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FIXED POINT THEOREMS FOR MULTI-VALUED MAPPINGS IN PROBABILISTIC METRIC SPACES

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Abstract. In this paper we shall introduce the notion of a compact probabilistic metric space and prove some fixed point theorems for multi-valued mappings in a compact probabilistic metric space.

Keywords: probabilistic metric space; fixed point; Menger space; t-norm; H-type.

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1. INTRODUCTION

The concept of an abstract metric space, introduced by M. Frechet in 1906 [5], furnishes the common idealization of a large number of mathematical, physical and other scientific constructs in which the notion of a "distance" appears. What matters is the possibility of associating a non-negative real number with each ordered pair of elements of a certain set, and that the numbers associated with pairs and triples of such elements satisfy certain conditions.

In 1942,K. Menger [6] was first who thought about distance distribution function in metric space and introduced the concept of probabilistic metric space. He replaced distance function $d(x,y)$, the distance between two point x, y by distance distribution function $\mathscr{F}_{x,y}(t)$ where the

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value of $\mathscr{F}_{x,y}(t)$ is interpreted as probability that the distance between x, y is less than t, $t > 0$. The history of probabilistic metric spaces is brief In the original paper, Menger gave postulates for the distribution functions $\mathscr{F}_{x,y}$ These included a generalized triangle inequality. In addition, he constructed a theory of betweeness and indicated possible fields of application. Such a probabilistic generalization of metric spaces appears to be well adapted for the investigation of physical quantities and physiological thresholds. It has also important applications in nonlinear analysis [1].

In 1943, shortly after the appearance of Menger's paper, A. Wald published a paper [7] in which he criticized Menger's generalized triangle inequality and proposed an alternative one. On the basis of this new inequality, A. Wald constructed a theory of betweeness having certain advantages over Menger's theory [8].

In 1951, Menger continued his study of probabilistic metric spaces in a paper [15] devoted to a resume of the earlier work, the construction of several specific examples and further considerations of the possible applications of the theory. In this paper, K. Menger adopted Wald's version of the triangle inequality.

PM-spaces have nice topological properties. Many different topological structures may be defined on a PM-space. The one that has received the most attention to date is the strong topology and it is the principal tool of this study. The convergence with respect to this topology is called strong convergence. Since the strong topology is first countable and Hausdorff, it can be completely specified in terms of the strong convergence of sequences.

Fixed point theory in probabilistic metric spaces can be considered as a part of Probabilistic Analysis, which is a very dynamic area of mathematical research. It is necessary to mention that fixed point theorems are main tools for mathematicians to study the problem of existence of a solution for a system of differential equations in probabilistic metric space. The first result about the existence of a fixed point of a mapping which is defined on a Menger space is obtained by Sehgel and Barucha-Reid. Schweizer and Sklar [2] developed the study of fixed point theory in probabilistic metric spaces. In 1966, Sehgal [10] initiated the study of contraction mapping theorem in probabilistic metric spaces. Several interesting and elegant result have been proved by various author in probabilistic metric spaces. In 2005, Mihet [9] proved a fixed point theorem concerning probabilistic contractions satisfying an implicit relation. Egbert [3] defined the notion of the distance between two sets in a Menger PM-space, i.e., the so-called Menger-Hausdorff metric. In 2006, Mustafa and Sims [11] introduced the concept of a generalized metric space, and many fixed point results have been obtained by many authors.

Our aim in this paper is to study the existence of fixed points for nonempty multi-valued mappings defined on a compact Menger probabilistic metric space $(X,\mathscr{F},\Delta)$, where Δ is a t-norm.

2. Preliminaries

The introduction of the general concept of statistical metric spaces is due to Karl Menger (1942), who dealt with probabilistic geometry. The new theory of fundamental probabilistic structures was developed later on by many authors. In this section, we start by recalling some basic concepts from Menger probabilistic metric spaces. For more details on such spaces, we refer to ([1]-[17]).

Definition 2.1. [2] A mapping $\mathscr{F}:\mathbb{R}\longrightarrow\mathbb{R}^+$ is called a distribution function if it is non-decreasing and left continuous with $sup_{x\in\mathbb{R}}\mathscr{F}(x)=1$ and $inf_{x\in\mathbb{R}}\mathscr{F}(x)=0$. We shall denote by D the set of all distribution functions while H will always denote the specific distribution function defined by

$$H(x)=\begin{cases}0, & x\leq 0;\\ 1, & x>0.\end{cases}$$

Example 2.2. Let $G:\mathbb{R}\longrightarrow\mathbb{R}^+$ be a mapping defined by

$$G(x)=\begin{cases}0, & x\leq 0;\\ a, & 0<x\leq k;\\ b, & k<x\leq 3k;\\ 1, & 3k<x.\end{cases}$$

Where $0<a\leq b<1$ and k is any positive number. It is clear that G is non-decreasing and left continuous with $inf_{x\in\mathbb{R}}G(x)=0$ and $sup_{x\in\mathbb{R}}G(x)=1$, then G is called a distribution function.

Definition 2.3. [1] A probabilistic metric space (briefly, a PM-space) is an ordered pair $(X,\mathscr{F})$, where X is an abstract set and $\mathscr{F}$ is a mapping of $X\times X$ on to the set of all distributions function, i.e, $\mathscr{F}$ associates a distribution function $\mathscr{F}(p,q)$ with every pair (p,q) of points in X. we shall denote the distribution function $\mathscr{F}(p,q)$ by $\mathscr{F}_{p,q}$, whence the symbol $\mathscr{F}_{p,q}(x)$ will denote the value of $F_{p,q}$ for the real argument x. the function $F_{p,q}$ are assumed to satisfy the following condition:
(PM-1) $\mathscr{F}_{p,q}(x)=1$ for all $x>0$ if and only if $p=q$,
(PM-2) $\mathscr{F}_{p,q}(0)=0$,
(PM-3) $\mathscr{F}_{p,q}=\mathscr{F}_{p,q}$,
(PM-4) if $\mathscr{F}_{p,q}(x)=1$ and $\mathscr{F}_{p,q}(y)=1$, then $\mathscr{F}_{p,q}(x+y)=1$.

In view of Condition (PM-2), which evidently implies that $\mathscr{F}_{p,q}(0)=0$ for all $x\leq 0$, the Condition (PM-1) is equivalent to the statement: $p=q$ if and only if $\mathscr{F}_{p,q}=H$.
Note that every metric space may be regarded as an PM-space of a special kind if we have only to set $\mathscr{F}_{p,q}(x)=H((x-d(p,q))$ for every pair of points (p,q) in the metric space.

Example 2.4. Let X be a set of all real numbers, and define:

$$\mathscr{F}_{p,q}(x)=\begin{cases} 0, & x\leq 0;\\ 1-e^{-\left(\frac{x}{d(p,q)}\right)}, & x>0,\end{cases}$$

is a distribution function for all $p,q\in X$, where $d(p,q)=|p-q|$ for all $p,q\in X$. By verifying that $\mathscr{F}_{p,q}$ satisfy the axioms (PM-1) to (PM-4), then $(X,\mathscr{F},\Delta)$ is probabilistic metric space.

Definition 2.5. [13] A mapping $\Delta:[0,1]\times[0,1]\longrightarrow[0,1]$ is called a triangular norm (for short, a $t-$ norm) if the following conditions are satisfied:
$(\Delta-1)$ $\Delta(a,1)=a$ and $\Delta(0,0)=0$,
$(\Delta-2)$ $\Delta(a,b)=\Delta(b,a)$,
$(\Delta-3)$ $\Delta(a,c)\Delta(b,d)$ for $a\geq b$, $c\geq d$,
$(\Delta-4)$ $\Delta(a,\Delta(b,c))=\Delta(\Delta(a,b),c)$.

Remark 2.6. From $(\Delta-4)$, it is not difficult to find that

$$\Delta(\Delta(a,b),\Delta(c,d))=\Delta(\Delta(\Delta(a,b),c),d)=\Delta(\Delta(\Delta(a,c),b),d)=\Delta(\Delta(a,c),\Delta(b,d))=...$$

Example 2.7. [1] The following are the three basic t-norms.

(1)The minimum $t-$norm: $\Delta_m(a,b) = min\{a,b\}$.

(2)The product $t-$norm: $\Delta_p(a,b) = a.b$.

(3)The Lukasiewicz $t-$norm: $\Delta_L(a,b) = max\{a+b1,0\}$.

In respect of above mentioned $t-$norms, we have the following ordering:

$$\Delta_L < \Delta_p < \Delta_m.$$

Definition 2.8. [1] A Menger PM-space is a tripled $(X,\mathscr{F},\Delta)$ where $(X,\mathscr{F})$ is a PM-space and Δ is a $t-$norm such that the inequality $\mathscr{F}_{p,r}(x+y) \geq \Delta(\mathscr{F}_{p,q}(x),\mathscr{F}_{q,r}(y))$ holds for all $p,q,r \in X$ and $x,y \geq 0$.

Lemma 2.9. [14] *Let $(X,\mathscr{F},\Delta)$ be a Menger PM-space. If there exists a constant $k \in (0,1)$ such that $\mathscr{F}_{p,q}(kt) \geq \mathscr{F}_{p,q}(t)$ for all $p,q \in X$ and $t > 0$, then $p = q$.*

Definition 2.10. [13] A sequence of points $\{P_n\}$ in an PM-space is said to be converge to a point p in X (and we write $P_n \longrightarrow p$) if and only if for every $\varepsilon > 0$ and every $\lambda \in (0,1)$, there exists an integer $M_{\varepsilon,\lambda}$ such that $P_n \in \{N_p(\varepsilon,\lambda)\}$ i.e., $\mathscr{F}_{p_n,p}(\varepsilon) > 1-\lambda$, whenever $n > M_{\varepsilon,\lambda}$.

Definition 2.11. [13] A sequence of points $\{P_n\}$ in an PM-space is said to be Cauchy sequence if for each $\varepsilon > 0$ and $\lambda \in (0,1)$, there exist is an integer $M_{\varepsilon,\lambda}$ such that $\mathscr{F}_{p_n,p_m}(\varepsilon) > 1-\lambda$, for all $n,m \geq M_{\varepsilon,\lambda}$.

Lemma 2.12. [16] *Let $\{P_n\}$ be a sequence in Menger space $(X,\mathscr{F},\Delta)$ where Δ is continuous and $\Delta(a,a) \geq a$ for all $a \in [0.1]$. If there exists a constant $k \in (0,1)$ such that $a > 0$ and $n \in N$, $\mathscr{F}_{P_n,P_{n+1}}(ka) \geq \mathscr{F}_{P_{n-1},P_n}(a)$, then $\{P_n\}$ is a Cauchy sequence.*

Definition 2.13. [13] A Menger space $(X,\mathscr{F},\Delta)$ is said to be complete, if every Cauchy sequence in X converges to a point x in X.

Definition 2.14. [18] Let $(X,\mathscr{F},\Delta)$ be a Menger PM-space, A subset A of X is closed if every convergent sequence in A converges to an element of A.

Definition 2.15. [17] Let $(X,\mathscr{F},\Delta)$ is a Menger PM-space with $\sup_{0<t<1}\Delta(t,t) = 1$.

(1) A sequence of points $\{P_n\}$ in X is said to $\tau-$converge to a point p in X (and we write $P_n \longrightarrow$

p) if for every $\varepsilon > 0$ and every $\lambda \in (0,1)$, there exists an integer $M_{\varepsilon,\lambda}$ such that $\mathscr{F}_{p_n,p}(\varepsilon) > 1-\lambda$, whenever $n > M_{\varepsilon,\lambda}$.

(2) A sequence of points $\{P_n\}$ in X is said to be $\tau-$Cauchy sequence if for each $\varepsilon > 0$ and $\lambda \in (0,1)$, there exist is an integer $M_{\varepsilon,\lambda}$ such that $\mathscr{F}_{p_n,p_m}(\varepsilon) > 1-\lambda$, for all $n,m \geq M_{\varepsilon,\lambda}$.

(3) A Menger PM-space $(X,\mathscr{F},\Delta)$ is said to be $\tau-$complete, if each -Cauchy sequence in X is $\tau-$converges to some point x in X.

Lemma 2.16. [1] *If $P_n \longrightarrow p$, then $\mathscr{F}_{p_n,p} \longrightarrow \mathscr{F}_{p,p} = H$, i.e., for all x, $\mathscr{F}_{p_n,p}(x) \longrightarrow \mathscr{F}_{p,p}(x) = H(x)$, and conversely.*

Definition 2.17. [3] Let $(X,\mathscr{F},\Delta)$ is a Menger PM-space with a continuous $t-$norm Δ and A be a nonempty subset of X. The function D_A defined by

$$D_A(x) = \sup_{s<x} \inf_{p,q\in A} \mathscr{F}_{p,q}(s),$$

will be called the probabilistic diameter of A.

Definition 2.18. [3] Let $(X,\mathscr{F},\Delta)$ is a Menger PM space with a continuous $t-$norm Δ. A nonempty subset A of X is called a probabilistically bounded set if $\sup_{x>0} D_A(x) = 1$.

Theorem 2.19. [3] *Let $(X,\mathscr{F},\Delta)$ is a Menger PM space with a continuous $t-$norm Δ. If A is a nonempty subset of X, then $D_A = H$ if and only if A consists of a single point.*

Theorem 2.20. [3] *Let $(X,\mathscr{F},\Delta)$ is a Menger PM space with a continuous $t-$norm Δ. If A and B are nonempty subsets of X and $A \subseteq B$, then $D_A \geq D_B$.*

Theorem 2.21. [3] *Let $(X,\mathscr{F},\Delta)$ is a Menger PM space with a continuous $t-$norm Δ. If A and B are two nonempty subsets of X such that $A\bigcap B = \emptyset$, then $D_{A\bigcup B}(x+y) \geq \Delta(D_A(x), D_B(y))$.*

Proposition 2.22. [1] *Let $(X,\mathscr{F},\Delta)$ is a Menger PM-space with a continuous $t-$norm Δ.*

(1) If A is a probabilistically bounded set, then $D_A(x)$ is a distribution function.

(2) If $A,B \subset X$ are any two probabilistically bounded sets, then $A\bigcup B$ is also is a probabilistically bounded set.

Definition 2.23. [1] Let $(X,\mathscr{F},\Delta)$ be a Menger PM-space with a continuous $t-$norm Δ and Ω is the family of all nonempty $\tau-$closed probabilistically bounded set. We defined a mapping $\hbar$ as follows (we denote $\hbar(A,B)$ by $h_{A,B}$ and the value of $h_{A,B}$ at $t\in R$ by $h_{A,B}(t)$):

$$h_{A,B}(x)=\sup_{t<x}\Delta(\inf_{p\in A}\sup_{q\in B}\mathscr{F}_{p,q}(t),\inf_{q\in B}\sup_{p\in A}\mathscr{F}_{p,q}(t)),\qquad A,B\in\Omega.$$

Then $\hbar$ is called the Menger-Hausdorff metric induced by F.

Proposition 2.24. [1] *Let $(\Omega,\hbar,\Delta)$ be a Menger PM-space. Then a mapping $\hbar$ from $\Omega\times\Omega$ into D satisfying the following condition:*

(1) $h_{A,B}(x)=1$ for all $x>0$ if and only if $A=B$,

(2) $h_{A,B}(0)=0$,

(3) $h_{A,B}=h_{B,A}$,

(4) $h_{A,B}(x+y)\geq\Delta(h_{A,C}(x),h_{C,B}(y))$ for all $A,B,C\in\Omega$ and $x,y\geq 0$.

Definition 2.25. [1] Let $(X,\mathscr{F},\Delta)$ be a Menger PM-space with a continuous $t-$norm Δ and Ω is the family of all nonempty $\tau-$closed probabilistically bounded set. The probabilistic distance between a point $x\in X$ and $A\in\Omega$ is the function $\mathscr{F}_{x,A}$ defined by

$$\mathscr{F}_{x,A}(t)=\sup_{s<t}\sup_{p\in A}\mathscr{F}_{x,p}(s),\qquad t\geq 0.$$

Proposition 2.26. [1] *Let $A\in\Omega$, and x,y be arbitrary points of X. Then*

(1) $\mathscr{F}_{x,A}(t)=1$ for all $t>0$ if and only if $x\in A$,

(2) $\mathscr{F}_{x,A}(t_1+t_2)\geq\Delta(\mathscr{F}_{x,y}(t_1),\mathscr{F}_{y,A}(t_2))$ for all $t_1,t_2\geq 0$,

(3) For any A,B and $x\in A$,

$$\mathscr{F}_{x,B}(t)\geq h_{A,B}(t),\qquad t\geq 0.$$

Definition 2.27. [1] Let $(X,\mathscr{F},\Delta)$ be a Menger PM-space, $A,B\in CB(X)$ ($CB(X)$ be the family of all nonempty $\tau-$closed subsets of X) and $x\in X$. we defined

(1) The probabilistic distance between A and B is the function $\mathscr{F}_{A,B}$ defined by

$$\mathscr{F}_{A,B}(t)=\sup_{s<t}\Delta(\inf_{x\in A}\sup_{y\in B}\mathscr{F}_{x,y}(s),\inf_{y\in B}\sup_{x\in A}\mathscr{F}_{x,y}(s)), forall\quad t\in R.$$

(2) The probabilistic distance between x and A is the function $\mathscr{F}_{x,A}$ defined by

$$\mathscr{F}_{x,A}(t) = \sup_{y\in A} \mathscr{F}_{x,y}(t), forall \quad t \in R.$$

Lemma 2.28. [1] *Let $(X, \mathscr{F}, \Delta)$ be a Menger PM-space, Δ be a left-continuous $t-$norm, $A \in CB(X)$ and $x, y \in X$. Then we have the following:*

(1) for any $B \in CB(X)$, $x \in A$ and $t \in R$,

$$\inf_{x\in A} \sup_{y\in B} \mathscr{F}_{x,y}(t) \leq \mathscr{F}_{x,B}(t),$$

(2) $\mathscr{F}_{x,A}(t) = 1$ for all $t > 0$ if and only if $x \in A$,

(3) $\mathscr{F}_{x,A}(t_1 + t_2) \geq \Delta(\mathscr{F}_{x,y}(t_1), \mathscr{F}_{y,A}(t_2))$ for all $t_1, t_2 \geq 0$,

(4) $\mathscr{F}_{x,A}(t)$ is a left-continuous functions at t.

Definition 2.29. [19] A topological space X is compact if every open cover of X has a finite subcover, i.e. if whenever $X = \bigcup_{i\in I} u_i$, for a collection of open sets $\{u_i : i \in I\}$ then we also have $X = \bigcup_{i\in A} u_i$, for some finite subset A of I.

Theorem 2.30. [19] *Let X be a compact topological space and let M be a closed subset of X. Then M is a compact topological space.*

Theorem 2.31. [19] *Suppose X is a Hausdorff topological space and that $M \subset X$ is a compact subspace. Then M is closed.*

Definition 2.32. [4] A PM-space $(X, \mathscr{F}, \Delta)$ is called precompact if for each $r > 0$, there is a finite subset A of X such that $X = \bigcup_{x\in A} u_x(r)$. in this case, we say that A is precompact probabilistic metric on X.

Theorem 2.33. [4] *Let $(X, \mathscr{F}, \Delta)$ be a PM-space and Δ be continuous. A PM-space $(X, \mathscr{F}, \Delta)$ is precompact if and only if every sequence has a Cauchy subsequence.*

Theorem 2.34. [4] *Let $(X, \mathscr{F}, \Delta)$ be a PM-space and Δ be continuous. If a Cauchy sequence clusters to a point $x \in X$, Then the sequence converges to x.*

Theorem 2.35. [4] *Let $(X, \mathscr{F}, \Delta)$ be a PM-space and Δ be continuous. A PM-space $(X, \mathscr{F}, \Delta)$ is compact if and only if is precompact and complete.*

3. Main results

In this section we shall introduce the notion of a compact probabilistic metric space and prove some fixed point theorems for multi-valued mappings in a compact probabilistic metric space.

Theorem 3.1. *Let $(X,\mathscr{F},\Delta)$ be a $\tau-$compact Menger PM-space where Δ be a continuous $t-$norm. Let μ be the family of all nonempty $\tau-$compact probabilistically bounded subsets and $\{T_i\}_{i\in N}$ be a sequence of multi-valued mappings $T_i : X \longrightarrow \mu$, $i = 1,2,\dots$. Suppose that there exists a constant $k > 1$ such that for any $i, j \in \mathbb{Z}^+$, $i \neq j$, and any $x,y \in X$,*

$$h_{T_ix,T_jy}(t) \geq min\{\mathscr{F}_{x,T_ix}(kt),\mathscr{F}_{y,T_jy}(kt)\}, \qquad t \geq 0.$$

Suppose further that for any $i \in \mathbb{Z}^+$, $x \in X$ and $u \in T_nx$, $n = 1,2,3,\dots$, there exists a point $v \in T_{n+1}u$ such that

$$\mathscr{F}_{u,v}(t) \geq \mathscr{F}_{v,T_iv}(t), \qquad t \geq 0.$$

Then the family $\{T_i : i = 1,2,..\}$ of multi-valued mappings has a common fixed point $x_* \in X$, i.e., there exists $x_* \in \bigcup_{i=1}^{\infty} T_ix_*$.

Proof. For any $x_0 \in X$, take $x_1 \in T_1x_0 \in \mu$. By the assumptions, there exists $x_2 \in T_2x_1 \in \mu$ such that

$$\mathscr{F}_{x_1,x_2}(t) \geq \mathscr{F}_{x_2,T_ix_2}(t), \qquad t \geq 0.$$

Similarly, there exists a point $x_3 \in T_3x_2 \in \mu$ such that

$$\mathscr{F}_{x_2,x_3}(t) \geq \mathscr{F}_{x_3,T_ix_3}(t), \qquad t \geq 0.$$

Continuing this procedure, we can obtain a sequence $\{x_n\}_{n\in N}$ in X satisfying the following conditions:

(i) $x_n \in T_nx_{n-1}, n = 1,2,\dots$.

(ii) $\mathscr{F}_{x_{n-1},x_n}(t) \geq \mathscr{F}_{x_n,T_ix_n}(t)$, for all $t \geq 0$.

From Theorem 2.35, we have $(X,\mathscr{F},\Delta)$ is $\tau-$precompact and $\tau-$complete. Now by Theorem 2.33, we have $\{x_n\}$ has a Cauchy subsequence $\{x_{\varphi(n)}\}_{n\in N}$ such that $x_{\varphi(n)} = x_{m_n} \in T_{m_n}x_{m_n-1}$ for some $m_n \in N$. From the $\tau-$completeness of $(X,\mathscr{F},\Delta)$, we can suppose a Cauchy subsequence $\tau-$converges to a point x_*.

Now we prove that x_* is a common fixed point of $\{T_i\}_{i=1}^{\infty}$. In fact, it follows from Proposition 2.26 (2) and (3) that

$$\begin{aligned}
\mathscr{F}_{x_*,T_ix_*}(t) &\geq \Delta\left(\mathscr{F}_{x_*,x_{\varphi(n)}}\left(\left(1-\frac{1}{\beta}\right)t\right), \mathscr{F}_{x_{\varphi(n)},T_ix_*}\left(\frac{t}{\beta}\right)\right) \\
&\geq \Delta\left(\mathscr{F}_{x_*,x_{\varphi(n)}}\left(\left(1-\frac{1}{\beta}\right)t\right), \mathscr{F}_{h_{T_{m_n}x_{m_n-1}},T_ix_*}\left(\frac{t}{\beta}\right)\right) \\
&\geq \Delta\left(\mathscr{F}_{x_*,x_{\varphi(n)}}\left(\left(1-\frac{1}{\beta}\right)t\right),\right. \\
&\qquad \left.\min\left\{\mathscr{F}_{x_{m_n-1},T_{m_n-1}x_{m_n-1}}\left(\frac{kt}{\beta}\right), \mathscr{F}_{x_*,T_ix_*}\left(\frac{kt}{\beta}\right)\right\}\right). \qquad (1)
\end{aligned}$$

where $\beta < k$ is constant and $k > \frac{\beta^2+1}{\beta}$. In addition, by Proposition 2.26 (1) and (2), we have

$$\begin{aligned}
\mathscr{F}_{x_{m_n-1},T_{m_n}x_{m_n-1}}\left(\frac{kt}{\beta}\right) &\geq \Delta\left(\mathscr{F}_{x_{m_n-1},x_{m_n}}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right), \mathscr{F}_{x_{m_n},T_{m_n}x_{m_n-1}}\left(\frac{t}{\beta^2}\right)\right) \\
&= \mathscr{F}_{x_{m_n-1},x_{m_n}}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right) \\
&\geq \mathscr{F}_{x_{m_n},T_ix_{m_n}}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right) \\
&= \mathscr{F}_{x_{\varphi(n)},T_ix_{\varphi(n)}}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right).
\end{aligned}$$

Substituting the above inequality into (1) and letting $n \longrightarrow \infty$, we have, by the continuity of Δ,

$$\mathscr{F}_{x_*,T_ix_*}(t) \geq \mathscr{F}_{x_*,T_ix_*}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right) \geq \ldots \geq \mathscr{F}_{x_*,T_ix_*}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)^m t\right), m = 1,2,3....$$

Letting $m \longrightarrow \infty$ on the right, we have for all $t > 0$ and $i = 1,2,...$,

$$\mathscr{F}_{x_*,T_ix_*}(t) = 1.$$

And so we have $x_* \in T_ix_*, i = 1,2,....$ Therefore, by Proposition 2.26 (1), we have

$$x_* \in \bigcup_{i=1}^{\infty} T_ix_*.$$

This is complete proof. □

Example 3.2. Let (X,d) be a metric space defined by $d(x,y)=|x-y|$ for all $x,y\in X$, where $X=[0,1]$ and defined $\mathscr{F}_{p,q}$ by

$$\mathscr{F}_{p,q}(t)=H(t-d(p,q))=\begin{cases} 0, & t\le d(p,q);\\ 1, & t>d(p,q).\end{cases}$$

Let $\{T_i\}_{i\in N}$ be a sequence of multi-valued mappings $T_i:X\longrightarrow\mu$, $i=1,2,...$, defined as $T_ix=[0,\frac{x}{2i}]$, $i=1,2,3,\dots$. It is clear that $(X,\mathscr{F},\Delta_m)$ is a $\tau-$compact Menger PM-space under a continuous $t-$norm Δ_m. By verifying that $\sup_{t>0}D_{T_ix}=1$ for all $x\in[0,1]$ and $i\in N$, then T_ix is a probabilistically bounded subset for all $x\in[0,1]$ and $i\in N$. Now by Theorem 2.30, we have μ be the family of nonempty $\tau-$compact probabilistically bounded subsets.

Now By verifying that there exists a constant $k>1$ such that for any $i,j\in\mathbb{Z}^+$, $i\neq j$, and any $x,y\in X$, $h_{T_ix,T_jy}(t)=1$ and $min\{\mathscr{F}_{x,T_ix}(kt),\mathscr{F}_{y,T_jy}(kt)\}=1$ for all $t>0$, then we have

$$h_{T_ix,T_jy}(t)\ge min\{\mathscr{F}_{x,T_ix}(kt),\mathscr{F}_{y,T_jy}(kt)\},\qquad t\ge 0.$$

By verifying that for any $i\in\mathbb{Z}^+$, $x\in X$ and $u\in T_nx$, $n=1,2,3,...$, there exists a point $v\in T_{n+1}u$ such that

$$\mathscr{F}_{u,v}(t)\ge\mathscr{F}_{v,T_iv}(t),\qquad t\ge 0.$$

Then the family $\{T_i:i=1,2,..\}$ of multi-valued mappings has a common fixed point.

Theorem 3.3. *Let $(X,\mathscr{F},\Delta)$ be a $\tau-$compact Menger PM-space where Δ be a continuous $t-$norm and $\Delta(a,a)\ge a$ for all $a\in[0,1]$. Let μ be the family of all nonempty $\tau-$compact probabilistically bounded sets and $\{T_i\}_{i\in N}$ be a sequence of multi-valued mappings $T_i:X\longrightarrow\mu$, $i=1,2,\dots$. Suppose that there exists a constant β, $0<\beta<1$ such that for any $i,j\in\mathbb{Z}^+$, $i\neq j$, and any $x,y\in X$,*

$$h_{T_ix,T_jy}(t)\ge min\left\{\mathscr{F}_{x,T_ix}\left(\frac{2}{\beta}t\right),\mathscr{F}_{y,T_jy}\left(\frac{2}{\beta}t\right),\mathscr{F}_{y,T_ix}\left(\frac{2}{\beta}t\right),\mathscr{F}_{x,T_jy}\left(\frac{2}{\beta}t\right)\right\},\qquad t\ge 0.$$

Suppose further that for any $i\in\mathbb{Z}^+$, $x\in X$ and $u\in T_nx$, $n=1,2,3,...$, there exists a point $v\in T_{n+1}u$ such that

$$\mathscr{F}_{u,v}(t)\ge\mathscr{F}_{v,T_iv}(t),\qquad t\ge 0.$$

Then the family $\{T_i:i=1,2,..\}$ of multi-valued mappings has a common fixed point $x_*\in X$, i.e., there exists $x_*\in\bigcup_{i=1}^{\infty}T_ix_*$.

Proof. For any $x_0 \in X$, take $x_1 \in T_1 x_0 \in \mu$. By the assumptions, there exists $x_2 \in T_2 x_1 \in \mu$ such that

$$\mathscr{F}_{x_1,x_2}(t) \geq \mathscr{F}_{x_2,T_i x_2}(t), \qquad t \geq 0.$$

Similarly, there exists a point $x_3 \in T_3 x_2 \in \mu$ such that

$$\mathscr{F}_{x_2,x_3}(t) \geq \mathscr{F}_{x_3,T_i x_3}(t), \qquad t \geq 0.$$

Continuing this procedure, we can obtain a sequence $\{x_n\}_{n\in N}$ in X satisfying the following conditions:

(i) $x_n \in T_n x_{n-1}, n = 1, 2, \dots$.

(ii) $\mathscr{F}_{x_{n-1},x_n}(t) \geq \mathscr{F}_{x_n,T_i x_n}(t)$, for all $t \geq 0$.

From Theorem 2.35, we have $(X, \mathscr{F}, \Delta)$ is $\tau-$precompact and $\tau-$complete. Now by Theorem 2.33, we have $\{x_n\}$ has a Cauchy subsequence $\{x_{\varphi(n)}\}_{n\in N}$ such that $x_{\varphi(n)} = x_{m_n} \in T_{m_n} x_{m_n-1}$ for some $m_n \in N$. From the $\tau-$completeness of $(X, \mathscr{F}, \Delta)$, we can suppose a Cauchy subsequence $\tau-$converges to a point x_*.

Now we prove that x_* is a common fixed point of $\{T_i\}_{i=1}^{\infty}$. In fact, it follows from Proposition 2.26 (2) and (3) that

$$\begin{aligned}
\mathscr{F}_{x_*,T_i x_*}(t) &\geq \Delta\left(\mathscr{F}_{x_*,x_{\varphi(n)}}\left(\left(1-\frac{1}{\beta}\right)t\right), \mathscr{F}_{x_{\varphi(n)},T_i x_*}\left(\frac{t}{\beta}\right)\right)\\
&\geq \Delta\left(\mathscr{F}_{x_*,x_{\varphi(n)}}\left(\left(1-\frac{1}{\beta}\right)t\right), h_{T_{m_n}x_{m_n-1},T_i x_*}\left(\frac{t}{\beta}\right)\right)\\
&\geq \Delta\left(\mathscr{F}_{x_*,x_{\varphi(n)}}\left(\left(1-\frac{1}{\beta}\right)t\right),\right.\\
&\left.\min\left\{\mathscr{F}_{x_{m_n-1},T_{m_n}x_{m_n-1}}\left(\frac{2t}{\beta^2}\right), \mathscr{F}_{x_*,T_i x_*}\left(\frac{2t}{\beta}\right), \mathscr{F}_{x_*,T_{m_n}x_{m_n-1}}\left(\frac{2t}{\beta^2}\right), \mathscr{F}_{x_{m_{n-1}},T_i x_*}\left(\frac{2t}{\beta^2}\right)\right\}\right).
\end{aligned} \tag{2}$$

In addition, by Proposition 2.26 (1) and (2), we have

$$\begin{aligned}
\mathscr{F}_{x_{m_n-1},T_{m_n}x_{m_n-1}}\left(\frac{2t}{\beta^2}\right) &\geq \Delta\left(\mathscr{F}_{x_{m_n-1},x_{m_n}}\left(\frac{t}{\beta^2}\right), \mathscr{F}_{x_{m_n},T_{m_n}x_{m_n-1}}\left(\frac{t}{\beta^2}\right)\right)\\
&= \mathscr{F}_{x_{m_n-1},x_{m_n}}\left(\frac{t}{\beta^2}\right)\\
&\geq \mathscr{F}_{x_{m_n},T_i x_{m_n}}\left(\frac{t}{\beta^2}\right)\\
&= \mathscr{F}_{x_{\varphi(n)},T_i x_{\varphi(n)}}\left(\frac{t}{\beta^2}\right).
\end{aligned}$$

Also,

$$\begin{aligned}\mathscr{F}_{x_*,T_{m_n}x_{m_n-1}}\left(\frac{2t}{\beta_2}\right) &\geq \Delta\left(\mathscr{F}_{x_*,x_{m_n}}\left(\frac{t}{\beta^2}\right),\mathscr{F}_{x_{m_n},T_{m_n}x_{m_n-1}}\left(\frac{t}{\beta^2}\right)\right)\\ &= \mathscr{F}_{x_*,x_{m_n}}\left(\frac{t}{\beta^2}\right)\\ &= \mathscr{F}_{x_*,x_{\varphi(n)}}\left(\frac{t}{\beta^2}\right).\end{aligned}$$

Also,

$$\begin{aligned}\mathscr{F}_{x_{m_n-1},T_ix_*}\left(\frac{2t}{\beta^2}\right) &\geq \Delta\left(\mathscr{F}_{x_{m_n-1},x_{m_n}}\left(\frac{t}{\beta^2}\right),\mathscr{F}_{x_{m_n},T_ix_*}\left(\frac{t}{\beta^2}\right)\right)\\ &\geq \Delta\left(\mathscr{F}_{x_{m_n},T_ix_{m_n}}\left(\frac{t}{\beta^2}\right),\mathscr{F}_{x_{m_n},T_ix_*}\left(\frac{t}{\beta^2}\right)\right)\\ &= \Delta\left(\mathscr{F}_{x_{\varphi(n)},T_ix_{\varphi(n)}}\left(\frac{t}{\beta^2}\right),\mathscr{F}_{x_{\varphi(n)},T_ix_*}\left(\frac{t}{\beta^2}\right)\right).\end{aligned}$$

Substituting the above inequalities into (2) and letting $n \longrightarrow \infty$, we have, by the continuity of Δ_m and $\Delta(a,a) \geq a$,

$$\begin{aligned}\mathscr{F}_{x_*,T_ix_*}(t) &\geq \Delta\left(1,\min\left\{\mathscr{F}_{x_*,T_ix_*}\left(\frac{t}{\beta^2}\right),\mathscr{F}_{x_*,T_ix_*}\left(\frac{2t}{\beta}\right),1,\mathscr{F}_{x_*,T_ix_*}\left(\frac{t}{\beta^2}\right)\right\}\right)\\ &= \mathscr{F}_{x_*,T_ix_*}\left(\frac{t}{\beta^2}\right) \geq \ldots \geq \mathscr{F}_{x_*,T_ix_*}\left(\left(\frac{1}{\beta^2}\right)^m t\right), m = 1,2,3....\end{aligned}$$

Letting $m \longrightarrow \infty$ on the right, we have for all $t > 0$ and $i = 1,2,...$,

$$\mathscr{F}_{x_*,T_ix_*}(t) = 1.$$

And so we have $x_* \in T_ix_*, i = 1,2,....$ Therefore, by Proposition 2.29 (1), we have

$$x_* \in \bigcup_{i=1}^{\infty} T_ix_*.$$

This is complete proof. □

Theorem 3.4. *Let $(X,\mathscr{F},\Delta)$ be a $\tau-$compact Menger PM-space where Δ be a continuous $t-$norm and $\Delta(a,a)\geq a$ for all $a\in[0,1]$. Let μ be the family of all nonempty $\tau-$compact probabilistically bounded subsets and $\{T_i\}_{i\in N}$ be a sequence of multi-valued mappings $T_i : X\longrightarrow\mu$, $i=1,2,\dots$. Suppose that there exists a constant $k>1$, such that for any $i,j\in\mathbb{Z}^+$, $i\neq j$, and any $x,y\in X$,*

$$h_{T_ix,T_jy}(t)\geq\Delta\left(\mathscr{F}_{x,T_ix}(kt),\mathscr{F}_{y,T_jy}(kt)\right),\qquad t\geq 0.$$

Suppose further that for any $i\in\mathbb{Z}^+$, $x\in X$ and $u\in T_nx$, $n=1,2,3,\dots$, there exists a point $v\in T_{n+1}u$ such that

$$\mathscr{F}_{u,v}(t)\geq\mathscr{F}_{v,T_iv}(t),\qquad t\geq 0.$$

Then the family $\{T_i : i=1,2,..\}$ of multi-valued mappings has a common fixed point $x_*\in X$, i.e., there exists $x_*\in\bigcup_{i=1}^{\infty}T_ix_*$.

Proof. For any $x_0\in X$, take $x_1\in T_1x_0\in\mu$. By the assumptions, there exists $x_2\in T_2x_1\in\mu$ such that

$$\mathscr{F}_{x_1,x_2}(t)\geq\mathscr{F}_{x_2,T_ix_2}(t),\qquad t\geq 0.$$

Similarly, there exists a point $x_3\in T_3x_2\in\mu$ such that

$$\mathscr{F}_{x_2,x_3}(t)\geq\mathscr{F}_{x_3,T_ix_3}(t),\qquad t\geq 0.$$

Continuing this procedure, we can obtain a sequence $\{x_n\}_{n\in N}$ in X satisfying the following conditions:

(i) $x_n\in T_nx_{n-1}, n=1,2,\dots$.

(ii) $\mathscr{F}_{x_{n-1},x_n}(t)\geq\mathscr{F}_{x_n,T_ix_n}(t)$, for all $t\geq 0$.

From Theorem 2.35, we have $(X,\mathscr{F},\Delta)$ is $\tau-$precompact and $\tau-$complete. Now by Theorem 2.33, we have $\{x_n\}$ has a Cauchy subsequence $\{x_{\varphi(n)}\}_{n\in N}$ such that $x_{\varphi(n)}=x_{m_n}\in T_{m_n}x_{m_n-1}$ for some $m_n\in N$. From the $\tau-$completeness of $(X,\mathscr{F},\Delta)$, we can suppose a Cauchy subsequence $\tau-$converges to a point x_*.

Now we prove that x_* is a common fixed point of $\{T_i\}_{i=1}^{\infty}$. In fact, it follows from Proposition

2.26 (2) and (3) that

$$
\begin{aligned}
\mathscr{F}_{x_*,T_ix_*}(t) &\geq \Delta\left(\mathscr{F}_{x_*,x_{\varphi(n)}}\left(\left(1-\frac{1}{\beta}\right)t\right),\mathscr{F}_{x_{\varphi(n)},T_ix_*}\left(\frac{t}{\beta}\right)\right)\\
&\geq \Delta\left(\mathscr{F}_{x_*,x_{\varphi(n)}}\left(\left(1-\frac{1}{\beta}\right)t\right),\mathscr{F}_{h_{T_{m_n}x_{m_n-1}},T_ix_*}\left(\frac{t}{\beta}\right)\right)\\
&\geq \Delta\left(\mathscr{F}_{x_*,x_{\varphi(n)}}\left(\left(1-\frac{1}{\beta}\right)t\right),\Delta\left(\mathscr{F}_{x_{m_n-1},T_{m_n-1}x_{m_n-1}}\left(\frac{kt}{\beta}\right),\mathscr{F}_{x_*,T_ix_*}\left(\frac{kt}{\beta}\right)\right)\right). \qquad (3)
\end{aligned}
$$

where $\beta < k$ is constant and $k > \frac{\beta^2+1}{\beta}$. In addition, by Proposition 2.26 (1) and (2), we have

$$
\begin{aligned}
\mathscr{F}_{x_{m_n-1},T_{m_n}x_{m_n-1}}\left(\frac{kt}{\beta}\right) &\geq \Delta\left(\mathscr{F}_{x_{m_n-1},x_{m_n}}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right),\mathscr{F}_{x_{m_n},T_{m_n}x_{m_n-1}}\left(\frac{t}{\beta^2}\right)\right)\\
&= \mathscr{F}_{x_{m_n-1},x_{m_n}}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right)\\
&\geq \mathscr{F}_{x_{m_n},T_ix_{m_n}}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right)\\
&= \mathscr{F}_{x_{\varphi(n)},T_ix_{\varphi(n)}}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right).
\end{aligned}
$$

Substituting the above inequality into (3) and letting $n \longrightarrow \infty$, we have, by the continuity of Δ,

$$
\begin{aligned}
\mathscr{F}_{x_*,T_ix_*}(t) &\geq \Delta\left(1,\Delta\left(\mathscr{F}_{x_*,T_ix_*}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right),\mathscr{F}_{x_*,T_ix_*}\left(\frac{kt}{\beta}\right)\right)\right)\\
&\geq \Delta\left(1,\Delta\left(\mathscr{F}_{x_*,T_ix_*}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right),\mathscr{F}_{x_*,T_ix_*}\mathscr{F}_{x_*,T_ix_*}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right)\right)\right)\\
&= \mathscr{F}_{x_*,T_ix_*}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right) \geq \ldots \geq \mathscr{F}_{x_*,T_ix_*}\left(\left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)^m t\right), m = 1,2,3....
\end{aligned}
$$

Letting $m \longrightarrow \infty$ on the right, we have for all $t > 0$ and $i = 1,2,...,$

$$\mathscr{F}_{x_*,T_ix_*}(t) = 1.$$

And so we have $x_* \in T_ix_*, i = 1,2,....$ Therefore, by Proposition 2.26 (1), we have

$$x_* \in \bigcup_{i=1}^{\infty} T_ix_*.$$

This is complete proof. □

Theorem 3.5. *Let $(X,\mathscr{F},\Delta)$ be a $\tau-$compact Menger PM-space where Δ be a continuous $t-$norm and $\Delta(a,a)\geq a$ for all $a\in[0,1]$. Let μ be the family of all nonempty $\tau-$compact probabilistically bounded subsets and $\{T_i\}_{i\in N}$ be a sequence of multi-valued mappings $T_i : X\longrightarrow \mu$, $i=1,2,\dots$. Suppose that there exists a constant β, $0<\beta<1$ such that for any $i,j\in\mathbb{Z}^+$, $i\neq j$, and any $x,y\in X$,*

$$h_{T_ix,T_jy}(t)\geq\Delta\left(\mathscr{F}_{y,T_ix}\left(\frac{2}{\beta}t\right),\mathscr{F}_{x,T_jy}\left(\frac{2}{\beta}t\right)\right),\qquad t\geq 0.$$

Suppose further that for any $i\in\mathbb{Z}^+$, $x\in X$ and $u\in T_nx$, $n=1,2,3,\ldots$, there exists a point $v\in T_{n+1}u$ such that

$$\mathscr{F}_{u,v}(t)\geq\mathscr{F}_{v,T_iv}(t),\qquad t\geq 0.$$

Then the family $\{T_i : i=1,2,..\}$ of multi-valued mappings has a common fixed point $x_*\in X$, i.e., there exists $x_*\in\bigcup_{i=1}^{\infty}T_ix_*$.

Proof. For any $x_0\in X$, take $x_1\in T_1x_0\in\mu$. By the assumptions, there exists $x_2\in T_2x_1\in\mu$ such that

$$\mathscr{F}_{x_1,x_2}(t)\geq\mathscr{F}_{x_2,T_ix_2}(t),\qquad t\geq 0.$$

Similarly, there exists a point $x_3\in T_3x_2\in\mu$ such that

$$\mathscr{F}_{x_2,x_3}(t)\geq\mathscr{F}_{x_3,T_ix_3}(t),\qquad t\geq 0.$$

Continuing this procedure, we can obtain a sequence $\{x_n\}_{n\in N}$ in X satisfying the following conditions:

(i) $x_n\in T_nx_{n-1}, n=1,2,\dots$.

(ii) $\mathscr{F}_{x_{n-1},x_n}(t)\geq\mathscr{F}_{x_n,T_ix_n}(t)$, for all $t\geq 0$.

From Theorem 2.35, we have $(X,\mathscr{F},\Delta)$ is $\tau-$precompact and $\tau-$complete. Now by Theorem 2.33, we have $\{x_n\}$ has a Cauchy subsequence $\{x_{\varphi(n)}\}_{n\in N}$ such that $x_{\varphi(n)}=x_{m_n}\in T_{m_n}x_{m_n-1}$ for some $m_n\in N$. From the $\tau-$completeness of $(X,\mathscr{F},\Delta)$, we can suppose a Cauchy subsequence $\tau-$converges to a point x_*.

Now we prove that x_* is a common fixed point of $\{T_i\}_{i=1}^{\infty}$. In fact, it follows from Proposition

2.26 (2) and (3) that

$$\begin{aligned}\mathscr{F}_{x_*,T_ix_*}(t) &\geq \Delta\left(\mathscr{F}_{x_*,x_{\varphi(n)}}\left(\left(1-\frac{1}{\beta}\right)t\right), \mathscr{F}_{x_{\varphi(n)},T_ix_*}\left(\frac{t}{\beta}\right)\right)\\ &\geq \Delta\left(\mathscr{F}_{x_*,x_{\varphi(n)}}\left(\left(1-\frac{1}{\beta}\right)t\right), h_{T_{m_n}x_{m_n-1},T_ix_*}\left(\frac{t}{\beta}\right)\right)\\ &\geq \Delta\left(\mathscr{F}_{x_*,x_{\varphi(n)}}\left(\left(1-\frac{1}{\beta}\right)t\right), \Delta\left(\mathscr{F}_{x_*,T_{m_n}x_{m_n-1}}\left(\frac{2t}{\beta^2}\right),\right.\right.\\ &\qquad \left.\left.\mathscr{F}_{x_{m_n-1},T_ix_*}\left(\frac{2t}{\beta^2}\right)\right)\right). \qquad (4)\end{aligned}$$

In addition, by Proposition 2.26 (1) and (2),we have

$$\begin{aligned}\mathscr{F}_{x_*,T_{m_n}x_{m_n-1}}\left(\frac{2t}{\beta^2}\right) &\geq \Delta\left(\mathscr{F}_{x_*,x_{m_n}}\left(\frac{t}{\beta^2}\right), \mathscr{F}_{x_{m_n},T_{m_n}x_{m_n-1}}\left(\frac{t}{\beta^2}\right)\right)\\ &= \mathscr{F}_{x_*,x_{m_n}}\left(\frac{t}{\beta^2}\right)\\ &= \mathscr{F}_{x_*,x_{\varphi(n)}}\left(\frac{t}{\beta^2}\right).\end{aligned}$$

Also,

$$\begin{aligned}\mathscr{F}_{x_{m_n-1},T_ix_*}\left(\frac{2t}{\beta^2}\right) &\geq \Delta\left(\mathscr{F}_{x_{m_n-1},x_{m_n}}\left(\frac{t}{\beta^2}\right), \mathscr{F}_{x_{m_n},T_{m_n}x_*}\left(\frac{t}{\beta^2}\right)\right)\\ &\geq \Delta\left(\mathscr{F}_{x_{m_n},T_ix_{m_n}}\left(\frac{t}{\beta^2}\right), \mathscr{F}_{x_{m_n},T_ix_*}\left(\frac{t}{\beta^2}\right)\right)\\ &= \Delta\left(\mathscr{F}_{x_{\varphi(n)},T_ix_{\varphi(n)}}\left(\frac{t}{\beta^2}\right), \mathscr{F}_{x_{\varphi(n)},T_ix_*}\left(\frac{t}{\beta^2}\right)\right)\end{aligned}$$

Substituting the above inequalities into (4) and letting $n \longrightarrow \infty$, we have, by the continuity of Δ and $\Delta(a,a) \geq a$,

$$\mathscr{F}_{x_*,T_ix_*}(t) \geq \mathscr{F}_{x_*,T_ix_*}\left(\frac{1}{\beta^2}t\right) \geq ... \geq \mathscr{F}_{x_*,T_ix_*}\left(\left(\frac{1}{\beta^2}\right)^m t\right), m = 1,2,3...$$

Letting $m \longrightarrow \infty$ on the right, we have for all $t > 0$ and $i = 1,2,...,$

$$\mathscr{F}_{x_*,T_ix_*}(t) = 1.$$

And so we have $x_* \in T_i x_*, i = 1, 2, \ldots$. Therefore, by Proposition 2.26 (1), we have

$$x_* \in \bigcup_{i=1}^{\infty} T_i x_*.$$

This is complete proof. □

Theorem 3.6. *Let $(X, \mathscr{F}, \Delta)$ be a $\tau-$compact Menger PM-space where Δ be a continuous $t-$norm and $\Delta(a,a) \geq a$ for all $a \in [0,1]$. Let μ be the family of all nonempty $\tau-$compact probabilistically bounded sets and $\{T_i\}_{i\in N}$ be a sequence of multi-valued mappings $T_i : X \longrightarrow \mu$, $i = 1, 2, \ldots$. Suppose that there exists a constant β, $0 < \beta < 1$ such that for any $i, j \in \mathbb{Z}^+$, $i \neq j$, and any $x, y \in X$,*

$$h_{T_ix,T_jy}(t) \geq \Delta\left(\Delta\left(\mathscr{F}_{x,T_ix}\left(\frac{2}{\beta}t\right), \mathscr{F}_{y,T_jy}\left(\frac{2}{\beta}t\right)\right), \Delta\left(\mathscr{F}_{y,T_ix}\left(\frac{2}{\beta}t\right), \mathscr{F}_{x,T_jy}\left(\frac{2}{\beta}t\right)\right)\right), \qquad t \geq 0.$$

Suppose further that for any $i \in \mathbb{Z}^+$, $x \in X$ and $u \in T_n x$, $n = 1, 2, 3, \ldots$, there exists a point $v \in T_{n+1}u$ such that

$$\mathscr{F}_{u,v}(t) \geq \mathscr{F}_{v,T_iv}(t), \qquad t \geq 0.$$

Then the family $\{T_i : i = 1, 2, ..\}$ of multi-valued mappings has a common fixed point $x_* \in X$, i.e., there exists $x_* \in \bigcup_{i=1}^{\infty} T_i x_*$.

Proof. For any $x_0 \in X$, take $x_1 \in T_1 x_0 \in \mu$. By the assumptions, there exists $x_2 \in T_2 x_1 \in \mu$ such that

$$\mathscr{F}_{x_1,x_2}(t) \geq \mathscr{F}_{x_2,T_ix_2}(t), \qquad t \geq 0.$$

Similarly, there exists a point $x_3 \in T_3 x_2 \in \mu$ such that

$$\mathscr{F}_{x_2,x_3}(t) \geq \mathscr{F}_{x_3,T_ix_3}(t), \qquad t \geq 0.$$

Continuing this procedure, we can obtain a sequence $\{x_n\}_{n\in N}$ in X satisfying the following conditions:

(i) $x_n \in T_n x_{n-1}, n = 1, 2, \ldots$.

(ii) $\mathscr{F}_{x_{n-1},x_n}(t) \geq \mathscr{F}_{x_n,T_ix_n}(t)$, for all $t \geq 0$.

From Theorem 2.35, we have $(X, \mathscr{F}, \Delta)$ is $\tau-$precompact and $\tau-$complete. Now by Theorem 2.33, we have $\{x_n\}$ has a Cauchy subsequence $\{x_{\varphi(n)}\}_{n\in N}$ such that $x_{\varphi(n)} = x_{m_n} \in T_{m_n} x_{m_n-1}$ for some $m_n \in N$. From the $\tau-$completeness of $(X, \mathscr{F}, \Delta)$, we can suppose a Cauchy subsequence

$\tau-$converges to a point x_*.

Now we prove that x_* is a common fixed point of $\{T_i\}_{i=1}^{\infty}$. In fact, it follows from Proposition 2.26 (2) and (3) that

$$\begin{aligned}
\mathscr{F}_{x_*,T_ix_*}(t) &\geq \Delta\left(\mathscr{F}_{x_*,x_{\varphi(n)}}\left(\left(1-\frac{1}{\beta}\right)t\right), \mathscr{F}_{x_{\varphi(n)},T_ix_*}\left(\frac{t}{\beta}\right)\right) \\
&\geq \Delta\left(\mathscr{F}_{x_*,x_{\varphi(n)}}\left(\left(1-\frac{1}{\beta}\right)t\right), h_{T_{m_n}x_{m_n-1},T_ix_*}\left(\frac{t}{\beta}\right)\right) \\
&\geq \Delta\left(\mathscr{F}_{x_*,x_{\varphi(n)}}\left(\left(1-\frac{1}{\beta}\right)t\right), \Delta\left(\Delta\left(\mathscr{F}_{x_{m_n-1},T_{m_n}x_{m_n-1}}\left(\frac{2t}{\beta^2}\right),\right.\right.\right. \\
&\left.\left.\left.\mathscr{F}_{x_*,T_ix_*}\left(\frac{2t}{\beta}\right)\right), \Delta\left(\mathscr{F}_{x_*,T_{m_n}x_{m_n-1}}\left(\frac{2t}{\beta^2}\right), \mathscr{F}_{x_{m_{n-1}},T_ix_*}\left(\frac{2t}{\beta^2}\right)\right)\right)\right). \qquad (5)
\end{aligned}$$

In addition, by Proposition 2.26 (1) and (2),we have

$$\begin{aligned}
\mathscr{F}_{x_{m_n-1},T_{m_n}x_{m_n-1}}\left(\frac{2t}{\beta^2}\right) &\geq \Delta\left(\mathscr{F}_{x_{m_n-1},x_{m_n}}\left(\frac{t}{\beta^2}\right), \mathscr{F}_{x_{m_n},T_{m_n}x_{m_n-1}}\left(\frac{t}{\beta^2}\right)\right) \\
&= \mathscr{F}_{x_{m_n-1},x_{m_n}}\left(\frac{t}{\beta^2}\right) \\
&\geq \mathscr{F}_{x_{m_n},T_ix_{m_n}}\left(\frac{t}{\beta^2}\right) \\
&= \mathscr{F}_{x_{\varphi(n)},T_ix_{\varphi(n)}}\left(\frac{t}{\beta^2}\right).
\end{aligned}$$

Also,

$$\begin{aligned}
\mathscr{F}_{x_*,T_{m_n}x_{m_n-1}}\left(\frac{kt}{\beta}\right) &\geq \Delta\left(\mathscr{F}_{x_*,x_{m_n}}\left(\frac{t}{\beta^2}\right), \mathscr{F}_{x_{m_n},T_{m_n}x_{m_n-1}}\left(\frac{t}{\beta^2}\right)\right) \\
&= \mathscr{F}_{x_*,x_{m_n}}\left(\frac{t}{\beta^2}\right) \\
&= \mathscr{F}_{x_*,x_{\varphi(n)}}\left(\frac{t}{\beta^2}\right).
\end{aligned}$$

Also,

$$\begin{aligned}\mathscr{F}_{x_{mn-1},T_ix_*}\left(\frac{2t}{\beta^2}\right) &\geq \Delta\left(\mathscr{F}_{x_{mn-1},x_{mn}}\left(\frac{t}{\beta^2}\right),\mathscr{F}_{x_{mn},T_ix_*}\left(\frac{t}{\beta^2}\right)\right)\\ &\geq \Delta\left(\mathscr{F}_{x_{mn},T_ix_{mn}}\left(\frac{t}{\beta^2}\right),\mathscr{F}_{x_{mn},T_ix_*}\left(\frac{t}{\beta^2}\right)\right)\\ &= \Delta\left(\mathscr{F}_{x_{\varphi(n)},T_ix_{\varphi(n)}}\left(\frac{t}{\beta^2}\right),\mathscr{F}_{x_{\varphi(n)},T_ix_*}\left(\frac{t}{\beta^2}\right)\right).\end{aligned}$$

Substituting the above inequalities into (5) and letting $n \longrightarrow \infty$, we have, by the continuity of Δ_m and $\Delta(a,a) \geq a$,

$$\begin{aligned}\mathscr{F}_{x_*,T_ix_*}(t) &\geq \Delta\left(1,\Delta\left(\Delta\left(\mathscr{F}_{x_*,T_ix_*}\left(\frac{t}{\beta^2}\right),\mathscr{F}_{x_*,T_ix_*}\left(\frac{t}{\beta}\right)\right),\Delta\left(1,\mathscr{F}_{x_*,T_ix_*}\left(\frac{t}{\beta^2}\right)\right)\right)\right)\\ &\geq \mathscr{F}_{x_*,T_ix_*}\left(\frac{t}{\beta^2}\right) \geq \ldots \geq \mathscr{F}_{x_*,T_ix_*}\left(\left(\frac{1}{\beta^2}\right)^m t\right), m = 1,2,3....\end{aligned}$$

Letting $m \longrightarrow \infty$ on the right, we have for all $t > 0$ and $i = 1,2,...$,

$$\mathscr{F}_{x_*,T_ix_*}(t) = 1.$$

And so we have $x_* \in T_ix_*, i = 1,2,....$ Therefore, by Proposition 2.26 (1), we have

$$x_* \in \bigcup_{i=1}^{\infty} T_ix_*.$$

This is complete proof. □

Conflict of Interests

The authors declare that there is no conflict of interests.

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