\phi_1-\phi_0||_{E_0},||\psi_1-\psi_0||_{E_0}\}),\\ \qquad\qquad\qquad\qquad\qquad\qquad \phi(max\{||\phi_1-\phi_0||_{E_0},||\psi_1-\psi_0||_{E_0}\}))\\ \qquad\qquad\qquad\qquad\qquad \text{and}\\ \psi(||F(\psi_0,\phi_0)-F(\psi_1,\phi_1)||_E)\leq G(\psi(max\{||\phi_1-\phi_0||_{E_0},||\psi_1-\psi_0||_{E_0}\}),\\ \qquad\qquad\qquad\qquad\qquad\qquad \phi(max\{||\phi_1-\phi_0||_{E_0},||\psi_1-\psi_0||_{E_0}\})).\end{array}\right. \tag{3.4}$$

Since F has c-mixed monotone property, we have

$$\left\{\begin{array}{c}\phi_1(c)=F(\phi_0,\psi_0)\preceq F(\phi_1,\psi_0)\preceq F(\phi_1,\psi_1)\\ \text{and}\\ F(\psi_1,\phi_1)\preceq F(\psi_0,\phi_1)\preceq F(\psi_0,\phi_0)=\psi_1(c).\end{array}\right. \tag{3.5}$$

Again from (iv), there exist $\phi_2,\psi_2\in R_c$ such that $F(\phi_1,\psi_1)=\phi_2(c)$ and $F(\psi_1,\phi_1)=\psi_2(c)$.

Clearly $\phi_1(c)\preceq\phi_2(c)$ and $\psi_2(c)\preceq\psi_1(c)$.

Since F is α_c-admissible mapping, from (3.3), we have

$$\left\{\begin{array}{c}\alpha((F(\phi_1,\psi_1),F(\psi_1,\phi_1)),(F(\phi_0,\psi_0),F(\psi_0,\phi_0)))\geq 1\\ \text{and}\\ \alpha((F(\psi_0,\phi_0),F(\phi_0,\psi_0)),(F(\psi_1,\phi_1),F(\phi_1,\psi_1)))\geq 1.\end{array}\right. \tag{3.6}$$

Therefore

$$\alpha((\phi_2(c),\psi_2(c)),(\phi_1(c),\psi_1(c)))\geq 1 \quad \text{and} \quad \alpha((\psi_1(c),\phi_1(c)),(\psi_2(c),\phi_2(c)))\geq 1.$$

Now, from (3.1), we have

$$\left\{\begin{array}{l}\psi(||F(\phi_2,\psi_2)-F(\phi_1,\psi_1)||_E)\leq G(\psi(max\{||\phi_2-\phi_1||_{E_0},||\psi_2-\psi_1||_{E_0}\}),\\ \qquad\qquad\qquad\qquad\qquad\qquad \phi(max\{||\phi_2-\phi_1||_{E_0},||\psi_2-\psi_1||_{E_0}\}))\\ \qquad\qquad\qquad\qquad\qquad \text{and}\\ \psi(||F(\psi_1,\phi_1)-F(\psi_2,\phi_2)||_E)\leq G(\psi(max\{||\phi_2-\phi_1||_{E_0},||\psi_2-\psi_1||_{E_0}\}),\\ \qquad\qquad\qquad\qquad\qquad\qquad \phi(max\{||\phi_2-\phi_1||_{E_0},||\psi_2-\psi_1||_{E_0}\})).\end{array}\right. \tag{3.7}$$

On continuing this process, we get a sequence $\{\phi_n\}$ and $\{\psi_n\}$ in R_c such that the following conditions are hold.

$$
(3.8) \qquad \begin{cases}
\text{(i)}\ \ \phi_0(c) \preceq \phi_1(c) \preceq \phi_2(c) \preceq .... \ \text{ and} \\
\qquad\qquad ..... \preceq \psi_3(c) \preceq \psi_2(c) \preceq \psi_1 \preceq \psi_0(c), \\
\text{(ii)}\ \ F(\phi_n,\psi_n) = \phi_{n+1}(c) \quad \text{ and } \quad F(\psi_n,\phi_n) = \psi_{n+1}(c), \\
\text{(iii)}\ \ \psi(||\phi_{n+2}-\phi_{n+1}||_{E_0}) = \psi(||\phi_{n+2}(c)-\phi_{n+1}(c)||_E) \\
\qquad\qquad = \psi(||F(\phi_{n+1},\psi_{n+1}) - F(\phi_n,\psi_n)||_E) \\
\qquad\qquad \leq G(\psi(max\{||\phi_{n+1}-\phi_n||_{E_0}, ||\psi_{n+1}-\psi_n||_{E_0}\}), \\
\qquad\qquad\qquad \phi(max\{||\phi_{n+1}-\phi_n||_{E_0}, ||\psi_{n+1}-\psi_n||_{E_0}\})) \\
\qquad\qquad \text{and} \\
\text{(iv)}\ \ \psi(||\psi_{n+1}-\psi_{n+2}||_{E_0}) = \psi(||\psi_{n+1}(c)-\psi_{n+2}(c)||_E) \\
\qquad\qquad = \psi(||F(\psi_n,\phi_n) - F(\psi_{n+1},\phi_{n+1})||_E) \\
\qquad\qquad \leq G(\psi(max\{||\phi_{n+1}-\phi_n||_{E_0}, ||\psi_{n+1}-\psi_n||_{E_0}\}), \\
\qquad\qquad\qquad \phi(max\{||\phi_{n+1}-\phi_n||_{E_0}, ||\psi_{n+1}-\psi_n||_{E_0}\})).
\end{cases}
$$

By Lemma 2.22, for any $m,n \in \mathbb{N}\cup\{0\}$ with $m<n$, we have

$$
(3.9) \qquad \begin{cases}
\alpha((\phi_n(c),\psi_n(c)),(\phi_m(c),\psi_m(c))) \geq 1 \\
\qquad\qquad \text{and} \\
\alpha((\psi_m(c),\phi_m(c)),(\psi_n(c),\phi_n(c))) \geq 1.
\end{cases}
$$

If $\phi_{n+1} = \phi_n$ and $\psi_{n+1} = \psi_n$ for some $n \in \mathbb{N}\cup\{0\}$, then the result is trivial.

Suppose that either $\phi_{n+1} \neq \phi_n$ or $\psi_{n+1} \neq \psi_n$ for all $n \in \mathbb{N}\cup\{0\}$.

From (3.8), we have

$$
(3.10) \qquad \begin{cases}
\psi(max\{||\phi_{n+2}-\phi_{n+1}||_{E_0}, ||\psi_{n+2}-\psi_{n+1}||_{E_0}\}) \\
\qquad = max\{\psi(||\phi_{n+2}-\phi_{n+1}||_{E_0}), \psi(||\psi_{n+2}-\psi_{n+1}||_{E_0})\} \\
\qquad \leq G(\psi(max\{||\phi_{n+1}-\phi_n||_{E_0}, ||\psi_{n+1}-\psi_n||_{E_0}\}), \\
\qquad\qquad \phi(max\{||\phi_{n+1}-\phi_n||_{E_0}, ||\psi_{n+1}-\psi_n||_{E_0}\})) \\
\qquad \leq \psi(max\{||\phi_{n+1}-\phi_n||_{E_0}, ||\psi_{n+1}-\psi_n||_{E_0}\}).
\end{cases}
$$

Let $d_n = max\{||\phi_{n+1}-\phi_n||_{E_0}, ||\psi_{n+1}-\psi_n||_{E_0}\}$.

Then the sequence $\{d_n\}$ is a decreasing sequence in $\mathbb{R}^+$ and hence it is convergent.

Let $\lim_{n\to\infty} d_n = d$.

On taking limits as $n \to \infty$ to (3.10), we get

$\psi(d) \leq G(\psi(d), \phi(d)) \leq \psi(d)$ and which implies that $G(\psi(d), \phi(d)) = \psi(d)$.

Therefore either $\psi(d) = 0$ or $\phi(d) = 0$ and hence $d = 0$, that is

$$\lim_{n\to\infty} max\{||\phi_{n+1} - \phi_n||_{E_0}, ||\psi_{n+1} - \psi_n||_{E_0}\} = 0. \tag{3.11}$$

We now show that $\{\phi_n\}$ and $\{\psi_n\}$ are Cauchy sequences.

Suppose that at least one of the sequences $\{\phi_n\}$ or $\{\psi_n\}$ is not a Cauchy sequence.

Then there exists $\varepsilon > 0$ and two subsequences $\{\phi_{n_k}\}, \{\phi_{m_k}\}$ of $\{\phi_n\}$ and $\{\psi_{n_k}\}, \{\psi_{m_k}\}$ of $\{\psi_n\}$ with $m_k > n_k > k$ such that $max\{||\phi_{n_k} - \phi_{m_k}||_{E_0}, ||\psi_{n_k} - \psi_{m_k}||_{E_0}\} \geq \varepsilon$ and

$max\{||\phi_{n_k} - \phi_{m_k-1}||_{E_0}, ||\psi_{n_k} - \psi_{m_k-1}||_{E_0}\} < \varepsilon$.

By applying the triangular inequality, we have

$$\begin{aligned} ||\phi_{n_k} - \phi_{m_k}||_{E_0} &\leq ||\phi_{n_k} - \phi_{m_k-1}||_{E_0} + ||\phi_{m_k-1} - \phi_{m_k}||_{E_0} \\ &< \varepsilon + ||\phi_{m_k-1} - \phi_{m_k}||_{E_0} \end{aligned}$$

and

$$\begin{aligned} ||\psi_{n_k} - \psi_{m_k}||_{E_0} &\leq ||\psi_{n_k} - \psi_{m_k-1}||_{E_0} + ||\psi_{m_k-1} - \psi_{m_k}||_{E_0} \\ &< \varepsilon + ||\psi_{m_k-1} - \psi_{m_k}||_{E_0}. \end{aligned}$$

Therefore

$$\varepsilon \leq max\{||\phi_{n_k} - \phi_{m_k}||_{E_0}, ||\psi_{n_k} - \psi_{m_k}||_{E_0}\} < \varepsilon + max\{||\phi_{m_k-1} - \phi_{m_k}||_{E_0}, ||\psi_{m_k-1} - \psi_{m_k}||_{E_0}\}.$$

On applying limits as $k \to \infty$, we get

$\varepsilon \leq \lim_{k\to\infty} max\{||\phi_{n_k} - \phi_{m_k}||_{E_0}, ||\psi_{n_k} - \psi_{m_k}||_{E_0}\} \leq \varepsilon$ and hence

$$\lim_{k\to\infty} max\{||\phi_{n_k} - \phi_{m_k}||_{E_0}, ||\psi_{n_k} - \psi_{m_k}||_{E_0}\} = \varepsilon. \tag{3.12}$$

By the triangular inequality, we have

$$\begin{aligned} ||\phi_{n_k} - \phi_{m_k+1}||_{E_0} &\leq ||\phi_{n_k} - \phi_{m_k}||_{E_0} + ||\phi_{m_k} - \phi_{m_k+1}||_{E_0} \\ &\leq max\{||\phi_{n_k} - \phi_{m_k}||_{E_0}, ||\psi_{n_k} - \psi_{m_k}||_{E_0}\} + max\{||\phi_{m_k} - \phi_{m_k+1}||_{E_0}, ||\psi_{m_k} - \psi_{m_k+1}||_{E_0}\}. \end{aligned}$$

In a similar way, we obtain that

$$||\psi_{n_k} - \psi_{m_k+1}||_{E_0} \leq max\{||\phi_{n_k} - \phi_{m_k}||_{E_0}, ||\psi_{n_k} - \psi_{m_k}||_{E_0}\} + max\{||\phi_{m_k} - \phi_{m_k+1}||_{E_0}, ||\psi_{m_k} - \psi_{m_k+1}||_{E_0}\}.$$

Therefore

$max\{||\phi_{n_k} - \phi_{m_k+1}||_{E_0}, ||\psi_{n_k} - \psi_{m_k+1}||_{E_0}\} \leq max\{||\phi_{n_k} - \phi_{m_k}||_{E_0}, ||\psi_{n_k} - \psi_{m_k}||_{E_0}\}$

$$+\,max\{||\phi_{m_k}-\phi_{m_k+1}||_{E_0},||\psi_{m_k}-\psi_{m_k+1}||_{E_0}\}.$$

On applying limit superior as $k\to\infty$, we get

$$\limsup_{k\to\infty} max\{||\phi_{n_k}-\phi_{m_k+1}||_{E_0},||\psi_{n_k}-\psi_{m_k+1}||_{E_0}\}\leq\varepsilon. \tag{3.13}$$

By the trianglular inequality, we have

$$\begin{aligned}||\phi_{n_k}-\phi_{m_k}||_{E_0}&\leq||\phi_{n_k}-\phi_{m_k+1}||_{E_0}+||\phi_{m_k+1}-\phi_{m_k}||_{E_0}\\&\leq max\{||\phi_{n_k}-\phi_{m_k+1}||_{E_0},||\psi_{n_k}-\psi_{m_k+1}||_{E_0}\}\\&\qquad+max\{||\phi_{m_k+1}-\phi_{m_k}||_{E_0},||\psi_{m_k+1}-\psi_{m_k}||_{E_0}\}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}||\psi_{n_k}-\psi_{m_k}||_{E_0}\leq\,&max\{||\phi_{n_k}-\phi_{m_k+1}||_{E_0},||\psi_{n_k}-\psi_{m_k+1}||_{E_0}\}\\&+max\{||\phi_{m_k+1}-\phi_{m_k}||_{E_0},||\psi_{m_k+1}-\psi_{m_k}||_{E_0}\}.\end{aligned}$$

Therefore

$$\begin{aligned}max\{||\phi_{n_k}-\phi_{m_k}||_{E_0},||\psi_{n_k}-\psi_{m_k}||_{E_0}\}\leq\,&max\{||\phi_{n_k}-\phi_{m_k+1}||_{E_0},||\psi_{n_k}-\psi_{m_k+1}||_{E_0}\}\\&+max\{||\phi_{m_k+1}-\phi_{m_k}||_{E_0},||\psi_{m_k+1}-\psi_{m_k}||_{E_0}\}.\end{aligned}$$

On applying limit inferior as $k\to\infty$, by Proposition 1.1, we get

$$\varepsilon\leq\liminf_{k\to\infty} max\{||\phi_{n_k}-\phi_{m_k+1}||_{E_0},||\psi_{n_k}-\psi_{m_k+1}||_{E_0}\}. \tag{3.14}$$

From (3.13) and (3.14), we have

$$\lim_{k\to\infty} max\{||\phi_{n_k}-\phi_{m_k+1}||_{E_0},||\psi_{n_k}-\psi_{m_k+1}||_{E_0}\}=\varepsilon. \tag{3.15}$$

Similarly, we have

$$\lim_{k\to\infty} max\{||\phi_{m_k}-\phi_{n_k+1}||_{E_0},||\psi_{m_k}-\psi_{n_k+1}||_{E_0}\}=\varepsilon. \tag{3.16}$$

We now show that $\lim\limits_{k\to\infty} max\{||\phi_{m_k+1}-\phi_{n_k+1}||_{E_0},||\psi_{m_k+1}-\psi_{n_k+1}||_{E_0}\}=\varepsilon$.
By the triangular inequality, we have

$$\begin{aligned}||\phi_{m_k+1}-\phi_{n_k+1}||_{E_0}&\leq||\phi_{m_k+1}-\phi_{n_k}||_{E_0}+||\phi_{n_k}-\phi_{n_k+1}||_{E_0}\\&\leq max\{||\phi_{m_k+1}-\phi_{n_k}||_{E_0},||\psi_{m_k+1}-\psi_{n_k}||_{E_0}\}\\&\qquad+max\{||\phi_{n_k}-\phi_{n_k+1}||_{E_0},||\psi_{n_k}-\psi_{n_k+1}||_{E_0}\}\end{aligned}$$

and

$$||\psi_{m_k+1}-\psi_{n_k+1}||_{E_0}\leq max\{||\phi_{m_k+1}-\phi_{n_k}||_{E_0},||\psi_{m_k+1}-\psi_{n_k}||_{E_0}\}$$

$$+max\{||\phi_{n_k}-\phi_{n_k+1}||_{E_0},||\psi_{n_k}-\psi_{n_k+1}||_{E_0}\}.$$

Therefore

$$max\{||\phi_{m_k+1}-\phi_{n_k+1}||_{E_0},||\psi_{m_k+1}-\psi_{n_k+1}||_{E_0}\}\leq max\{||\phi_{m_k+1}-\phi_{n_k}||_{E_0},$$
$$||\psi_{m_k+1}-\psi_{n_k}||_{E_0}\}+max\{||\phi_{n_k}-\phi_{n_k+1}||_{E_0},||\psi_{n_k}-\psi_{n_k+1}||_{E_0}\}.$$

On applying limit superior as $k\to\infty$ on both sides, we get

$$\limsup_{k\to\infty} max\{||\phi_{m_k+1}-\phi_{n_k+1}||_{E_0},||\psi_{m_k+1}-\psi_{n_k+1}||_{E_0}\}\leq\varepsilon. \tag{3.17}$$

By the triangular inequality, we have

$$||\phi_{m_k}-\phi_{n_k+1}||_{E_0}\leq||\phi_{m_k}-\phi_{m_k+1}||_{E_0}+||\phi_{m_k+1}-\phi_{n_k+1}||_{E_0}$$
$$\leq max\{||\phi_{m_k}-\phi_{m_k+1}||_{E_0},||\psi_{m_k}-\psi_{m_k+1}||_{E_0}\}$$
$$+max\{||\phi_{m_k+1}-\phi_{n_k+1}||_{E_0},||\psi_{m_k+1}-\psi_{n_k+1}||_{E_0}\}$$

and

$$||\psi_{m_k}-\psi_{n_k+1}||_{E_0}\leq max\{||\phi_{m_k}-\phi_{m_k+1}||_{E_0},||\psi_{m_k}-\psi_{m_k+1}||_{E_0}\}$$
$$+max\{||\phi_{m_k+1}-\phi_{n_k+1}||_{E_0},||\psi_{m_k+1}-\psi_{n_k+1}||_{E_0}\}.$$

Therefore, from (3.16), we have

$$max\{||\phi_{m_k}-\phi_{n_k+1}||_{E_0},||\psi_{m_k}-\psi_{n_k+1}||_{E_0}\}\leq max\{||\phi_{m_k}-\phi_{m_k+1}||_{E_0},||\psi_{m_k}-\psi_{m_k+1}||_{E_0}\}$$
$$+max\{||\phi_{m_k+1}-\phi_{n_k+1}||_{E_0},||\psi_{m_k+1}-\psi_{n_k+1}||_{E_0}\}.$$

On applying limit inferior as $k\to\infty$ on both sides, by Proposition 1.1, we get

$$\varepsilon\leq\liminf_{k\to\infty} max\{||\phi_{m_k+1}-\phi_{n_k+1}||_{E_0},||\psi_{m_k+1}-\psi_{n_k+1}||_{E_0}\}. \tag{3.18}$$

From (3.17) and (3.18), we have

$$\lim_{k\to\infty} max\{||\phi_{m_k+1}-\phi_{n_k+1}||_{E_0},||\psi_{m_k+1}-\psi_{n_k+1}||_{E_0}\}=\varepsilon. \tag{3.19}$$

Since $n_k<m_k$, from (3.8) we have

$\phi_{n_k}(c)\preceq\phi_{m_k}(c)$ and $\psi_{m_k}(c)\preceq\psi_{n_k}(c)$.

From (3.9), we have

$$\begin{cases}\alpha((\phi_{m_k}(c),\psi_{m_k}(c)),(\phi_{n_k}(c),\psi_{n_k}(c)))\geq 1\\ \qquad\qquad\text{and}\\ \alpha((\psi_{n_k}(c),\phi_{n_k}(c)),(\psi_{m_k}(c),\phi_{m_k}(c)))\geq 1.\end{cases} \tag{3.20}$$

From (3.1) and (3.8), we have

$$\begin{aligned}\psi(||\phi_{m_k+1}-\phi_{n_k+1}||_{E_0}) &= \psi(||F(\phi_{m_k},\psi_{m_k})-F(\phi_{n_k},\psi_{n_k})||_E)\\ &\leq G(\psi(max\{||\phi_{m_k}-\phi_{n_k}||_{E_0},||\psi_{m_k}-\psi_{n_k}||_{E_0}\}),\\ &\qquad \phi(max\{||\phi_{m_k}-\phi_{n_k}||_{E_0},||\psi_{m_k}-\psi_{n_k}||_{E_0}\}))\end{aligned}$$

and

$$\begin{aligned}\psi(||\psi_{n_k+1}-\psi_{m_k+1}||_{E_0}) &= \psi(||F(\psi_{n_k},\psi_{n_k})-F(\psi_{m_k},\psi_{m_k})||_E)\\ &\leq G(\psi(max\{||\phi_{m_k}-\phi_{n_k}||_{E_0},||\psi_{m_k}-\psi_{n_k}||_{E_0}\}),\\ &\qquad \phi(max\{||\phi_{m_k}-\phi_{n_k}||_{E_0},||\psi_{m_k}-\psi_{n_k}||_{E_0}\})).\end{aligned}$$

Therefore

$$\begin{aligned}&\psi(max\{||\phi_{m_k+1}-\phi_{n_k+1}||_{E_0},||\psi_{m_k+1}-\psi_{n_k+1}||_{E_0}\})\\ &\qquad = max\{\psi(||\phi_{m_k+1}-\phi_{n_k+1}||_{E_0}),\psi(||\psi_{m_k+1}-\psi_{n_k+1}||_{E_0})\}\\ &\qquad \leq G(\psi(max\{||\phi_{m_k}-\phi_{n_k}||_{E_0},||\psi_{m_k}-\psi_{n_k}||_{E_0}\}),\\ &\qquad\qquad \phi(max\{||\phi_{m_k}-\phi_{n_k}||_{E_0},||\psi_{m_k}-\psi_{n_k}||_{E_0}\}))\\ &\qquad \leq \psi(max\{||\phi_{m_k}-\phi_{n_k}||_{E_0},||\psi_{m_k}-\psi_{n_k}||_{E_0}\}).\end{aligned}$$

On applying limits as $k\to\infty$, we get $\psi(\varepsilon)\leq G(\psi(\varepsilon),\phi(\varepsilon))\leq\psi(\varepsilon)$ and hence $G(\psi(\varepsilon),\phi(\varepsilon))=\psi(\varepsilon)$. Therefore either $\psi(\varepsilon)=0$ or $\phi(\varepsilon)=0$ and hence $\varepsilon=0$, a contradiction.

Therefore the sequences $\{\phi_n\}$ and $\{\psi_n\}$ are Cauchy sequences in $R_c\subseteq E_0$.

Since E_0 is complete, there exists ϕ^*,ψ^* in E_0 such that $\phi_n\to\phi^*$ and $\psi_n\to\psi^*$ as $n\to\infty$.

Since R_c is topologically closed, we have $\phi^*,\psi^*\in R_c$.

From (3.8), we have $F(\phi_n,\psi_n)=\phi_{n+1}(c)$ and $F(\psi_n,\phi_n)=\psi_{n+1}(c)$.

On applying limits as $n\to\infty$ on both sides, we get

$F(\phi^*,\psi^*)=\phi^*(c)$ and $F(\psi^*,\phi^*)=\psi^*(c)$.

Therefore (ϕ^*,ψ^*) is a PPF dependent coupled fixed point of F.

We now suppose that $(E,\preceq)$ is a totally ordered set.

We define $\alpha: E^2\times E^2\to\mathbb{R}^+$ by

$$\alpha((a,b),(c,d))=\begin{cases}1 & \text{if } c\preceq a \text{ and } b\preceq d\\ 0 & \text{otherwise}\end{cases}$$

for any $a,b,c,d\in E$.

<u>Case (i):</u> Suppose that $\phi^*(c)\preceq\psi^*(c)$.

From (3.1), we have

$$\begin{aligned}\psi(||\psi^*-\phi^*||_{E_0}) = \psi(||\psi^*(c)-\phi^*(c)||_E) &= \psi(||F(\psi^*,\phi^*)-F(\phi^*,\psi^*)||_E)\\ &\le G(\psi(||\psi^*-\phi^*||_{E_0}),\phi(||\phi^*-\psi^*||_{E_0}))\\ &\le \psi(||\psi^*-\phi^*||_{E_0}).\end{aligned}$$

Therefore $G(\psi(||\psi^*-\phi^*||_{E_0}),\phi(||\phi^*-\psi^*||_{E_0})) = \psi(||\psi^*-\phi^*||_{E_0})$ and hence $\phi^*=\psi^*$.

Case (ii): Suppose that $\psi^*(c)\preceq\phi^*(c)$.

From (3.1), we have

$$\begin{aligned}\psi(||\phi^*-\psi^*||_{E_0}) = \psi(||\phi^*(c)-\psi^*(c)||_E) &= \psi(||F(\phi^*,\psi^*)-F(\psi^*,\phi^*)||_E)\\ &\le G(\psi(||\phi^*-\psi^*||_{E_0}),\phi(||\phi^*-\psi^*||_{E_0}))\\ &\le \psi(||\phi^*-\psi^*||_{E_0}).\end{aligned}$$

Therefore $G(\psi(||\phi^*-\psi^*||_{E_0}),\phi(||\psi^*-\phi^*||_{E_0})) = \psi(||\phi^*-\psi^*||_{E_0})$ and hence $\psi^*=\phi^*$. □

Theorem 3.2. *Let $c\in I$. Let $(E,\preceq)$ be a partially ordered Banach space. Let $F:E_0\times E_0\to E$ be a function having the mixed c-monotone property. Let $G:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ be a C-class function. Assume that*

(i) *the conditions* (i), (ii), (iii), (iv) *and* (v) *of Theorem 3.1,*

(ii) *if $\{\phi_n\}$ and $\{\psi_n\}$ are sequences in E_0 such that $\phi_n\to\phi'$ and $\psi_n\to\psi'$ as $n\to\infty$ then*

a) $\phi'(c)\preceq\phi_n(c)$ and $\psi_n(c)\preceq\psi'(c)$, and

b) for every $n\in\mathbb{N}$,

$$\alpha((\phi_n(c),\psi_n(c)),(\phi_{n-1}(c),\psi_{n-1}(c)))\ge 1 \implies \alpha((\phi_n(c),\psi_n(c)),(\phi'(c),\psi'(c)))\ge 1$$

and

$$\alpha((\psi_{n-1}(c),\phi_{n-1}(c)),(\psi_n(c),\phi_n(c)))\ge 1 \implies \alpha((\psi'(c),\phi'(c)),(\psi_n(c),\phi_n(c)))\ge 1.$$

Then F has a PPF dependent coupled fixed point in $R_c\times R_c$,

i.e., there exist $\phi^*,\psi^*\in R_c$ such that $F(\phi^*,\psi^*)=\phi^*(c)$ and $F(\psi^*,\phi^*)=\psi^*(c)$.

Proof. As in the proof of Theorem 3.1, we get two sequences $\{\phi_n\}$ and $\{\psi_n\}$ such that $\phi_n\to\phi^*$ and $\psi_n\to\psi^*$ as $n\to\infty$, where $\phi^*,\psi^*\in R_c$.

By Lemma 2.22, we have for every $n \in \mathbb{N}$

$$\tag{3.21} \begin{cases} \alpha((\phi_n(c),\psi_n(c)),(\phi_{n-1}(c),\psi_{n-1}(c))) \geq 1 \\ \qquad\qquad\qquad\text{and} \\ \alpha((\psi_{n-1}(c),\phi_{n-1}(c)),(\psi_n(c),\phi_n(c))) \geq 1. \end{cases}$$

From the assumption (ii), we have

$$\tag{3.22} \begin{aligned} &\alpha((\phi_n(c),\psi_n(c)),(\phi^*(c),\psi^*(c))) \geq 1, \alpha((\psi^*(c),\phi^*(c)),(\psi_n(c),\phi_n(c))) \geq 1, \\ &\phi^*(c) \preceq \phi_n(c) \ \text{ and } \ \psi_n(c) \preceq \psi^*(c). \end{aligned}$$

We consider

$$\begin{aligned} \psi(||\phi_{n+1}(c) - F(\phi^*,\psi^*)||_E) &= \psi(||F(\phi_n,\psi_n) - F(\phi^*,\psi^*)||_E) \\ &\leq G(\psi(max\{||\phi_n - \phi^*||_{E_0}, ||\psi_n - \psi^*||_{E_0}\}), \\ &\qquad \phi(max\{||\phi_n - \phi^*||_{E_0}, ||\psi_n - \psi^*||_{E_0}\})) \\ &\leq \psi(max\{||\phi_n - \phi^*||_{E_0}, ||\psi_n - \psi^*||_{E_0}\}). \end{aligned}$$

On applying limits as $n \to \infty$, we get

$\psi(||\phi^*(c) - F(\phi^*,\psi^*)||_E) \leq 0$ and hence $F(\phi^*,\psi^*) = \phi^*(c)$.

Similarly,

$$\begin{aligned} \psi(||F(\psi^*,\phi^*) - \psi_{n+1}(c)||_E) &= \psi(||F(\psi^*,\phi^*) - F(\psi_n,\phi_n)||_E) \\ &\leq G(\psi(max\{||\phi_n - \phi^*||_{E_0}, ||\psi_n - \psi^*||_{E_0}\}), \\ &\qquad \phi(max\{||\phi_n - \phi^*||_{E_0}, ||\psi_n - \psi^*||_{E_0}\})) \\ &\leq \psi(max\{||\phi_n - \phi^*||_{E_0}, ||\psi_n - \psi^*||_{E_0}\}). \end{aligned}$$

On applying limits as $n \to \infty$, we get

$\psi(||F(\psi^*,\phi^*) - \psi^*(c)||_E) \leq 0$ and hence $F(\psi^*,\phi^*) = \psi^*(c)$.

Therefore (ϕ^*,ψ^*) is a PPF dependent coupled fixed point of F. □

Corollary 3.3. *Let $c \in I$. Let $(E, \preceq)$ be a partially ordered Banach space. Let $F : E_0 \times E_0 \to E$ be a continuous function having the mixed c-monotone property.*

Assume that

(i) *there exist $\phi, \psi \in \Psi$ such that for any $\phi_1, \phi_2, \psi_1, \psi_2 \in E_0$ with*

$\phi_1(c) \preceq \phi_2(c), \psi_2(c) \preceq \psi_1(c)$ *and* $\alpha((\phi_2(c), \psi_2(c)), (\phi_1(c), \psi_1(c))) \geq 1 \implies$

$$\begin{cases} \psi(||F(\phi_2, \psi_2) - F(\phi_1, \psi_1)||_E) \leq \psi(max\{||\phi_2 - \phi_1||_{E_0}, ||\psi_2 - \psi_1||_{E_0}\}) \\ \qquad\qquad -\phi(max\{||\phi_2 - \phi_1||_{E_0}, ||\psi_2 - \psi_1||_{E_0}\}), \end{cases} \tag{3.23}$$

(ii) *F is a triangular α_c-admissible mapping,*

(iii) *R_c is algebraically closed with respect to the difference,*

(iv) *$F(E_0 \times E_0) \subseteq R_c(c)$, where $R_c(c) = \{\phi(c) \mid \phi \in R_c\}$,*

(v) *there exist $\phi_0, \psi_0 \in R_c$ such that*

$$\begin{cases} \phi_0(c) \preceq F(\phi_0, \psi_0),\ F(\psi_0, \phi_0) \preceq \psi_0(c), \\ \alpha((F(\phi_0, \psi_0), F(\psi_0, \phi_0)), (\phi_0(c), \psi_0(c))) \geq 1 \\ \qquad\qquad \text{and} \\ \alpha((\psi_0(c), \phi_0(c)), (F(\psi_0, \phi_0), F(\phi_0, \psi_0))) \geq 1. \end{cases} \tag{3.24}$$

Then F has a PPF dependent coupled fixed point in $R_c \times R_c$,

i.e., there exist $\phi^*, \psi^* \in R_c$ such that $F(\phi^*, \psi^*) = \phi^*(c)$ and *$F(\psi^*, \phi^*) = \psi^*(c)$.*

Proof. Follows by choosing $G(s,t) = s - t$ in Theorem 3.1. □

Corollary 3.4. *Let $c \in I$. Let $(E, \preceq)$ be a partially ordered Banach space. Let $F : E_0 \times E_0 \to E$ be a function having the mixed c-monotone property.*

Assume that

(i) *the conditions* (i), (ii), (iii), (iv) *and* (v) *of Corollary 3.3,*

(ii) *if $\{\phi_n\}$ and $\{\psi_n\}$ are sequences in E_0 such that $\phi_n \to \phi'$ and $\psi_n \to \psi'$ as $n \to \infty$ then*

a) $\phi'(c) \preceq \phi_n(c)$ and $\psi_n(c) \preceq \psi'(c)$, and

b) for every $n \in \mathbb{N}$,

$\alpha((\phi_n(c), \psi_n(c)), (\phi_{n-1}(c), \psi_{n-1}(c))) \geq 1 \implies \alpha((\phi_n(c), \psi_n(c)), (\phi'(c), \psi'(c))) \geq 1$

and

$\alpha((\psi_{n-1}(c), \phi_{n-1}(c)), (\psi_n(c), \phi_n(c))) \geq 1 \implies \alpha((\psi'(c), \phi'(c)), (\psi_n(c), \phi_n(c))) \geq 1.$

Then F has a PPF dependent coupled fixed point in $R_c \times R_c$,

i.e., there exist $\phi^*, \psi^* \in R_c$ such that $F(\phi^*, \psi^*) = \phi^*(c)$ and $F(\psi^*, \phi^*) = \psi^*(c)$.

Proof. Follows by choosing $G(s,t) = s - t$ in Theorem 3.2. □

Corollary 3.5. *Let $c \in I$. Let $(E, \preceq)$ be a partially ordered Banach space. Let $F : E_0 \times E_0 \to E$ be a continuous function having the mixed c-monotone property.*

Assume that

(i) *there exist $k \in [0,1)$ such that for any $\phi_1, \phi_2, \psi_1, \psi_2 \in E_0$ with*

$\phi_1(c) \preceq \phi_2(c),\ \psi_2(c) \preceq \psi_1(c)$ *and* $\alpha((\phi_2(c), \psi_2(c)), (\phi_1(c), \psi_1(c))) \geq 1 \implies$

$$|| \, F(\phi_2, \psi_2) - F(\phi_1, \psi_1) \, ||_E \leq \frac{k}{2} \, [||\phi_2 - \phi_1||_{E_0} + ||\psi_2 - \psi_1||_{E_0}], \tag{3.25}$$

(ii) *F is a triangular α_c-admissible mapping,*

(iii) *R_c is algebraically closed with respect to the difference,*

(iv) *$F(E_0 \times E_0) \subseteq R_c(c)$, where $R_c(c) = \{\phi(c) \mid \phi \in R_c\}$,*

(v) *there exist $\phi_0, \psi_0 \in R_c$ such that*

$$\begin{cases} \phi_0(c) \preceq F(\phi_0, \psi_0),\ F(\psi_0, \phi_0) \preceq \psi_0(c), \\ \alpha((F(\phi_0, \psi_0), F(\psi_0, \phi_0)), (\phi_0(c), \psi_0(c))) \geq 1 \\ \qquad\qquad\qquad \text{and} \\ \alpha((\psi_0(c), \phi_0(c)), (F(\psi_0, \phi_0), F(\phi_0, \psi_0))) \geq 1. \end{cases} \tag{3.26}$$

Then F has a PPF dependent coupled fixed point in $R_c \times R_c$,

i.e., there exist $\phi^*, \psi^* \in R_c$ such that $F(\phi^*, \psi^*) = \phi^*(c)$ and $F(\psi^*, \phi^*) = \psi^*(c)$.

Proof. If F satisfies (3.25) , then F satisfies (3.23) with $\psi(t) = t$ and $\phi(t) = (1-k)t$ where $k \in [0,1)$. Then the result follows from Corollary 3.3. $\square$

Corollary 3.6. *Let $c \in I$. Let $(E, \preceq)$ be a partially ordered Banach space. Let $F : E_0 \times E_0 \to E$ be a function having the mixed c-monotone property.*

Assume that

(i) *the conditions* (i), (ii), (iii), (iv) *and* (v) *of Corollary 3.5,*

(ii) *if $\{\phi_n\}$ and $\{\psi_n\}$ are sequences in E_0 such that $\phi_n \to \phi'$ and $\psi_n \to \psi'$ as $n \to \infty$ then*

a) $\phi'(c) \preceq \phi_n(c)$ and $\psi_n(c) \preceq \psi'(c)$, and

b) for every $n \in \mathbb{N}$,

$\alpha((\phi_n(c), \psi_n(c)), (\phi_{n-1}(c), \psi_{n-1}(c))) \geq 1 \implies \alpha((\phi_n(c), \psi_n(c)), (\phi'(c), \psi'(c))) \geq 1$

and

$\alpha((\psi_{n-1}(c), \phi_{n-1}(c)), (\psi_n(c), \phi_n(c))) \geq 1 \implies \alpha((\psi'(c), \phi'(c)), (\psi_n(c), \phi_n(c))) \geq 1.$

Then F has a PPF dependent coupled fixed point in $R_c \times R_c$,
i.e., there exist $\phi^*, \psi^* \in R_c$ such that $F(\phi^*, \psi^*) = \phi^*(c)$ and $F(\psi^*, \phi^*) = \psi^*(c)$.

Proof. If F satisfies (3.25) , then F satisfies (3.23) with $\psi(t) = t$ and $\phi(t) = (1-k)t$ where $k \in [0,1)$. Then the result follows from Corollary 3.4. □

Corollary 3.7. *Let $c \in I$. Let $(E, \preceq)$ be a partially ordered Banach space. Let $F : E_0 \times E_0 \to E$ be a continuous function having the mixed c-monotone property.*
Assume that

(i) *there exist $k \in [0,1)$ such that for any $\phi_1, \phi_2, \psi_1, \psi_2 \in E_0$ with*
$\phi_1(c) \preceq \phi_2(c),\ \psi_2(c) \preceq \psi_1(c) \implies$

$$|| F(\phi_2, \psi_2) - F(\phi_1, \psi_1) ||_E \le \frac{k}{2} [||\phi_2 - \phi_1||_{E_0} + ||\psi_2 - \psi_1||_{E_0}], \tag{3.27}$$

(ii) *R_c is algebraically closed with respect to the difference,*
(iii) *$F(E_0 \times E_0) \subseteq R_c(c)$, where $R_c(c) = \{\phi(c) \mid \phi \in R_c\}$,*
(iv) *there exist $\phi_0, \psi_0 \in R_c$ such that $\phi_0(c) \preceq F(\phi_0, \psi_0),\ F(\psi_0, \phi_0) \preceq \psi_0(c)$.*

Then F has a PPF dependent coupled fixed point in $R_c \times R_c$,
i.e., there exist $\phi^*, \psi^* \in R_c$ such that $F(\phi^*, \psi^*) = \phi^*(c)$ and *$F(\psi^*, \phi^*) = \psi^*(c)$.*

Proof. We define $\alpha : E^2 \times E^2 \to \mathbb{R}^+$ by $\alpha(x,y) = 1$ for any $x, y \in E^2$.
Then from Corollary 3.5 the result follows. □

Corollary 3.8. *Let $c \in I$. Let $(E, \preceq)$ be a partially ordered Banach space. Let $F : E_0 \times E_0 \to E$ be a function having the mixed c-monotone property.*
Assume that

(i) *the conditions* (i), (ii), (iii) *and* (iv) *of Corollary 3.7,*
(ii) *if $\{\phi_n\}$ and $\{\psi_n\}$ are sequences in E_0 such that $\phi_n \to \phi'$ and $\psi_n \to \psi'$ as $n \to \infty$ then*
$\phi'(c) \preceq \phi_n(c)$ and $\psi_n(c) \preceq \psi'(c)$.

Then F has a PPF dependent coupled fixed point in $R_c \times R_c$,
i.e., there exist $\phi^*, \psi^* \in R_c$ such that $F(\phi^*, \psi^*) = \phi^*(c)$ and $F(\psi^*, \phi^*) = \psi^*(c)$.

Proof. We define $\alpha : E^2 \times E^2 \to \mathbb{R}^+$ by $\alpha(x,y) = 1$ for any $x, y \in E^2$.
Then from Corollary 3.6 the result follows. □

Example 3.9. Let $E=\mathbb{R},\ c=1\in I=[\frac{1}{2},2]\subseteq\mathbb{R},\ E_0=C(I,E)$.

We define a partial order relation $\preceq$ on E by $a\preceq b \iff a\leq b$ for any $a,b\in E$.

For any $n\in\mathbb{R}$, we define $\phi_n: I\to E$ by

$$\phi_n(x)=\begin{cases} nx^2 & \text{if } x\in[\frac{1}{2},1] \\ \frac{n}{x^2} & \text{if } x\in[1,2]. \end{cases}$$

Clearly $\phi_n\in E_0, ||\phi_n||_{E_0}=||\phi_n(c)||_E$ and hence $\phi_n\in R_c$ for any $n\in\mathbb{R}$.

Let $\mathrm{F}_0=\{\phi_n \mid n\in\mathbb{R}\}$.

Then F_0 is algebraically closed with respect to the difference and $\mathrm{F}_0\subseteq R_c$.

Therefore $\mathbb{R}=\{n\mid n\in\mathbb{R}\}=\{\phi_n(c)\mid n\in\mathbb{R}\}=\mathrm{F}_0(c)\subseteq R_c(c)$.

Clearly $R_c(c)\subseteq\mathbb{R}$ and hence $R_c(c)=\mathbb{R}$.

We define $F: E_0\times E_0\to E$ by $F(\eta,\mu)=\frac{\eta(c)-\mu(c)}{10}$ for any $\eta,\mu\in E_0$.

Clearly F is continuous and $F(E_0\times E_0)\subseteq\mathbb{R}=R_c(c)$.

We now show that F has mixed c-monotone property.

Let $\eta_1,\eta_2\in E_0$ be such that $\eta_1(c)\preceq\eta_2(c)$.

Then for any $\eta\in E_0$ we have

$$F(\eta_1,\eta)=\frac{\eta_1(c)-\eta(c)}{10}\preceq\frac{\eta_2(c)-\eta(c)}{10}=F(\eta_2,\eta).$$

Let $\psi_1,\psi_2\in E_0$ be such that $\psi_1(c)\preceq\psi_2(c)$.

Then for any $\eta\in E_0$ we have

$$F(\eta,\psi_2)=\frac{\eta(c)-\psi_2(c)}{10}\preceq\frac{\eta(c)-\psi_1(c)}{10}=F(\eta,\psi_1).$$

Therefore F has mixed c-monotone property.

We define $\alpha: E^2\times E^2\to\mathbb{R}^+$ by

$$\alpha((a,b),(c,d))=\begin{cases} 2 & \text{if } c\preceq a \text{ and } b\preceq d \\ \frac{3}{4} & \text{otherwise} \end{cases}$$

for any $a,b,c,d\in E$.

Clearly F is a triangular α_c-admissible mapping.

We define $G:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}$ and $\phi,\psi:\mathbb{R}^+\to\mathbb{R}^+$ by

$G(s,t)=s-t, \psi(t)=t$ and $\phi(t)=\frac{t}{10}$ for any $s,t\in\mathbb{R}^+$. Clearly $G\in\zeta$ and $\phi,\psi\in\Psi$.

Let $\eta_1,\eta_2,\psi_1,\psi_2\in E_0$ be such that $\eta_1(c)\preceq\eta_2(c)$ and $\psi_2(c)\preceq\psi_1(c)$.

Then, from the definition of α, we get $\alpha((\eta_2(c),\psi_2(c)),(\eta_1(c),\psi_1(c)))\geq 1$.

We consider

$$\begin{aligned}
\psi(||F(\eta_2,\psi_2)-F(\eta_1,\psi_1)||_E) &= ||F(\eta_2,\psi_2)-F(\eta_1,\psi_1)||_E\\
&= ||\tfrac{\eta_2(c)-\psi_2(c)}{10}-\tfrac{\eta_1(c)-\psi_1(c)}{10}||_E\\
&= \tfrac{1}{10}||[\eta_2(c)-\eta_1(c)]+[\psi_1(c)-\psi_2(c)]||_E\\
&\leq \tfrac{1}{10}[||\eta_2(c)-\eta_1(c)||_E+||\psi_1(c)-\psi_2(c)||_E]\\
&\leq \tfrac{1}{10}[||\eta_2-\eta_1||_{E_0}+||\psi_1-\psi_2||_{E_0}]\\
&\leq \tfrac{1}{5}max\{||\eta_2-\eta_1||_{E_0},||\psi_2-\psi_1||_{E_0}\}\\
&\leq max\{||\eta_2-\eta_1||_{E_0},||\psi_2-\psi_1||_{E_0}\}\\
&\qquad -\tfrac{1}{10}max\{||\eta_2-\eta_1||_{E_0},||\psi_2-\psi_1||_{E_0}\}\\
&= \psi(max\{||\eta_2-\eta_1||_{E_0},||\psi_2-\psi_1||_{E_0}\})\\
&\qquad -\phi(max\{||\eta_2-\eta_1||_{E_0},||\psi_2-\psi_1||_{E_0}\})\\
&= G(\psi(max\{||\eta_2-\eta_1||_{E_0},||\psi_2-\psi_1||_{E_0}\}),\\
&\qquad \phi(max\{||\eta_2-\eta_1||_{E_0},||\psi_2-\psi_1||_{E_0}\})).
\end{aligned}$$

Therefore the inequality (3.1) is satisfied.

We choose $\eta_0=\phi_{-2},\ \psi_0=\phi_{\frac{5}{4}}\in R_c$ so that

$$\eta_0(c)=\phi_{-2}(c)=-2\preceq\tfrac{-13}{40}=\tfrac{-8-5}{40}=\tfrac{-2-\frac{5}{4}}{10}=\frac{\phi_{-2}(c)-\phi_{\frac{5}{4}}(c)}{10}=\tfrac{\eta_0(c)-\psi_0(c)}{10}=F(\eta_0,\psi_0)$$

and

$$F(\psi_0,\eta_0)=F(\phi_{\frac{5}{4}},\phi_{-2})=\frac{\phi_{\frac{5}{4}}(c)-\phi_{-2}(c)}{10}=\frac{\frac{5}{4}+2}{10}=\tfrac{13}{40}\preceq\tfrac{5}{4}=\phi_{\frac{5}{4}}(c)=\psi_0(c).$$

Also $\alpha((F(\eta_0,\psi_0),F(\psi_0,\eta_0)),(\eta_0(c),\psi_0(c)))=\alpha((\frac{-13}{40},\frac{13}{40}),(-2,\frac{5}{4}))\geq 1$

and

$$\alpha((\psi_0(c),\eta_0(c)),(F(\psi_0,\eta_0),F(\eta_0,\psi_0)))=\alpha((\tfrac{5}{4},-2),(\tfrac{13}{40},\tfrac{-13}{40}))\geq 1.$$

Therefore condition (3.2) is satisfied.

Hence F satisfies all the hypotheses of Theorem 3.1 and (ϕ_0,ϕ_0) in $R_c\times R_c$ is a PPF dependent coupled fixed point of F.

Conflict of Interests

The authors declare that there is no conflict of interests.

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