Available online at http://jfpt.scik.org
J. Fixed Point Theory, 2019, 2019:8
ISSN: 2052-5338

GENERALIZATION OF DARBO'S FIXED POINT THEOREM VIA NEW CONTRACTION

MERYEME ELHARRAK*, AHMED HAJJI

LabMIA-SI Laboratory, Department of Mathematics, Mohammed V University in Rabat, Rabat, Morocco

Abstract. In the present paper, using a new contraction on Banach space, we generalize Darbo's fixed point theorem and also some fixed point theorems which were recently proved by some authors [2, 11].

Keywords: fixed point; measure of noncompactness.

2010 AMS Subject Classification: 47H10, 47H09.

1. Introduction

In 1955, Darbo [7] proved the fixed point theorem for α-set contraction (i.e., $\alpha(T(A)) \leq k\alpha(A)$ with $k \in [0,1)$) on a closed, bounded and convex subset of a Banach space in terms of the measure of noncompactness. Since then many generalizations and extensions of this theorem have appeared; see for example [2, 3, 4, 6, 8, 9, 10, 11, 12].

The aim of this paper is to generalize Darbo's fixed point theorem and also a result of Aghajani et al [2] using the following new contraction:

$$\varphi(\overline{\text{conv}}(TA)) \leq \varphi(A) - \phi(\sigma(A)) \ \forall A \in P(\Omega),$$

*Corresponding author

E-mail address: mr.elharrak7@gmail.com

Received March 18, 2019

where T is a continuous mapping from Ω to itself, with Ω is a closed, bounded and convex subset of a Banach space, $P(\Omega)$ is the set of subsets of Ω, $\overline{\text{conv}}$ is the closed convex hull, $\sigma, \varphi \colon P(\Omega) \to \mathbb{R}^+$ are mappings, such that σ verifies some properties of a measure of non-compactness and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semicontinuous function such that $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Moreover, we derive from this theorem some consequences which are a generalizations of Darbos fixed point theorem and a Samadi-Ghaemi's result [11].

2. Preliminaries

At the begining we provide notations and some auxilary facts which will be needed in the sequel. To this end, assume that X is a given Banach space. If B is a subset of X then the symbols $\overline{B}$ and $\text{conv}(B)$ stand for the closure and the convex hull of B, respectively. Moreover, let $\mathfrak{M}_X$ be the family of all nonempty and bounded subsets of X and $\mathfrak{N}_X$ be its subfamily consisting of all relatively compact sets.

We mention the following definition of the measure of noncompactness, given in [5].

Definition 2.1. A mapping $\mu : \mathfrak{M}_X \to [0, \infty)$ is said to be a measure of noncompactness in X if it satisfies the following conditions:

(i) The family $ker\mu = \{A \in \mathfrak{M}_X : \mu(A) = 0\}$ is nonempty and $\ker \mu \subseteq \mathfrak{N}_X$.

(ii) $A \subseteq B \Rightarrow \mu(A) \leq \mu(B)$.

(iii) $\mu(\overline{A}) = \mu(A) = \mu(\text{conv}(A))$.

(iv) $\mu(\lambda A + (1-\lambda)B) \leq \lambda\mu(A) + (1-\lambda)\mu(B)$, for $\lambda \in [0,1]$.

(v) If (A_n) is a sequence of closed sets from $\mathfrak{M}_X$ such that $A_{n+1} \subseteq A_n$ for $(n = 1, 2, \ldots)$ and if $\lim\limits_{n\to+\infty} \mu(A_n) = 0$, then the set $A_\infty = \cap_{n=1}^{\infty} A_n \neq \emptyset$.

The family $\ker \mu$ defined in axiom (i) is called the kernel of the measure of noncompactness. Darbos fixed point theorem is a generalization of Schauders fixed point theorem.

Theorem 2.2 (Schauder [1])**.** *Let Ω be a nonempty, compact and convex subset of a Banach space X. Then each continuous mapping $T \colon \Omega \to \Omega$ has at least one fixed point in Ω.*

In the following, we state a fixed point theorem of Darbo type proved by Banaś and Goebel [5].

Theorem 2.3. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let $T: \Omega \to \Omega$ be a continuous mapping. Assume that there exists a constant $k \in [0,1)$ such that*

$$\mu(TA) \le k\mu(A) \tag{1}$$

for any nonempty subset A of Ω, where μ is a measure of noncompactness defined in X. Then T has a fixed point in Ω.

3. Main results

The main result of the present paper is the following fixed point theorem which is a generalization of Darbo's fixed point theorem.

Theorem 3.1. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let $T: \Omega \to \Omega$ be a continuous mapping. Assume that :*

$$\varphi(\overline{\text{conv}}(TA)) \le \varphi(A) - \phi(\sigma(A)) \tag{2}$$

for any nonempty subset A of Ω, where $\sigma, \varphi\colon P(\Omega) \to [0,+\infty)$ are a mappings such that σ satisfies the axioms (i), (ii) and (v) in the Definition 2.1 and $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a lower semicontinuous function, $\phi(0) = 0$ and $\phi(t) > 0$ for all $t > 0$. Then T has a fixed point in Ω.

Proof. We define the sequence $(\Omega_n)_n$ as follows

$$\begin{cases} \Omega_0 := \Omega, \\ \Omega_n := \overline{\text{conv}}(T\Omega_{n-1}), \quad n = 1,2,... \end{cases} \tag{3}$$

by induction, we observe easily that

$$\Omega_{n+1} \subseteq \Omega_n, \quad n = 0,1,2,... \tag{4}$$

we put $a_n = \varphi(\Omega_n)$, from the inequality (2) we have $\varphi(\Omega_n) - \varphi(\overline{\text{conv}}(T\Omega_n) \ge 0$, and we obtain

$$a_{n+1} = \varphi(\Omega_{n+1}) = \varphi(\overline{\text{conv}}(T\Omega_n)) \le \varphi(\Omega_n) = a_n,$$

therefore $(a_n)_n$ and $(\sigma(\Omega_n))_n$ are a decreasing sequences in $\mathbb{R}^+$, then there exist a real numbers a and r such that $a_n \underset{n\to+\infty}{\longrightarrow} a$ and $\sigma(\Omega_n) \underset{n\to+\infty}{\longrightarrow} r$. On the other hand, we have

$$\varphi(\overline{\text{conv}}(T\Omega_n)) \leq \varphi(\Omega_n) - \phi(\sigma(\Omega_n)),$$

this implies that $a_{n+1} \leq a_n - \phi(\sigma(\Omega_n))$, we obtain

$$\limsup_{n\to+\infty} a_{n+1} \leq \limsup_{n\to+\infty} a_n - \liminf_{n\to+\infty} \phi(\sigma(\Omega_n)). \tag{5}$$

This yields $a \leq a - \phi(r)$. Consequently $\phi(r) = 0$ so $r = 0$.
Hence we deduce that

$$\lim_{n\to+\infty} \sigma(\Omega_n) = 0. \tag{6}$$

From the property (v) in the Definition 2.1, the set $\Omega_\infty := \cap_{n=1}^{\infty} \Omega_n$ is nonempty. Moreover, for every $n = 0, 1, 2, ...$, we have

$$\Omega_\infty \subseteq \Omega_n, \tag{7}$$

which implies that

$$\sigma(\Omega_\infty) \leq \sigma(\Omega_n), \quad n = 0, 1, 2, ...$$

Passing to the limit as $n \to \infty$ and using (6), we obtain

$$\sigma(\Omega_\infty) = 0, \tag{8}$$

which implies from the property (i) in the Definition 2.1 that $\overline{\Omega_\infty} = \Omega_\infty$ is compact.
Moreover, since each Ω_n is convex, Ω_∞ is convex, and

$$T\Omega_\infty \subseteq T\Omega_n \subseteq \overline{\text{conv}}(T\Omega_n) = \Omega_{n+1} \subseteq \Omega_n, \quad n = 0, 1, 2, ...$$

Then $T \colon \Omega_\infty \to \Omega_\infty$ is well defined. By Theorem 2.2, T has a fixed point in Ω. □

Remark 3.2. By taking ϕ the identity function, $\sigma = \mu$ and $\varphi = (\frac{1}{1-k})\mu$ where μ is a measure of noncompactness and $k \in [0, 1)$, in Theorem 3.1 we obtain Darbo's fixed point theorem.

As a consequence of Theorem 3.1, we have the following result due to Aghajani et al [3].

Theorem 3.3. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let $T:\ \Omega \to \Omega$ be a continuous operator such that*

$$\psi(\mu(TA)) \leq \psi(\mu(A)) - \phi(\mu(A)) \tag{9}$$

for any nonempty subset A of Ω, where μ is an arbitrary measure of noncompactness and $\phi, \psi : \mathbb{R}^+ \to \mathbb{R}^+$ are a given functions such that ϕ is lower semicontinuous and ψ is continuous on $\mathbb{R}^+$. Moreover, $\phi(0) = 0$ and $\phi(t) > 0$ for $t > 0$. Then T has at least one fixed point in Ω.

Proof. By taking $\sigma = \mu$ and $\varphi = \psi \circ \mu$ where μ is a measure of noncompactness and $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, in Theorem 3.1 we obtain Theorem 3.3. $\square$

Now, we give the following corollaries of Theorem 3.1.

Corollary 3.4. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let $T\colon\ \Omega \to \Omega$ be a continuous mapping. Assume that :*

$$\varphi(\overline{\text{conv}}(TA)) \leq \varphi(A) - \sigma(A) \tag{10}$$

for any nonempty subset A of Ω, where $\sigma, \varphi\colon\ P(\Omega) \to [0, +\infty)$ are a mappings such that σ satisfies the axioms (i), (ii) and (v) in the Definition 2.1. Then T has a fixed point in Ω.

Corollary 3.5. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let $T\colon\ \Omega \to \Omega$ be a continuous mapping such that*

$$\mu(TA) \leq \eta(\mu(A)) \tag{11}$$

for any nonempty subset A of Ω, where μ is a measure of noncompactness defined in X and $\eta\colon\ [0, +\infty) \to [0, +\infty)$ is a mapping such that $\eta(t) < t$, for each $t > 0$ and $\frac{\eta(t)}{t}$ is non-decreasing. Then T has a fixed point in Ω.

Proof. Taking $\sigma = \mu$ and $\varphi(A) = \frac{\mu(A)}{1 - \frac{\eta(\mu(A))}{\mu(A)}}$, if $\mu(A) \neq 0$ and otherwise $\varphi(A) = 0$. Then (11) schows that

First case, if $\mu(A) \neq 0$, we have

$$\mu(TA) \leq \mu(A) - \left[1 - \frac{\eta(\mu(A))}{\mu(A)}\right] \mu(A), \tag{12}$$

it means that

$$\frac{\mu(TA)}{1-\frac{\eta(\mu(A))}{\mu(A)}} \leq \frac{\mu(A)}{1-\frac{\eta(\mu(A))}{\mu(A)}} - \mu(A). \tag{13}$$

Since $\frac{\eta(t)}{t}$ is non-decreasing and $\mu(TA) < \mu(A)$, we have

a) If $\mu(TA) \neq 0$, then

$$\frac{\mu(TA)}{1-\frac{\eta(\mu(TA))}{\mu(TA)}} \leq \frac{\mu(A)}{1-\frac{\eta(\mu(A))}{\mu(A)}} - \mu(A), \tag{14}$$

so

$$\varphi(\overline{\text{conv}}TA) \leq \varphi(A) - \mu(A). \tag{15}$$

b) If $\mu(TA) = 0$, we have $\varphi(\overline{\text{conv}}(TA)) = \mu(\overline{\text{conv}}(TA)) = 0$ and

$$\begin{aligned}
-\frac{\eta(\mu(A))}{\mu(A)} \leq 0 \Rightarrow 1 - \frac{\eta(\mu(A))}{\mu(A)} &\leq 1 \\
&\Rightarrow 1 \leq \frac{1}{1-\frac{\eta(\mu(A))}{\mu(A)}} \\
&\Rightarrow \mu(A) \leq \frac{\mu(A)}{1-\frac{\eta(\mu(A))}{\mu(A)}} \\
&\Rightarrow 0 \leq \frac{\mu(A)}{1-\frac{\eta(\mu(A))}{\mu(A)}} - \mu(A) \\
&\Rightarrow \varphi(\overline{\text{conv}}(TA)) \leq \varphi(A) - \mu(A).
\end{aligned}$$

Seconde case, if $\mu(A) = 0$, from the properties (i), (ii) and the fact that T is continuous, we have $\mu(TA) = 0$, so

$$\varphi(\overline{\text{conv}}TA) \leq \varphi(A) - \mu(A).$$

Then by Corollary 3.4, T has a fixed point in Ω. □

Remark 3.6. By taking $\eta(t) = kt$ for all $t \in [0,\infty)$ with $k \in [0,1)$, then Corollary 3.5 is a generalization of Darbo's fixed point theorem.

Corollary 3.7. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let $T: \Omega \to \Omega$ be a continuous mapping such that*

$$\mu(TA) \leq \mu(A) - \theta(\mu(A)) \tag{16}$$

for any nonempty subset A of Ω, where μ is a measure of noncompactness defined in X and $\theta\colon (0,+\infty)\to(0,+\infty)$ is a mapping such that, $\frac{\theta(t)}{t}$ is non-increasing. Then T has a fixed point in Ω.

Proof. Let $\eta(t)=t-\theta(t)$, for each $t>0$. Then $\eta(t)<t$, for each $t>0$ and $\frac{\eta(t)}{t}=1-\frac{\theta(t)}{t}$ is non-decreasing. Thus, the result is obtained by Corollary 3.5. $\square$

Remark 3.8. By taking $\theta(t)=(1-k)t$ for all $t\in[0,\infty)$ with $k\in[0,1)$, then Corollary 3.7 is a generalization of Darbo's fixed point theorem.

The following corollary schows the same result due to Samadi and Ghaemi [11].

Corollary 3.9. *Let Ω be a nonempty, bounded, closed and convex subset of a Banach space X and let $T\colon\Omega\to\Omega$ be a continuous mapping such that*

$$\mu(TA)\le\phi(\mu(A))\mu(A) \tag{17}$$

for any nonempty subset A of Ω, where μ is a measure of noncompactness defined in X and $\phi\colon[0,+\infty)\to[0,1)$ is a non-decreasing mapping. Then T has a fixed point in Ω.

Proof. Let $\eta(t)=\phi(t)t$ for all $t>0$. Then $\eta(t)<t$ for all $t>0$ and $\frac{\eta(t)}{t}=\phi(t)$ is non-decreasing. By assumption $\mu(TA)\le\phi(\mu(A))\mu(A)=\eta(\mu(A))$ for all $A\subseteq\Omega$, therefore by Corollary 3.5, T has a fixed point. $\square$

Remark 3.10. By taking $\phi(t)=k$ for all $t\in[0,\infty)$ with $k\in[0,1)$, then Corollary 3.9 is a generalization of Darbo's fixed point theorem.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] R. P. Agarwal, D. ORegan, Fixed point theory and applications, Cambridge university press. 2004.

[2] A. Aghajani, M. Aliaskari, Generalization of Darbo's fixed point theorem and application, Int. J. Nonlinear Anal. Appl. 2 (2011), 86-95.

[3] A. Aghajani, J. Banaś, N. Sabzali, Some generalizations of Darbo fixed point theorem and applications, Bull. Belg. Math. Soc. Simon Stevin. 20 (2013), 345-358.

[4] R. Arab, Some generalizations of Darbo fixed point theorem and its application, Miskolc Math. Notes. 18 (2017), 595-610.

[5] J. Banaś, K. Goebel, Measures of Noncompactness in Banach Spaces, Lect. Notes Pure Appl. Math., Dekker, New York. 60 (1980).

[6] L. Cai, L. Liang, New generalizations of Darbo's fixed point theorem, Fixed Point Theory Appl. 2015 (2015), 156.

[7] G. Darbo, Punti uniti in transformazioni a condominio non compatto (in Italy, Rend. Sem. Math. Univ. Padova. 2 (1955), 84-92.

[8] A. Hajji, E. Hanebaly, Commuting mappings and α-compact type fixed point theorems in locally convex spaces, Int. J. Math. Anal. 1 (2007), 661-680.

[9] A. Hajji, A generalization of Darbo's fixed point and common solutions of equations in Banach spaces, Fixed Point Theory Appl. 2013 (2013), 62.

[10] N. Khodabakhshi, S. M. Vaezpour, Common fixed point theorems via measure of noncompactness, Fixed point theory. 17 (2016), 381-386.

[11] A. Samadi, M. B. Ghaemi, An Extension of Darbo's Theorem and Its Application, Abstr. Appl. Anal. 2014 (2014), 852324.

[12] F. N. Sarvestani, S. M. Vaezpour, M. Asadi, Some generalization of Darbo's fixed point, J. Math. Anal. 7 (2016), 78-94.