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APPROXIMATING FIXED POINTS OF A NON LINEAR MAPPING USING K ITERATION PROCESS ON COMPLETE METRIC SPACE

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Abstract. This paper presents some extensions of the result that has been proved in [5]. We also obtain some result on set theoretic structure of the fixed points of the mappings satisfying some conditions.

Keywords: complete metric space; condition (A); strict convexity; uniformly convex space; fixed point.

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1. INTRODUCTION

If $T : X \to X$ and p be a point in X such that $Tp = p$. Then p is said to be a fixed point of T. It was in the year 1922 when Banach introduced the ground breaking result known as Banach contraction Principle, which guarantees the existence of a fixed point. This completely initiated a new dimension of research in the field of non linear analysis. This result was done on a complete metric space. From then on many researchers have considered different spaces and have taken different contraction condition to prove the existence of a fixed point. However, finding the value of the fixed point is not that easy. So to solve this problem, we need an iterative processes that can give us the fixed point. It all began by the result given in Mann

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[10], Ishikawa [11], Agarwal [12], Noor [13], Abbas [14], Vatan Two-step [15] and so on. All the above process depends on choosing some initial point on the space that would generate a sequence obeying some iterative schemes and converges to the fixed point.

Recently a new type of iteration has been obtained by Hussian et.al.[9] known as K-iteration process. The author in [9] have considered the contraction condition and (C) condition also known as generalised non expansive mapping for approximation of fixed points.

Let X be a Banach space and K be a non-empty bounded closed convex subset of X. Let $T : K \to K$ such that

$$\|Tx - Ty\| \leq a_1\|x-y\| + a_2\|x-Tx\| + a_3\|y-Ty\| + a_4\|x-Ty\| + a_5\|y-Tx\|,$$

$\forall\, x, y \in K,\ a_i \geq 0,\ \sum_{i=1}^{5} a_i \leq 1, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$

In 1973, Goebel et.al.[4] have proved the existence of fixed points for the mapping satisfying the above condition (1) in a Uniformly Convex space. In 1976, Chakrabarty and Lahiri,[5] obtained the existence of fixed point in Reflexive Banach space for mappings satisfying the above condition (1).

2. Preliminaries

Let X be a Banach space and K be a non-empty bounded closed convex subset of X. Let $T : K \to K$. Then T is said to satisfy condition (A) if it satisfies condition (1) with $a_2 > 0$ or $a_3 > 0$.

The uniqueness of the fixed points is guarenteed by condition (A). Tiwary et.al. [3] in 1995 considered the following iteration scheme

$$x_n = (1-\alpha)x_{n-1} + \alpha T[(1-\beta)x_{n-1} + \beta T x_{n-1}], \alpha, \beta \in [0,1],$$

to prove some results on approximation of fixed points of those operators that satisfies condition (A).

Motivated by above, in this paper, we use K iteration process to obtain a sequence whose limit will approximate the fixed point of those operators which satisfies condition (A).

A Banach space X is called uniformly convex [ ] if for each $\varepsilon \in (0,2]$, there is a $\delta > 0$ such that for $x, y \in X$, such that

$$\|x\| \leq 1, \|y\| \leq 1, \|x-y\| > \varepsilon,$$

implies

$$\|\frac{x+y}{2}\| \leq \delta.$$

Let X be a non empty set. Then d is said to be a metric if it satisfies the following conditions:

(i) $d(x,y) \geq 0$, $\forall x, y \in X$ and $d(x,y) = 0$ if $x = y$.

(ii) $d(x,y) = d(y,x)$, $\forall x, y \in X$.

(iii) $d(x,y) \leq d(x,z) + d(z,y)$, $\forall x, y, z \in X$.

The ordered pair (X,d) is called a metric space.

Definition 2.1. *Let (X,d) be a metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for $\varepsilon > 0$ there is a natural number N such that for all $n > N$, $d(x_n, x) < \varepsilon$. Then $\{x_n\}$ is said to be convergent and x is the limit of $\{x_n\}$. We denote this by, $lim_{n\to\infty} x_n = x$.*

Definition 2.2. *: Let (X,d) be a metric space. Let $\{x_n\}$ be a sequence in X. If for any $\varepsilon > 0$ there is a positive integer N such that for all $m, n > N$, $d(x_n, x_m) < \varepsilon$. Then $\{x_n\}$ is said to be a Cauchy sequence in X.*

Definition 2.3. *: If every Cauchy sequence in X is convergent in X, then (X,d) is called a complete metric space.*

Tiwary et. al. [3] proved the following result in 1995.

Theorem 2.4. *Let X be a Uniformly convex Banach space. Let K be a nonempty closed bounded convex subset of X and $T : X \to X$ be a continuous mapping that satisfies Condition (A) in K. Let $\alpha, \beta \in [0,1]$ be two fixed numbers. Let*

$$T_{\alpha,\beta} x = (1-\alpha)x + \alpha T[(1-\beta)x + \beta T x]$$

and let there exist a $x_0 \in K$ such that the sequence $\{x_n\}$, where

$$x_n = (1-\alpha)x_{n-1} + \alpha T[(1-\beta)x_{n-1} + \beta T x_{n-1}]$$

has a subseuence x_{n_i} converging to $z \in K$. Then z is a unique fixed point of T in K and $x_n \to z$, provided $\|T_{\alpha,\beta}z - u\| < \|z-u\|$ if u is a fixed point of T in K and $z \neq u$.

3. Main results

Theorem 3.1. *Let X be a Uniformly convex Banach space. Let K be a nonempty closed bounded convex subset of X and $T : X \to X$ be a continuous mapping that satisfies Condition (A) in K. Let $\alpha, \beta \in [0,1]$ be two fixed numbers. Let*

$$T_{\alpha,\beta}x = T[T[(1-\alpha)Tx + \alpha T[(1-\beta)x + \beta Tx]]]$$

and let there exist a $x_0 \in K$ such that the sequence $\{x_n\}$, where

$$x_n = T[T[(1-\alpha)Tx_{n-1} + \alpha T[(1-\beta)x_{n-1} + \beta T x_{n-1}]]]$$

has a subseuence x_{n_i} converging to $z \in K$. Then z is a unique fixed point of T in K and $x_n \to z$, provided $\|T_{\alpha,\beta}z - u\| < \|z-u\|$ if u is a fixed point of T in K and $z \neq u$.

Proof : Existence of fixed point is guaranteed by the theorem of Geobel et.al. So we assume that u is the fixed point in K of T and it is unique because it satisfies condition (A).
Now $T_{\alpha,\beta}u = T[T[(1-\alpha)Tu + \alpha T[(1-\beta)u + \beta Tu]]]$.
$= T[T[(1-\alpha)u + \alpha T[(1-\beta)u + \beta u]]]$.
$= T[T[(1-\alpha)u + \alpha Tu$.
$= T[Tu] = u$.
Hence, u is the fixed point of $T_{\alpha,\beta}$. Now,
$T_{\alpha,\beta}x_n = T[T[(1-\alpha)Tx_n + \alpha T[(1-\beta)x_n + \beta T x_n]]] = x_{n+1}$.
So, if $x_n = u$. Then we have,
$x_{n+1} = T_{\alpha,\beta}x_n = T_{\alpha,\beta}(u) = u$.
Hence, $x_n = u$, for $j \geq n$. Consequently, $x_j \to u$ that is $x_{n_j} \to u$ so $z = u$.
Assume, $x_n \neq u, \forall n$. Then, if $x_n \to z$, by the continuity of T we get, $Tx_n \to Tz$
Now, $T_{\alpha,\beta}x_n = T[T[(1-\alpha)Tx_n + \alpha T[(1-\beta)x_n + \beta T x_n]]]$.

$\rightarrow T[T[(1-\alpha)Tz+\alpha T[(1-\beta)z+\beta Tz]]]$.

$\rightarrow T_{\alpha,\beta}z$.

Hence, $T_{\alpha,\beta}$ is also a continuous function.

$\|x_{n+1}-u\| = \|T[T[(1-\alpha)Tx_n+\alpha T[(1-\beta)x_n+\beta Tx_n]]]-Tu\|$.....................(A)

Let us begin with,

$\|z_n-u\|$, where $z_n=(1-\beta)x_n+\beta Tx_n$. So,

$\|z_n-u\| = \|(1-\beta)x_n+\beta Tx_n-u\|$.

$= \|(1-\beta)x_n+\beta Tx_n-(1-\beta+\beta)u\|$.

$= \|(1-\beta)(x_n-u)+\beta(Tx_n-u)\|$.

$\leq (1-\beta)\|(x_n-u)\|+\beta\|(Tx_n-u)\|$.......................(B)

Now, $\|Tx_n-u\| = \|Tx_n-Tu\|$.

$= a_1\|x_n-u\|+a_2\|x_n-Tx_n\|+a_3\|u-Tu\|+a_4\|x_n-Tu\|+a_5\|u-Tx_n\|$.

$= a_1\|x_n-u\|+a_2\|x_n-u+u-Tx_n\|+a_4\|x_n-u\|+a_5\|u-Tx_n\|$.

$\leq (a_1+a_2+a_4)\|x_n-u\|+(a_2+a_5)\|u-Tx_n\|$.

$(1-a_2-a_5)\|Tx_n-u\| \leq (a_1+a_2+a_4)\|x_n-u\|$.

$(1-a_2-a_5)\|Tx_n-u\| \leq (1-a_3-a_5)\|x_n-u\|$.............(1)

By, symmetry, we get,

$(1-a_3-a_4)\|Tx_n-u\| \leq (1-a_2-a_4)\|x_n-u\|$.............(2)

Addding (1) and (2), we get, $\|Tx_n-u\| \leq \|x_n-u\|$................(2A)

So, using (2A) in the equation (B), we get,

$\|z_n-u\| \leq (1-\beta)\|x_n-u\|+\beta\|x_n-u\| = \|x_n-u\|$.

$\|z_n-u\| \leq \|x_n-u\|$................(C)

Now we consider $y_n = T[(1-\alpha)Tx_n+\alpha Tz]$.

So,

$\|y_n-u\| = \|T[(1-\alpha)Tx_n+\alpha Tz_n]-Tu\|$.

$\leq a_1\|(1-\alpha)Tx_n+\alpha Tz_n-u\|+a_2\|T[(1-\alpha)Tx_n+\alpha Tz_n]-[(1-\alpha)Tx_n+\alpha Tz_n]\|+a_3\|u-$
$Tu\|+a_4\|(1-\alpha)Tx_n+\alpha Tz_n-u\|+a_5\|u-T[(1-\alpha)Tx_n+\alpha Tz_n]\|$......................(D)

For a_1 in equation (D),

$\|(1-\alpha)Tx_n+\alpha Tz_n-u\| \leq \|(1-\alpha)Tx_n+\alpha Tz_n-(1-\alpha+\alpha)u\|$.

$\leq (1-\alpha)\|Tx_n - u\| + \alpha\|Tz_n - u\|$.

By using equation (2A), we get,

$\|(1-\alpha)Tx_n + \alpha Tz_n - u\| \leq (1-\alpha)\|x_n - u\| + \alpha\|z_n - u\|$.

From (C), we get,

$\|(1-\alpha)Tx_n + \alpha Tz_n - u\| \leq (1-\alpha)\|x_n - u\| + \alpha\|x_n - u\|$.

$\|(1-\alpha)Tx_n + \alpha Tz_n - u\| \leq \|x_n - u\|$.................(E)

For a_2 in equation (D),

$\|T[(1-\alpha)Tx_n + \alpha Tz_n] - [(1-\alpha)Tx_n + \alpha Tz_n]\| \leq \|T[(1-\alpha)Tx_n + \alpha Tz_n] - u\| + \|[(1-\alpha)Tx_n + \alpha Tz_n] - u\|$.

By using Eq. (E), we get $\leq \|T[(1-\alpha)Tx_n + \alpha Tz_n] - u\| + \|x_n - u\|$.

Again by using Eq. (2A) and (E), we get, $\leq \|[(1-\alpha)Tx_n + \alpha Tz_n] - u\| + \|x_n - u\|$.

$\leq \|x_n - u\| + \|x_n - u\|$.

$\leq 2\|x_n - u\|$.

That is, $\|T[(1-\alpha)Tx_n + \alpha Tz_n] - [(1-\alpha)Tx_n + \alpha Tz_n]\| \leq 2\|x_n - u\|$.............(F)

For a_3 in equation (D), we get,

$\|u - Tu\| = 0$.

For, a_4 in equation (D), we get,

$\|(1-\alpha)Tx_n + \alpha Tz_n - u\| \leq \|x_n - u\|$.

For, a_5 in equation (D), we get,

$\|u - T[(1-\alpha)Tx_n + \alpha Tz_n]\| \leq \|x_n - u\|$.

Combining all the above result in Eq. (D), we get,

$\|y_n - u\| \leq a_1\|x_n - u\| + 2a_2\|x_n - u\| + a_3\|x_n - u\| + a_4\|x_n - u\| + a_5\|x_n - u\|$.

So,

$\|y_n - u\| \leq (a_1 + 2a_2 + a_4 + a_5)\|x_n - u\|$.................(3)

By symmetry, $\|y_n - u\| \leq (a_1 + 2a_3 + a_4 + a_5)\|x_n - u\|$.................(4)

Adding Eqs. (3) and (4), we get,

$2\|y_n - u\| \leq 2(a_1 + a_2 + a_3 + a_4 + a_5)\|x_n - u\|$.

$\|y_n - u\| \leq \|x_n - u\|$......................(G)

Now, we calculate Eq. (A), $\|x_{n+1} - u\|$, Where $x_{n+1} = Ty_n$. $\|x_{n+1} - u\| = \|Ty_n - u\|$.

By Eq. (2A) and (G), we get,

$\|x_{n+1} - u\| = \|y_n - u\| \leq \|x_n - u\|$............$(H)$

Hence, $\{\|x_n - u\|\}$, is a monotonically decreasing sequence and bounded, hence convergent. Also,

$\|x_{n_i+1} - u\| \leq \|x_{n_i+1} - T_{\alpha,\beta}z\| + \|T_{\alpha,\beta}z - u\|$....................(I)

we know that $T_{\alpha,\beta}$ is also continuous and by hypothesis $x_{n_i} \to z$, we have

$T_{\alpha,\beta}x_{n_i} \to T_{\alpha,\beta}z$.

Now,

$T_{\alpha,\beta}x_{n_i} = T[T[(1-\alpha)Tx_{n_i} + \alpha T[(1-\beta)x_{n_i} + \beta Tx_{n_i}]]] = x_{n_i+1}$.

Therefore, from (I) we get,

$\lim_{i\to\infty} \|x_{n_i+1} - u\| \leq \|T_{\alpha,\beta}z - u\|$..............(J)

By using hypothesis $x_{n_i} \to z$ we get,

$\lim_{n\to\infty} \|x_n - u\| = \lim_{i\to\infty} \|x_{n_i} - u\| = \|z - u\|$.

Hence, we have,

$\|z - u\| = \lim_{n\to\infty} \|x_n - u\| = \lim_{i\to\infty} \|x_{n_i+1} - u\| \leq \|T_{\alpha,\beta}z - u\|$.

Which is a contradiction to our hypothesis, we therefore conclude that $z = u$.

Now, $\lim_{n\to\infty} \|x_n - u\| = \lim_{i\to\infty} \|x_{n_i} - u\| = \lim_{i\to\infty} \|x_{n_i} - z\| = 0$,

and this completes the proof of the theorem.

Remark 3.2. *If the hypothesis $\|T_{\alpha,\beta}z - u\| < \|z - u\|$ where u is the fixed point of T, $z \neq u$, $x_{n_i} \to z$, $\{x_{n_i}\} \subset \{x_n\}$ of the Theorem 1 is replaces by $\|T_{\alpha,\beta}x - u\| < \|x - u\|$, u is a fixed point of T, $x \neq u$, $x \in K$, the Theorem can be proved in few lines as follows from [3]:*

Corollary 3.3. *Let X be a strictly convex Banach space and K be a nonempty bounded closed convex subset of X. Let T be a mapping of K into convex subset of K such that T satisfies the condition (A) in K and T is completely continuous. Then for $\alpha, \beta \in [0,1]$, the sequence $\{x_n\}$,*

$$x_n = T[T[(1-\alpha)Tx_n + \alpha T[(1-\beta)x_n + \beta Tx_n]]], x_0 \in T(K),$$

converges to the unique fixed point of T in K, provided $\|Tx - u\| < \|x - u\|$, where u is a fixed point of T, $x \neq u$ and $x \in T(K)$.

Proof : By Schauder's Theorem, since T is completely continuous, T has a fixed point u in K. Since T satisfies condition (A) the fixed point is unique.
Let $x \neq u$. We would like to prove that

$$\|T_{\alpha,\beta}x - u\| < \|x - u\|............(1)$$

We have,

$$\|T_{\alpha,\beta}x - u\| = \|T[T[(1-\alpha)Tx + \alpha T[(1-\beta)x + \beta Tx]]] - u\|, ...(2)$$

From the proof of the previous theorem using (A) we get,

$$\|T_{\alpha,\beta}x - u\| \leq \|x - u\|............(3)$$

If possible,

$$\|T_{\alpha,\beta}x - u\| = \|x - u\|............(4)$$

. Then
$\|x - u\| = \|T[T[(1-\alpha)x + \alpha T[(1-\beta)x + \beta Tx]]] - u\|$.
By using equation (2A), (C) from the proof of the previous theorem, we get,
$\leq \|x - u\| = \|T[(1-\alpha)Tx + \alpha T[(1-\beta)x + \beta T]] - u\|$.
$\leq \|(1-\alpha)Tx + \alpha T[(1-\beta)x + \beta Tx] - u\|$.
$\leq \|(1-\alpha)Tx + \alpha T[(1-\beta)x + \beta Tx] - (1-\alpha+\alpha)u\|$.
$\leq \|(1-\alpha)(Tx - u) + \alpha[T[(1-\beta)x + \beta Tx] - u]\|$.
$\leq \|(1-\alpha)(x - u) + \alpha[T[(1-\beta)x + \beta Tx] - u]\|$.
$\leq (1-\alpha)\|(Tx - u)\| + \alpha\|T[(1-\beta)x + \beta Tx] - u\|$.
$\leq (1-\alpha)\|(Tx - u)\| + \alpha\|(x - u)\|$.
$\leq (1-\alpha)\|(x - u)\| + \alpha\|(x - u)\|$.
$\leq \|(x - u)\|$.
Therefore, we have,
$\|(1-\alpha)(x - u) + \alpha[T[(1-\beta)x + \beta Tx] - u]\| = \|(1-\alpha)(x - u)\| + \|\alpha[T[(1-\beta)x + \beta Tx] - u]\|$...........(5).
and also $\|(1-\alpha)(x - u)\| + \|\alpha[T[(1-\beta)x + \beta Tx] - u]\| = \|x - u\|$...........(6)

From eq (5) and strict convexity, we have,

$(1-\alpha)(x-u) = \mu\alpha[T[(1-\beta)x+\beta Tx]-u], \mu > 0$............(7)

From Eq. (6) and (7), we get, $(1-\alpha)\|x-u\| + \frac{(1-\alpha)}{\mu}\|x-u\| = \|x-u\|$.

$$\mu = \frac{1-\alpha}{\alpha}\text{................(8)}$$

From, eq (7) and (8), we get,

$\frac{(1-\alpha)}{\alpha}(x-u) = \mu[T[(1-\beta)x+\beta Tx]-u]$.

$\mu(x-u) = \mu[T[(1-\beta)x+\beta Tx]-u]$.

$(x-u) = [T[(1-\beta)x+\beta Tx]-u]$.

$\|x-u\| = \|T[(1-\beta)x+\beta Tx]-u\|$.

$\leq \|[(1-\beta)x+\beta Tx]-u\|$.

$\leq \|[(1-\beta)x+\beta Tx]-(1-\beta+\beta)u\|$.

$\leq \|[(1-\beta)(x-u)+\beta(Tx-u)\|$.

$\leq (1-\beta)\|x-u\|+\beta\|Tx-u\|$.

$\leq (1-\beta)\|x-u\|+\beta\|x-u\|$.

$\leq \|x-u\|$.

So, we have, $(1-\beta)\|x-u\|+\beta\|Tx-u\| = \|x-u\|$.

$\beta\|Tx-u\| = (1-1+\beta)\|x-u\|$.

$\beta\|Tx-u\| = \beta\|x-u\|$.

$\|Tx-u\| = \|x-u\|$.

A contradiction to the fact that

$$\|Tx-u\| < \|x-u\|.$$

Therefore, $\|T_{\alpha,\beta}x-u\| < \|x-u\|$.

Thus because $T(K)$ is convex and compact and X is complete, the proof follows from the remark (2) above.

Theorem 3.4. *Let K be a nonempty bounded closed convex subset of a strictly convex Banach space X and let $T\colon K \to K$ be a mapping satisfying condition (A) and either*

$$(a) sup_{x,z\in F}\|Tx-z\| < \delta(F)$$

for every nonempty closed convex bounded subset F of K containing more than one element and is mapped into itself by T, and

$$(b) T : K \to K \text{ is continuous.}$$

Then the sequence $\{x_n\}, x_n = T[T[(1-\alpha)Tx_{n-1} + \alpha T[(1-\beta)x_{n-1} + \beta Tx_{n-1}]]], x_0 \in K, \alpha, \beta \in [0,1]$ *converges to the unique fixed point u of T provided the sequence* $\{x_n - Tx_n\}$ *has subsequence* $\{x_{n_k} - Tx_{n_k}\}$ *such that* $\|x_{n_k} - Tx_{n_k}\| \to 0$.

Proof : Since, a uniformly convex Banach space is reflexive, then the existence of a fixed point u of T in K is assured by a theorem of Chakraborty and Lahiri (1976) if (a) is satisfied and by a theorem of Goebel et.al (1973) if (b) is satisfied. The fixed point u of T in K is unique because a_2 or a_3. Following the proof of theorem (2) Tiwary et. al. we get, $x_{n_k} \to u$. From theorem 3.1, equation (H), we get,

$$\|x_n - u\| \leq \|x_{n-1} - u\|$$

and so,

$$\lim_{n\to} \|x_n - u\| = \lim_{k\to} \|x_{n_k} - u\| = 0.$$

This completes the proof of the theorem.

Here we discuss some set theoretic structure of the fixed points of $T_{\alpha,\beta}$ when T is linear. The following theorem has been proved by Tiwary et.al. [3].

Theorem 3.5. *Let X be a nonempty bounded closed convex subset of a strictly convex Banach space B and let* $T : X \to X$. *Let*

$$T_{\alpha,\beta}x = (1-\alpha)x + \alpha T[(1-\beta)x + \beta Tx], \alpha, \beta \in [0,1],$$

and let $F(T_{\alpha,\beta}) = \{x \in X : T_{\alpha,\beta}x = x\}$ *be nonempty. Then* $F(T_{\alpha,\beta})$ *is closed when* T *is linear and continuous.*

Inspired by this result we now present some results using K- iteration process.

Theorem 3.6. *Let X be a nonempty bounded closed convex subset of a strictly convex Banach space B and let* $T : X \to X$. *Let*

$$T_{\alpha,\beta}x = T[T[(1-\alpha)Tx + \alpha T[(1-\beta)x + \beta Tx]]], \alpha, \beta \in [0,1],$$

and let $F(T_{\alpha,\beta}) = \{x \in X : T_{\alpha,\beta}x = x\}$ be nonempty. Then $F(T_{\alpha,\beta})$ is closed when T is linear and continuous.

Proof : If $F(T_{\alpha,\beta})$ is a singleton set then the proof is obvious. Let us suppose that $F(T_{\alpha,\beta})$ contains more that one point. We choose a sequence $\{x_n\}$, $\{x_n\} \in F(T_{\alpha,\beta})$ such that $\lim_{n\to\infty} x_n = z$ for some $z \in B$. Now,

$\|T_{\alpha,\beta}z - z\| \leq \|T_{\alpha,\beta}z - x_n\| + \|x_n - z\|$.

$= \|T_{\alpha,\beta}z - T_{\alpha,\beta}x_n\| + \|x_n - z\|$.

$= \|T[T[(1-\alpha)Tx + \alpha T[(1-\beta)x + \beta Tx]]] - z\| + \|T[T[(1-\alpha)Tx_n + \alpha T[(1-\beta)x_n + \beta Tx_n]]] - z\| + \|x_n - z\|$.

By referring the proof of the previous corollary eq (1), we get,

$\leq \|T_{\alpha,\beta}z - z\| + \|T_{\alpha,\beta}x_n - z\| + \|x_n - z\|$.

$\leq \|z - z\| + \|x_n - z\| + \|x_n - z\|$.

Taking $\lim_{n\to\infty}$, we have,

$T_{\alpha,\beta}z = z$.

This implies, $z \in F(T_{\alpha,\beta})$ and $F(T_{\alpha,\beta})$ become closed.

Theorem 3.7. *Let E be a nonempty bounded closed convex subset of a strictly convex Banach space B and let $T : E \to E$. Let*

$$T_{\alpha,\beta}x = T[T[(1-\alpha)Tx + \alpha T[(1-\beta)x + \beta Tx]]], \alpha, \beta \in [0,1],$$

and let $F(T_{\alpha,\beta}) = \{x \in E : T_{\alpha,\beta}x = x\}$ be nonempty. Then $F(T_{\alpha,\beta})$ is convex when T is linear.

Proof : If $F(T_{\alpha,\beta})$ is a singleton set then we have nothing to prove. Let us suppose that $F(T_{\alpha,\beta})$ contains more than one point. Let $x, y (x \neq y) \in F(T_{\alpha,\beta})$ be arbitrary. Let also, for some $\lambda : 0 < \lambda < 1, z = \lambda x + (1-\lambda)y$. Then

By using the eq (1) of Corollary 3.3, we get,

$$\|T_{\alpha,\beta}z - x\| \leq \|z - x\|..............(1)$$

Similarly,

$$\|T_{\alpha,\beta}z - y\| \leq \|z - y\|................(2)$$

Thus, $\|x-y\| = \|x - T_{\alpha,\beta}z + T_{\alpha,\beta}z - y\|$.

$\leq \|x-z\| + \|z-y\|$.

$z = \lambda x + (1-\lambda)y$

$z - \lambda x = y - \lambda y$.

$z - y = \lambda(x-y)$.

$x - z = (x-y) + (y-z) = (x-y) - \lambda(x-y) = (x-y)(1-\lambda)$.

Therefore,

$\|x-y\| \leq (1-\lambda)\|x-y\| + \lambda\|x-y\| = \|x-y\|$.

Hence,

$\|x-y\| = \|x-z\| + \|z-y\|$.

Hence the proof follows from Theorem 5, of Tiwary et.al. [3].

Conflict of Interests

The authors declare that there is no conflict of interests.

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