Available online at http://jfpt.scik.org
J. Fixed Point Theory, 2019, 2019:12
ISSN: 2052-5338

APPLICATIONS TO FRACTIONAL EVOLUTION EQUATIONS USING KRASNOSELSKII TRIPLED FIXED POINT THEOREMS

ANIMESH GUPTA

H.No. 93/654, Ward No. 2 Gandhi Chowk Pachmarhi (M.P.) - India

Abstract. In this paper, we first prove some tripled fixed point theorems of Krasnoselskii type in partially ordered Φ-orbitally complete normed linear spaces and then apply the obtained fixed point theorems to a class of semi-linear evolution systems of fractional order for proving the existence of tripled mild solutions under some weaker monotone conditions. An example is given to illustrate the application of the abstract results.

Keywords: tripled fixed point theorem; nonlinear fractional evolution system; equi-continuous C_0-semigroup; tripled mild solution existence.

2010 AMS Subject Classification: 34A08, 47H10.

1. Introduction

We recall some definitions of the monotone mapping. A mapping $\Psi : E \to E$ is either monotone nondecreasing or monotone nonincreasing. Let $(E, \leq)$ is a partially ordered set and $\Psi : E \to E$. For $x, y \in E$, if $x \leq y$ implies $\Psi(x) \leq \Psi(y)$, Ψ is called a monotone nondecreasing mapping in E. Similarly, we can define a monotone nonincreasing mapping in E. If for $x_1, x_2 \in E$, $x_1 \leq x_2$ implies $\Psi(x_1, y) \leq \Psi(x_2, y)$ for all $y \in E$, while for $y_1, y_2 \in E$, $y_1 \leq y_2$ implies $\Psi(x, y_1) \geq \Psi(x, y_2)$ for all $x \in E$, then we say that Ψ is a mixed monotone mapping in E.

*Corresponding author

E-mail address: dranimeshgupta@gmail.com

Received April 25, 2019

The above mentioned fixed point theorems, see [1, 11, 16, 17, 18, 19], are all for the monotone mapping.

Most recently, the concept of a triple fixed point has been studied in partially ordered complete metric spaces for nonlinear contractions by Berinde and Borcut [2], who obtained the existence and uniqueness theorems for contractive type mappings in this setup which was later on studied by many authors. A large list of references can be found, for example, in [7, 8, 9, 10] studied some tripled fixed point theorems by using monotone property.

In our results, we neither assume that the mapping Ψ is mixed monotone, nor assume that it is a contraction. We divide the mapping Ψ into two parts, and assume that every part satisfies some conditions, by using an existing Krasnoselskii-type fixed point theorem, a tripled fixed point theorem for the mapping Ψ is proved. As applications, we apply the obtained tripled fixed point theorem to a certain abstract fractional evolution systems for proving the existence of tripled mild solutions.

The rest of this paper is organized as follows. In Section 2, some definitions are recalled and an existing Krasnoselskii-type fixed point theorem is introduced. In Section 3, tripled fixed point theorems are proved. In Section 4, we apply the obtained tripled fixed point theorem to a certain abstract fractional evolution systems. A specific example is given in Section 5 to illustrate the abstract results.

2. Preliminaries

Let E be a partially ordered normed linear space with partial order $\leq$ and the norm $\|\cdot\|_E$. If two elements $x, y \in E$ satisfy either $x \leq y$ or $x \geq y$, we say that they are comparable. If E is complete with respect to the norm $\|\cdot\|_E$, we called it a partially ordered complete normed linear space. We pick definitions 1—5 which are found in [4, 5].

Definition 1. *Let $\Psi : E \to E$ be a mapping. For any $x \in E$, we define an orbit $\Phi(x;\Psi)$ by*

$$\Phi(x;\Psi) = \{x, \Psi x, \Psi^2 x, \ldots, \Psi^n x, \ldots\}.$$

If for any sequence $\{x_n\} \subset \Phi(x;\Psi)$, $x_n \to x^*$ implies $\Psi x_n \to \Psi x^*$ for each $x \in E$, Ψ is said to be $\Phi-$orbitally continuous in E. A normed linear space $(E, \|\cdot\|_E)$ is called $\Phi-$orbitally complete if every Cauchy sequence $\{x_n\} \subset \Phi(x;\Psi)$ converges to a point x^* in E.

Definition 2. *A function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ is called a $\mathfrak{D}$-function if it is a monotone nondecreasing and upper semi-continuous function satisfying $\phi(0) = 0$.*

Definition 3. *A mapping $\Psi : E \to E$ is called partially nonlinear $\mathfrak{D}$-Lipschitz if for all comparable elements $x, y \in E$, there is a $\mathfrak{D}$-function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that*

$$\|\Psi x - \Psi y\|_E \leq \phi\big(\|x-y\|_E\big).$$

Furthermore, if $\phi(r) < r$ for $r > 0$, Ψ is called a partially nonlinear $\mathfrak{D}$-contraction in E.

Definition 4. *A mapping $\Psi : E \to E$ is said to be partially compact if for all totally ordered sets or chains $C \subset E$, $\Psi(C)$ is a relatively compact subset of E.*

Definition 5. *The norm $\|\cdot\|_E$ and the order relation $\leq$ on a partially ordered normed linear space $(E, \leq, \|\cdot\|_E)$ are said to be compatible if for any monotone sequence $\{x_n\}$ in E, subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x^*implies that the whole sequence $\{x_n\}$ converges to x^*.*

Remark 6. *From Definition 1, if the normed linear space $(E, \|\cdot\|_E)$ is complete, then it is $\Phi-$orbitally complete. But the converse expression may not be true.*

Remark 7. *The $\mathfrak{D}$-functions and the partially nonlinear $\mathfrak{D}$-Lipschitz conditions are much useful in research of solutions for nonlinear differential equations via fixed point theorems; see* [6].

The following Krasnoselskii-type fixed point theorem was proved by Dhage in [4].

Lemma 8. *Let E be a partially ordered complete normed linear space with the partial order $\leq$ and the norm $\|\cdot\|_E$ such that $\leq$ and $\|\cdot\|_E$ are compatible. Suppose that $A_1, A_2 : E \to E$ are two monotone nondecreasing mappings satisfying:*

(a) A_1 is continuous and a partially nonlinear $\mathfrak{D}$-contraction,

(b) A_2 is continuous and partially compact,

(c) there is an element $v_0 \in E$ satisfying $v_0 \leq A_1 v_0 + A_2 y$ for all $y \in E$, and

(d) every pair of elements has an upper and a lower bound in E.

Then $x = A_1 x + A_2 x$ has a solution in E.

3. Fixed point theorems

Let $(E, \|\cdot\|_E)$ be a -orbitally complete normed linear space. Define a positive cone K in E by

$$K = \{x \in E : x \geq 0\}.$$

Then E becomes now a partially ordered $\Phi-$orbitally complete normed linear space with the partial order $\leq$ induced by K. It is well known that the partially order $\leq$ and the norm $\|\cdot\|_E$ are compatible if cone K is normal. By Lemma 8, we first prove the following fixed point theorem.

Theorem 9. *Let E be a partially ordered $\Phi-$orbitally complete normed linear space with the norm $\|\cdot\|_E$ and the partial order $\leq$, whose positive cone K is normal, and let D be a nonempty closed subset of E. Assume that $A_1, A_2, A_3 : D \to D$ are three monotone nondecreasing mappings satisfying:*

(a) A_1 is $\Phi-$orbitally continuous and

$$\|A_1x - A_1y\|_E < \|x-y\|_E$$

for all comparable elements $x, y \in D$ with $x \not\equiv y$,

(b) A_2, A_3 are $\Phi-$orbitally continuous and partially compact,

(c) there exists an element $v_0 \in D$ such that $v_0 \leq A_1v_0 + A_2y +, A_3z$ for all $y, z \in D$, and

(d) every pair of elements in D has an upper and a lower bound.

Then the equation $x = A_1x + A_2x + A_3x$ has a solution in D.

Proof. By Remark 7 of [4] and the proof of Lemma 8, if the continuity of operators A_1, A_2 and $,A_3$ is replaced by the $\Phi-$orbitally continuity in conditions (a) and (b) of Lemma 8, the conclusion of Lemma 8 is still true. On the other hand, since D is a nonempty closed subset of E, it follows that D is $\Phi-$orbitally complete and has partial order $\leq$. For any comparable elements $x, y \in D$, $x \not\equiv y$, by the condition (a), it follows that there is $\tau \in (0,1)$ satisfying

$$\|A_1x - A_1y\|_E \leq \tau\|x-y\|_E < \|x-y\|_E.$$

Let $\phi(r) = \tau r$. Then Φ is a $\mathfrak{D}$-function and $\phi(r) < r$ for any $r > 0$. This implies that the condition (a) of Lemma 8 is satisfied. By Lemma 8, we obtain the desired conclusion. □

If the inequality given in assumption (c) of Lemma 8 is reverse, more precisely, the condition (c) of Lemma 8 is replaced by (c) there is $v_0 \in D$ satisfying $v_0 \geq A_1 v_0 + A_2 y$ for all $y \in D$. By Theorem **??** of [4], the conclusion of Lemma 8 is still true. Hence we can obtain the following fixed point theorem. Since the proof is similar to Theorem 9, we omit the details here.

Theorem 10. *Let E be a partially ordered $\Phi-$orbitally complete normed linear space with the norm $\|\cdot\|_E$ and the partial order $\leq$, whose positive cone K is normal, and let D be a nonempty closed subset of E. Assume that $A_1, A_2, A_3 : D \to D$ are three monotone nondecreasing mappings satisfying the conditions (a), (b), (c) and (d). Then $x = A_1 x + A_2 x +, A_3 x$ has a solution in D.*

Let $\widehat{E} := E \times E \times E$. Define a sum and a scalar multiplication in $\widehat{E}$ by

$$w + v = (w_1, w_2, w_3) + (v_1, v_2, v_3) = (w_1 + v_1, w_2 + v_2, w_3 + v_3),$$

$$\lambda w = \lambda(w_1, w_2, w_3) = (\lambda w_1, \lambda w_2, \lambda w_3)$$

for all $w = (w_1, w_2, w_3), v = (v_1, v_2, v_3) \in \widehat{E}$ and $\lambda \in \mathbb{R}$. And define a positive cone and a norm in $\widehat{E}$ by

$$K_{\widehat{E}} = \{w = (w_1, w_2, w_3) \in \widehat{E} : w_1, w_2, w_3 \in K\},$$

$$\|v\|_{\widehat{E}} = \|(v_1, v_2, v_3)\|_{\widehat{E}} = \|v_1\|_E + \|v_2\|_E + \|v_3\|_E, \quad v = (v_1, v_2, v_3) \in \widehat{E}.$$

Then $(\widehat{E}, \|\cdot\|_{\widehat{E}})$ is a partially ordered normed linear space with the order relation $\leq$ induced by $K_{\widehat{E}}$. Let D be a nonempty closed subset of E. Then $D \times D \times D$ is a nonempty closed subset of $\widehat{E}$. It is clear that cone $K_{\widehat{E}}$ is normal if K is normal. Let $Q : \widehat{E} \to E$. A pair of elements $(x, y, z) \in \widehat{E}$ is called a tripled fixed point of $Q : \widehat{E} \to E$ if and only if it satisfies

$$x = Q(x, y, z), \qquad y = Q(y, x, y), \qquad z = Q(z, y, x).$$

By Theorem 9, the following tripled fixed point theorem is obtained.

Theorem 11. *Let E be a partially ordered $\Phi-$orbitally complete normed linear space with the partial order $\leq$ and the norm $\|\cdot\|_E$, whose positive cone K is normal, and let D be a nonempty closed subset of E. Assume that $A_1, A_2, A_3 : D \to D$ are three monotone nondecreasing mappings satisfying*

(i) A_1 is $\Phi-$orbitally continuous and a partially nonlinear $\mathfrak{D}$-contraction,

(ii) A_2 *and* A_3 *are* $\Phi-$*orbitally continuous and partially compact,*

(iii) there is an element $v_0 \in D$ *satisfying* $v_0 \leq A_1 v_0 + A_2 y + A_3 z$ *for all* $y, z \in D$*, and*

(iv) every pair of elements in D has an upper and a lower bound.

Then $Q(x,y,z) = A_1 x + A_2 y + A_3 z$ *has a tripled fixed point in* $\widehat{E}$*.*

Proof. Since $(E, \leq, \|\cdot\|_E)$ is a partially ordered $\Phi-$orbitally complete normed linear space, and positive cone K is normal, it follows that $(\widehat{E}, \leq, \|\cdot\|_{\widehat{E}})$ is a partially ordered $\Phi-$orbitally complete normed linear space and positive cone $K_{\widehat{E}}$ is normal. Since D is a nonempty closed subset of E, it follows that $D \times D \times D$ is a nonempty closed subset of $\widehat{E}$.

Let $\widehat{D} = D \times D \times D$. Define two operators $\widehat{A}_1, \widehat{A}_2, \widehat{A}_3 : \widehat{D} \to \widehat{D}$ by

$$\widehat{A}_1 u = (A_1 x, A_1 y, A_1 z), \qquad \widehat{A}_2 u = (A_2 y, A_2 x, A_2 y), \qquad \widehat{A}_3 u = (A_3 z, A_3 y, A_3 x),$$

for all $u = (x,y,z) \in \widehat{D}$.

If the operator equation $u = \widehat{A}_1 u + \widehat{A}_2 u + \widehat{A}_3 u$ has a solution $u = (x,y,z) \in \widehat{D}$, namely,

$$(x,y,z) = u = \widehat{A}_1 u + \widehat{A}_2 u + \widehat{A}_3 u = \widehat{A}_1 (x,y,z) + \widehat{A}_2 (x,y,z) + + \widehat{A}_3 (x,y,z),$$

then we obtain

$$\begin{aligned}
(x,y,z) &= \widehat{A}_1 u + \widehat{A}_2 u + \widehat{A}_3 u \\
&= (A_1 x, A_1 y, A_1 z) + (A_2 y, A_2 x, A_2 y) + (A_3 z, A_3 y, A_3 x) \\
&= (A_1 x + A_2 y + A_3 z, A_1 y + A_2 x + A_3 y, A_1 z + A_2 y + A_3 x) \\
&= \big(Q(x,y,z), Q(y,x,y), Q(z,y,x)\big).
\end{aligned}$$

This implies that the operator $Q(x,y,z)$ has a tripled fixed point in $\widehat{E}$. We will apply Theorem 9 to prove that the operator equation $u = \widehat{A}_1 u + \widehat{A}_2 u + \widehat{A}_3 u$ has a solution in $\widehat{E}$. The proof will be given in several steps.

Step I. $\widehat{A}_1$ is $\Phi-$orbitally continuous and

$$\|\widehat{A}_1 w - \widehat{A}_1 v\|_{\widehat{E}} < \|w - v\|_{\widehat{E}},$$

for all comparable elements $w, v \in \widehat{D}$ with $w \not\equiv v$. Since A_1 is $\Phi-$orbitally continuous, by the definition of $\widehat{A}_1$, it is easy to see that $\widehat{A}_1$ is $\Phi-$orbitally continuous.

For all comparable elements $w=(w_1,w_2,w_3), v=(v_1,v_2,v_3)\in\widehat{D}$ with $w\not\equiv v$, we have

$$\begin{aligned}\|\widehat{A}_1w-\widehat{A}_1v\|_{\widehat{E}} &= \big\|(A_1w_1,A_1w_2,A_1w_3)-(A_1v_1,A_1v_2,A_1v_3)\big\|_{\widehat{E}}\\ &= \big\|(A_1w_1-A_1v_1),(A_1w_2-A_1v_2),(A_1w_3-A_1v_3)\big\|_{\widehat{E}}\\ &= \|A_1w_1-A_1v_1\|_E+\|A_1w_2-A_1v_2\|_E+\|A_1w_3-A_1v_3\|_E\\ &\le \varphi\big(\|w_1-v_1\|_E\big)+\varphi\big(\|w_2-v_2\|_E\big)+\varphi\big(\|w_3-v_3\|_E\big)\\ &< \|w_1-v_1\|_E+\|w_2-v_2\|_E+\|w_3-v_3\|_E\\ &= \big\|(w_1-v_1,w_2-v_2,w_3-v_3)\big\|_{\widehat{E}}\\ &= \big\|(w_1,w_2,w_3)-(v_1,v_2,v_3)\big\|_{\widehat{E}}\\ &= \|w-v\|_{\widehat{E}}.\end{aligned}$$

Hence, we obtain

$$\|\widehat{A}_1w-\widehat{A}_1v\|_{\widehat{E}}<\|w-v\|_{\widehat{E}}$$

for all comparable elements $w,v\in\widehat{D}$ with $w\not\equiv v$.

Step II. $\widehat{A}_2$ and $\widehat{A}_3$ are $\Phi-$orbitally continuous and partially compact. Since A_2 and A_3 are $\Phi-$orbitally continuous, by the definition of $\widehat{A}_2$ and $\widehat{A}_3$, it is easy to see that $\widehat{A}_2$ and $\widehat{A}_3$ are $\Phi-$orbitally continuous.

Let $C\subset D$ be a bounded chain. Since $A_2$and A_3 are partially compact in D, it follows that $A_2(C)$ and $A_3(C)$ are equi-continuous and uniformly bounded in D. Set $\widehat{C}=C\times C\times C$. Then $\widehat{C}$ is a bounded chain in $\widehat{D}$.

Next, we claim that $\widehat{A}_2(\widehat{C}), \widehat{A}_3(\widehat{C})$ are equi-continuous and uniformly bounded in $\widehat{D}$.

Since $A_2(C)\subset D$ and $A_3(C)\subset D$ uniformly bounded, there is a constant $\overline{M}>0$ satisfying $\|A_2p\|_E\le\overline{M}$ and $\|A_3p\|_E\le\overline{M}$ for any $p\in C$. For any $P\in\widehat{A}_2(\widehat{C})$ and $P\in\widehat{A}_3(\widehat{C})$, there are $x,y,z\in C$ satisfying $u=(x,y,z)\in\widehat{C}$ such that $P=\widehat{A}_2u$ and $P=\widehat{A}_3u$. Thus,

$$\begin{aligned}\|P\|_{\widehat{E}}=\|\widehat{A}_2u\|_{\widehat{E}} &= \big\|(A_2y,A_2x,A_2y)\big\|_{\widehat{E}}\\ &= \|A_2y\|_E+\|A_2x\|_E+\|A_2y\|_E\\ &\le 3\overline{M}.\end{aligned}$$

and

$$\begin{aligned}\|P\|_{\widehat{E}} &= \|\widehat{A}_3 u\|_{\widehat{E}} = \big\|(A_3 z, A_3 y, A_3 x)\big\|_{\widehat{E}} \\ &= \|A_3 z\|_E + \|A_3 y\|_E + \|A_3 x\|_E \\ &\leq 3\overline{M}.\end{aligned}$$

This implies that $\widehat{A}_2(\widehat{C})$ and $\widehat{A}_3(\widehat{C})$ are uniformly bounded in $\widehat{D}$. Since $A_2(C)$ and $A_3(C)$ are equi-continuous in D, for any $p \in C$ and $t_2 > t_1$, we have

$$\big\|(A_2 p)(t_2) - (A_2 p)(t_1)\big\|_E \to 0$$

and

$$\big\|(A_3 p)(t_2) - (A_3 p)(t_1)\big\|_E \to 0$$

as $t_2 - t_1 \to 0$. Hence for any $P = \widehat{A}_2 u = \widehat{A}_2(x,y,z) \in \widehat{A}_2(\widehat{C})$, we have

$$\begin{aligned}&\big\|P(t_2) - P(t_1)\big\|_{\widehat{E}} = \big\|(\widehat{A}_2 u)(t_2) - (\widehat{A}_2 u)(t_1)\big\|_{\widehat{E}} \\ &= \Big\|\big((A_2 y)(t_2), (A_2 x)(t_2), (A_2 y)(t_2)\big) - \big((A_2 y)(t_1), (A_2 x)(t_1), (A_2 y)(t_1)\big)\Big\|_{\widehat{E}} \\ &= \big\|(A_2 y)(t_2) - (A_2 y)(t_1), (A_2 x)(t_2) - (A_2 x)(t_1), (A_2 y)(t_2) - (A_2 y)(t_1)\big\|_{\widehat{E}} \\ &= \big\|(A_2 y)(t_2) - (A_2 y)(t_1)\big\|_E + \big\|(A_2 x)(t_2) - (A_2 x)(t_1)\big\|_E + \big\|(A_2 y)(t_2) - (A_2 y)(t_1)\big\|_E \\ &\to 0\end{aligned}$$

and for $P = \widehat{A}_3 u = \widehat{A}_3(x,y,z) \in \widehat{A}_3(\widehat{C})$, we have

$$\begin{aligned}&\big\|P(t_2) - P(t_1)\big\|_{\widehat{E}} = \big\|(\widehat{A}_3 u)(t_2) - (\widehat{A}_3 u)(t_1)\big\|_{\widehat{E}} \\ &= \Big\|\big((A_3 z)(t_2), (A_3 y(t_2), (A_3 x)(t_2)\big) - \big((A_3 z)(t_1), (A_3 y)(t_1), (A_3 x)(t_1)\big)\Big\|_{\widehat{E}} \\ &= \big\|(A_3 z)(t_2) - (A_3 z)(t_1), (A_3 y)(t_2) - (A_3 y)(t_1), (A_3 x)(t_2) - (A_3 x)(t_1)\big\|_{\widehat{E}} \\ &= \big\|(A_3 z)(t_2) - (A_3 z)(t_1)\big\|_E + \big\|(A_3 y)(t_2) - (A_3 y)(t_1)\big\|_E + \big\|(A_3 x)(t_2) - (A_3 x)(t_1)\big\|_E \\ &\to 0\end{aligned}$$

as $t_2 - t_1 \to 0$. This implies that $\widehat{A}_2(\widehat{C})$ and $\widehat{A}_3(\widehat{C})$ are equi-continuous in $\widehat{D}$. Therefore, by the Arzela-Ascoli theorem, $\widehat{A}_2(\widehat{C}) \subset \widehat{D}$ and $\widehat{A}_3(\widehat{C}) \subset \widehat{D}$ are relatively compact. Consequently, $\widehat{A}_2 : \widehat{D} \to \widehat{D}$ and $\widehat{A}_3 : \widehat{D} \to \widehat{D}$ are partially compact.

Step III. There is an element $u_0 \in \widehat{D}$ satisfying

$$u_0 \leq \widehat{A}_1 u_0 + \widehat{A}_2 u + \widehat{A}_3 u$$

for all $u \in \widehat{D}$.

Let $u_0 = (v_0, v_0, v_0)$. For any $u = (x, y, z) \in \widehat{D}$, by the condition (iii), we have

$$\begin{aligned}
\widehat{A}_1 u_0 + \widehat{A}_2 u + \widehat{A}_3 u &= (A_1 v_0, A_1 v_0, A_1 v_0) + (A_2 y, A_2 x, A_2 y) + (A_3 z, A_3 y, A_3 x) \\
&= (A_1 v_0 + A_2 y + A_3 z, A_1 v_0 + A_2 x + A_3 y, A_1 v_0 + A_2 y + A_3 x) \\
&\geq (v_0, v_0, v_0) \\
&= u_0.
\end{aligned}$$

Hence we obtain the desired conclusion.

Step IV. Every pair of elements in $\widehat{D}$ has an upper and a lower bound. For every pair of elements $w = (w_1, w_2, w_3), v = (v_1, v_2, v_3) \in \widehat{D}$, by condition (iv), there exist $p_1, p_2, p_3, p_4, p_5, p_6 \in D$ such that

$$\begin{aligned}
&p_1 \leq w_1, \quad && p_1 \leq v_1, \quad && p_4 \geq w_1, \quad && p_4 \geq v_1, \\
&p_2 \leq w_2, && p_2 \leq v_2, && p_5 \geq w_2, && p_5 \geq v_2 \\
&p_3 \leq w_3, && p_3 \leq v_3, && p_6 \geq w_3, && p_6 \geq v_3.
\end{aligned}$$

Thus, we have

$$(p_1, p_2, p_3) \leq (w_1, w_2, w_3) \leq (p_4, p_5, p_6),$$

$$(p_1, p_2, p_3) \leq (v_1, v_2, v_3) \leq (p_4, p_5, p_6).$$

Consequently, every pair of elements $w, v \in \widehat{D}$ has an upper and a lower bound. Therefore, by Theorem 9, the operator equation $u = \widehat{A}_1 u + \widehat{A}_2 u + \widehat{A}_3 u$ has a solution in $\widehat{E}$.

□

By Theorems 10 and 11, the following tripled fixed point theorem is obtained. Because its proof is similar to Theorem 11, we omit the details.

Theorem 12. *Let E be a partially ordered Φ-orbitally complete normed linear space with the norm $\|\cdot\|_E$ and the order relation $\leq$, positive cone K be normal, and let D be a nonempty closed subset of E. Assume that $A_1, A_2, A_3 : D \to D$ are three monotone nondecreasing mappings satisfying* *(i)* A_1 *is* Φ*-orbitally continuous and a partially nonlinear* $\mathfrak{D}$*-contraction,*

(ii) A_2 *and* A_3 *are* Φ*-orbitally continuous and partially compact,*

(iii)' there is an element $v_0 \in D$ *satisfying* $v_0 \geq A_1 v_0 + A_2 y + A_3 z$ *for all* $y,z \in D$*, and*

(iv) every pair of elements in D has an upper and a lower bound.

Then $Q(x,y,z) = A_1 x + A_2 y + A_3 z$ *has a tripled fixed point in* $\widehat{E}$*.*

Remark 13. *The hypothesis (iv) of Theorems 11 and 12 holds if the partially ordered set E is a lattice. We known that the set* $C(J,X)$ *is a lattice, where* $C(J,X)$ *is the set of all continuous X-valued functions on* $J \in \mathbb{R}$*, X is a partially ordered set. For any* $x,y \in C(J,X)$*,* $\max\{x,y,z\}$ *and* $\min\{x,y,z\}$ *are the upper and lower bounds, respectively.*

Remark 14. *The assumptions of mixed monotone property and contractive property of the mapping Q are essential in* [13, 15, 20]. *But in Theorems 11 and 12, we neither assume that the mapping Q is mixed monotone, nor assume that the mapping Q is a contraction. We only suppose that the mapping Q is a nondecreasing mapping and a part of Q (namely, the operator* A_1*) is a partially nonlinear* $\mathscr{D}$*-contraction. Plus with other assumptions we obtain the tripled fixed point theorems. Hence Theorems 11 and 12 extend the main results of* [13, 15, 20].

4. Existence results for fractional evolution systems

Let $(X, \|\cdot\|)$ be a Φ-orbitally complete normed linear space. Define its positive cone as $\overline{K} = \{x \in X : x \geq 0\}$. Then X becomes a partially ordered Φ-orbitally complete normed linear space with the norm $\|\cdot\|$ and the partial order $\leq$ induced by the cone $\overline{K}$. In this section, we always assume that $\overline{K}$ is normal. Investigate the existence of tripled mild solutions to the initial value problem of the fractional hybrid evolution system

$$
\begin{aligned}
{}^{\mathrm{C}}D_t^{\sigma} x(t) + Ax(t) &= f_1(t,x(t)) + f_2(t,y(t)) + f_3(t,z(t)),\\
{}^{\mathrm{C}}D_t^{\sigma} y(t) + Ay(t) &= f_1(t,y(t)) + f_2(t,x(t)) + f_3(t,y(t)),\\
{}^{\mathrm{C}}D_t^{\sigma} z(t) + Az(t) &= f_1(t,z(t)) + f_2(t,y(t)) + f_3(t,x(t)), \quad t \in J,\\
x(0) &= \tau_0, \quad y(0) = \tau_0, \quad z(0) = \tau_0 \in X,
\end{aligned}
\tag{4.1}
$$

where $J=[0,b]$, $b>0$ is a constant, ${}^{\mathrm{C}}D_t^{\sigma}$ denotes the $\sigma\in(0,1)$ order Caputo fractional derivative, A generates a C_0- semigroup $S(t)$ $(t\geq 0)$ of uniformly bounded linear operator in X, f and h are given functions.

For the C_0-semigroup $S(t)$ $(t\geq 0)$, if $S(t)x\geq 0$ for all $x\geq 0$, it is called a positive C_0-semigroup. Throughout this section, we always assume that Agenerates a positive C_0-semigroup $S(t)$ $(t\geq 0)$ of uniformly bounded linear operator in X. Namely, there is a constant $M>0$ such that $\|S(t)\|\leq M$ for all $t\geq 0$.

Definitions 15 and 16 can be found in [12, 14, 21].

Definition 15. *The fractional integral of order* $\sigma>0$ *with the lower limits zero for a function* $f\in L^1(J,E)$ *is defined by*

$$I^{\sigma}f(t)=\frac{1}{\Gamma(\sigma)}\int_0^t\frac{f(s)}{(t-s)^{1-\sigma}}ds,\quad t>0,$$

where Γ *is the gamma function.*

Definition 16. *The Riemann-Liouville derivative of order* $n-1<\sigma<n$*,* $n\in\mathbb{N}$ *with the lower limits zero for a function* $f\in L^1(J,E)$ *can be defined as*

$${}^{\mathrm{L}}D_t^{\sigma}f(t)=\frac{1}{\Gamma(n-\sigma)}\frac{d^n}{dt^n}\int_0^t\frac{f(s)}{(t-s)^{\sigma+1-n}}ds,\quad t>0.$$

The Caputo fractional derivative of order $0<\sigma<1$ with the lower limits zero for a function $f\in L^1(J,E)$ can be defined as

$${}^{\mathrm{C}}D_t^{\sigma}f(t)={}^{L}D_t^{\sigma}\big(f(t)-f(0)\big),\quad t>0.$$

Define two operator families $\{\mathscr{U}_{\sigma}(t)\}_{t\geq 0}$ and $\{\mathscr{V}_{\sigma}(t)\}_{t\geq 0}$ as

$$\mathscr{U}_{\sigma}(t)x=\int_0^{\infty}\zeta_{\sigma}(\tau)S\big(t^{\sigma}\tau\big)x\,d\tau,$$
$$\mathscr{V}_{\sigma}(t)x=\sigma\int_0^{\infty}\tau\zeta_{\sigma}(\tau)S\big(t^{\sigma}\tau\big)x\,d\tau,\quad 0<\sigma<1,$$

where

$$\zeta_{\sigma}(\tau)=\frac{1}{\sigma}\tau^{-1-\frac{1}{\sigma}}\rho_{\sigma}\big(\tau^{-\frac{1}{\sigma}}\big),$$
$$\rho_{\sigma}(\tau)=\frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\tau^{-\sigma n-1}\frac{\Gamma(n\sigma+1)}{n!}\sin(n\pi\sigma),\quad \tau\in(0,\infty).$$

Lemma 17. *(i) For any $x \in X$ and fixed $t \geq 0$, one has*

$$\left\|\mathscr{U}_\sigma(t)x\right\| \leq M\|x\|, \qquad \left\|\mathscr{V}_\sigma(t)x\right\| \leq \frac{M}{\Gamma(\sigma)}\|x\|.$$

(ii) If $S(t)$ $(t \geq 0)$ is an equi-continuous semigroup, $\mathscr{V}_\sigma(t)$ is equi-continuous in X for $t > 0$.

(iii) If $S(t)$ $(t \geq 0)$ is a positive C_0-semigroup, $\mathscr{U}_\sigma(t)$ and $\mathscr{V}_\sigma(t)$ are positive operators for all $t \geq 0$.

Proof. (i) and (ii) can be found in reference [14, 21]. (iii) is easily seen from the definitions of $\mathscr{U}_\sigma(t)$ and $\mathscr{V}_\sigma(t)$. So, we omit the details here. □

Let $C(J,X)$ be a set of all continuous $X-$valued functions on the interval J and let

$$\overline{K}_C = \left\{x \in C(J,X) : x(t) \in \overline{K}, t \in J\right\}.$$

Then $C(J,X)$ is a partially ordered $\Phi-$orbitally complete normed linear space with the norm $\|x\|_C := \sup\{\|x(t)\| : t \geq 0\}$ and the partial order induced by $\overline{K}_C$. It is clear that $\overline{K}_C$ is normal because $\overline{K}_C$ is normal.

Let $E = C(J,X)$ and $K = \overline{K}_C$. Define three operators $A_1, A_2, A_3 : E \to E$ by

$$(A_1x)(t) = \mathscr{U}_\sigma(t)\tau_0 + \int_0^t (t-s)^{\sigma-1}\mathscr{V}_\sigma(t-s)f_1\big(s,x(s)\big)\,ds, \quad t \in J, \tag{4.2}$$

$$(A_2x)(t) = \int_0^t (t-s)^{\sigma-1}\mathscr{V}_\sigma(t-s)f_2\big(s,x(s)\big)\,ds, \quad t \in J. \tag{4.3}$$

$$(A_3x)(t) = \int_0^t (t-s)^{\sigma-1}\mathscr{V}_\sigma(t-s)f_3\big(s,x(s)\big)\,ds, \quad t \in J. \tag{4.4}$$

Definition 18. *An element $(x,y,z) \in E \times E \times E$ is called a tripled mild solution of the system 4.1 if and only if it satisfies the following system of operator equations:*

$$\begin{cases} x(t) = (A_1x)(t) + (A_2y)(t) + (A_3z)(t), & t \in J, \\ y(t) = (A_1y)(t) + (A_2x)(t) + (A_3y)(t), & t \in J \\ z(t) = (A_1z)(t) + (A_2y)(t) + (A_3x)(t), & t \in J. \end{cases} \tag{4.5}$$

We shall use Theorem 11 to prove that 18 has a tripled fixed point in $E \times E \times E$. For this purpose, we consider the following hypotheses:

(H1) The positive C_0-semigroup $S(t)$ $(t \geq 0)$ is equi-continuous.

(H2) The function $f : J \times X \to X$ is continuous in x for all $t \in J$ and there exist a constant $\rho \in \mathbb{R}$ with $0 < \rho < \frac{\Gamma(\sigma+1)}{Mb^\sigma}$ and a $\mathfrak{D}$-function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(r) < r$ for any $r > 0$ satisfying

$$0 \leq f_1(t,u) - f_1(t,v) \leq \rho\phi\left(\|u-v\|\right), \quad \forall u,v \in X, u \geq v$$

for all $t \in J$.

(H3) The function $f_2(t,x), f_3(t,x) : J \times X \to X$ is continuous, nondecreasing and bounded in x.

(H4) There is an element $\overline{x} \in E$ satisfying

$$\begin{cases} {}^{\mathrm{C}}D_t^\sigma \overline{x}(t) + A\overline{x}(t) \leq f_1(t,\overline{x}(t)) + f_2(t,y(t)) + f_3(t,z(t)), & t \in J, \\ \overline{x}(0) \leq \tau_0 \in X \end{cases}$$

for all $y,z \in E$.

Since, by (H3), $f_2(t,x)$ and $f_3(t,x)$ bounded in x for all $t \in J$, there is a constant $\widetilde{M} > 0$ such that $\|f_2(t,x)\| \leq \widetilde{M}$ and $\|f_3(t,x)\| \leq \widetilde{M}$ for all $t \in J$ and $x \in X$. For

$$r^* \geq \max\left\{\frac{M\widetilde{M}b^\sigma}{\Gamma(\sigma+1)}, \left(1 - \frac{M\rho b^\sigma}{\Gamma(\sigma+1)}\right)^{-1}\left(M\|\tau_0\| + \frac{MF^*b^\sigma}{\Gamma(\sigma+1)}\right)\right\} + 1, \tag{4.6}$$

where $F^* = \sup_{t\in J}\|f_1(t,0)\|$, we define an open ball $\mathfrak{B}(x_0,r)$ in E by

$$\mathfrak{B}(x_0,r) = \left\{x \in E : \|x_0 - x\|_C < r\right\},$$

where $r = \|x_0\|_C + r^*$. Let $D = \overline{\mathfrak{B}}(x_0,r)$. Then D is a closed and bounded subset in E. By virtue of the assumptions (H1)-(H3), we have the following lemmas.

Lemma 19. *Assume that the hypothesis (H2) holds. Then the operator $A_1 : D \to D$ is Φ-orbitally continuous, nondecreasing and a partially nonlinear $\mathfrak{D}$-contraction in E.*

Proof. By (H2), 4.2 and 4.6, for any $x \in E$ with $\|x\|_C \leq r^*$, we have

$$\begin{aligned}\big\|(A_1x)(t)\big\| &\leq M\|\tau_0\| + \frac{M}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\big[\big\|f_1\big(s,x(s)\big)-f_1(s,0)\big\| + \big\|f_1(s,0)\big\|\big]\,ds \\ &\leq M\|\tau_0\| + \frac{M}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\big[\rho\phi\big(\|x\|_C\big)+F^*\big]\,ds \\ &\leq M\|\tau_0\| + \frac{M\rho b^\sigma}{\Gamma(\sigma+1)}r^* + \frac{MF^*b^\sigma}{\Gamma(\sigma+1)} \\ &\leq r^*.\end{aligned}$$

This implies that $\|A_1x\|_C \leq r^*$ for any $x \in E$ with $\|x\|_C \leq r^*$. Moreover, we have

$$\|x_0 - A_1x\|_C \leq \|x_0\|_C + \|A_1x\|_C \leq \|x_0\|_C + r^* = r.$$

This implies that A_1 maps D into itself.

Since $S(t)$ $(t \geq 0)$ is a positive C_0-semigroup, by (H2) and Lemma 17, it follows that $A_1 : D \to D$ is nondecreasing. Take a sequence $\{x_n\} \subset \Phi(x;A_1)$ for any $x \in D$ with $x_n \to x^*$ as $n \to \infty$. Since f_1 is continuous in x for all $t \in J$, by assumption (H2) and dominated convergence theorem, we have

$$\begin{aligned}\lim_{n\to\infty}(A_1x_n)(t) &= \mathscr{U}_\sigma(t)\tau_0 + \int_0^t (t-s)^{\sigma-1}\mathscr{V}_\sigma(t-s)\lim_{n\to\infty} f_1\big(s,x_n(s)\big)\,ds \\ &= \mathscr{U}_\sigma(t)\tau_0 + \int_0^t (t-s)^{\sigma-1}\mathscr{V}_\sigma(t-s) f_1\big(s,x^*(s)\big)\,ds \\ &= \big(A_1x^*\big)(t), \quad t \in J.\end{aligned}$$

This implies that $A_1 : D \to D$ is Φ-orbitally continuous. For any comparable elements $x, y \in D$, without loss of generality, we assume that $x \geq y$. By (H2), for any $t \in J$, we have

$$\begin{aligned}\big\|(A_1x)(t)-(A_1y)(t)\big\| &\leq \int_0^t (t-s)^{\sigma-1}\big\|\mathscr{V}_\sigma(t-s)\big[f_1\big(s,x(s)\big)-f_1\big(s,y(s)\big)\big]\big\|\,ds \\ &\leq \frac{M\rho}{\Gamma(\sigma)}\int_0^t (t-s)^{\sigma-1}\phi\big(\big\|x(s)-y(s)\big\|\big)\,ds \\ &\leq \frac{Mb^\sigma\rho}{\Gamma(\sigma+1)}\phi\big(\|x-y\|_C\big).\end{aligned}$$

This implies that

$$\|A_1x - A_1y\|_C \leq \phi\big(\|x-y\|_C\big)$$

because of $0 < \rho < \frac{\Gamma(\sigma+1)}{Mb^\sigma}$. Hence $A_1 : D \to D$ is a partially nonlinear $\mathfrak{D}$-contraction in E. $\square$

Lemma 20. *Let the hypotheses (H1) and (H3) hold. Then the operator $A_2 : D \to D$ is Φ-orbitally continuous, nondecreasing and partially compact in E.*

Proof. By the assumption (H3), 4.3 and 4.6, for any $x \in E$ with $\|x\|_C \leq r^*$, we have

$$\big\|(A_2x)(t)\big\| \leq \frac{M}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \big\|f_2\big(s,x(s)\big)\big\|\, ds \leq \frac{M\widetilde{M}b^\sigma}{\Gamma(\sigma+1)} \leq r^*.$$

This follows that $\|A_2x\|_C \leq r^*$ for any $x \in E$ with $\|x\|_C \leq r^*$. Further, we have

$$\|x_0 - A_2x\|_C \leq \|x_0\|_C + \|A_2x\|_C \leq \|x_0\|_C + r^* = r.$$

This implies that A_2 maps D into itself. Since $S(t)$ $(t \geq 0)$ is a positive C_0-semigroup, by (H3) and Lemma 17, a similar proof as in Lemma 19 shows that $A_2 : D \to D$ is Φ-orbitally continuous and nondecreasing. For any $t_1, t_2 \in J$ with $t_1 < t_2$, denote

$$\begin{aligned}
G_1 &= \left\|\int_0^{t_1} \big[(t_2-s)^{\sigma-1} - (t_1-s)^{\sigma-1}\big] \mathscr{V}_\sigma(t_2-s) f_2\big(s,x(s)\big)\, ds\right\|,\\
G_2 &= \left\|\int_0^{t_1} (t_1-s)^{\sigma-1} \big[\mathscr{V}_\sigma(t_2-s) - \mathscr{V}_\sigma(t_1-s)\big] f_2\big(s,x(s)\big)\, ds\right\|,\\
G_3 &= \left\|\int_{t_1}^{t_2} (t_2-s)^{\sigma-1} \mathscr{V}_\sigma(t_2-s) f_2\big(s,x(s)\big)\, ds\right\|.
\end{aligned}$$

Then, by Lemma 17, we have

$$\begin{aligned}
G_1 &\leq \frac{M\widetilde{M}}{\Gamma(\sigma)} \int_0^{t_1} \big|(t_2-s)^{\sigma-1} - (t_1-s)^{\sigma-1}\big|\, ds,\\
G_3 &\leq \frac{M\widetilde{M}b^\sigma}{\Gamma(\sigma+1)} (t_2-t_1)^\sigma.
\end{aligned}$$

This implies that $G_1 \to 0$ and $G_3 \to 0$ as $t_2 - t_1 \to 0$. If $t_1 \equiv 0$ and $0 < t_2 \leq b$, it is clear that $G_2 \equiv 0$. For $t_1 > 0$ and $\delta \in (0,t_1)$ small enough, we have

$$\begin{aligned}
G_2 \leq& \left\|\int_0^{t_1-\delta} (t_1-s)^{\sigma-1} \big[\mathscr{V}_\sigma(t_2-s) - \mathscr{V}_\sigma(t_1-s)\big] f_2\big(s,x(s)\big)\, ds\right\|\\
&+ \left\|\int_{t_1-\delta}^{t_1} (t_1-s)^{\sigma-1} \big[\mathscr{V}_\sigma(t_2-s) - \mathscr{V}_\sigma(t_1-s)\big] f_2\big(s,x(s)\big)\, ds\right\|\\
\leq& \frac{\widetilde{M}(t_1^\sigma - \delta^\sigma)}{\sigma} \sup_{s \in [0,t_1-\delta]} \big\|\mathscr{V}_\sigma(t_2-s) - \mathscr{V}_\sigma(t_1-s)\big\|\\
&+ \frac{3M\widetilde{M}\delta^\sigma}{\Gamma(\sigma+1)}.
\end{aligned}$$

Hence $G_2 \to 0$ as $t_2 - t_1 \to 0$ and $\delta \to 0$because of (H1). Let $C \subset D$ be an arbitrary chain. For any $W_1 \in A_2(C)$, there is $x \in C$ such that $W_1(t) = (A_2x)(t)$ for all $t \in J$. So, for any $t_1, t_2 \in J$ with $t_1 < t_2$, by the definition of the operator A_2, the inequality

$$\big\|W_1(t_2) - W_1(t_1)\big\| = \big\|(A_2x)(t_2) - (A_2x)(t_1)\big\| \le G_1 + G_2 + G_3$$

implies that

$$\big\|W_1(t_2) - W_1(t_1)\big\| \to 0$$

as $t_2 - t_1 \to 0$. This further implies that $A_2(C)$ is equi-continuous on J. On the other hand, by (H3), we have

$$\big\|W_1(t)\big\| = \big\|(A_2x)(t)\big\| \le \int_0^t (t-s)^{\sigma-1} \big\|\mathscr{V}_\sigma(t-s) f_2\big(s, x(s)\big)\big\|\, ds \le \frac{M\widetilde{M}b^\sigma}{\Gamma(\sigma+1)}, \quad \forall t \in J.$$

It follows that $\|W_1\|_C \le \frac{M\widetilde{M}b^\sigma}{\Gamma(\sigma+1)}$. Hence $A_2(C)$ is uniformly bounded in E. By the Arzela-Ascoli theorem, $A_2 : D \to D$ is partially compact. □

Lemma 21. *Let the hypotheses (H1) and (H3) hold. Then the operator $A_3 : D \to D$ is Φ-orbitally continuous, nondecreasing and partially compact in E.*

Proof. By the assumption (H3), 4.4 and 4.6, for any $x \in E$ with $\|x\|_C \le r^*$, we have

$$\big\|(A_3x)(t)\big\| \le \frac{M}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} \big\|f_3\big(s, x(s)\big)\big\|\, ds \le \frac{M\widetilde{M}b^\sigma}{\Gamma(\sigma+1)} \le r^*.$$

This follows that $\|A_3x\|_C \le r^*$ for any $x \in E$ with $\|x\|_C \le r^*$. Further, we have

$$\|x_0 - A_3x\|_C \le \|x_0\|_C + \|A_3x\|_C \le \|x_0\|_C + r^* = r.$$

This implies that A_3 maps D into itself. Since $S(t)$ $(t \ge 0)$ is a positive C_0-semigroup, by (H3) and Lemma 17, a similar proof as in Lemma 19 shows that $A_3 : D \to D$ is Φ-orbitally continuous and nondecreasing. For any $t_1, t_2 \in J$ with $t_1 < t_2$, denote

$$G_4 = \left\| \int_0^{t_1} \big[(t_2-s)^{\sigma-1} - (t_1-s)^{\sigma-1}\big] \mathscr{V}_\sigma(t_2-s) f_3\big(s, x(s)\big)\, ds \right\|,$$

$$G_5 = \left\| \int_0^{t_1} (t_1-s)^{\sigma-1} \big[\mathscr{V}_\sigma(t_2-s) - \mathscr{V}_\sigma(t_1-s)\big] f_3\big(s, x(s)\big)\, ds \right\|,$$

$$G_6 = \left\| \int_{t_1}^{t_2} (t_2-s)^{\sigma-1} \mathscr{V}_\sigma(t_2-s) f_3\big(s, x(s)\big)\, ds \right\|.$$

Then, by Lemma 17, we have

$$G_4 \leq \frac{M\widetilde{M}}{\Gamma(\sigma)} \int_0^{t_1} \left|(t_2-s)^{\sigma-1}-(t_1-s)^{\sigma-1}\right| ds,$$

$$G_6 \leq \frac{M\widetilde{M}b^{\sigma}}{\Gamma(\sigma+1)}(t_2-t_1)^{\sigma}.$$

This implies that $G_4 \to 0$ and $G_6 \to 0$ as $t_2-t_1 \to 0$. If $t_1 \equiv 0$ and $0 < t_2 \leq b$, it is clear that $G_5 \equiv 0$. For $t_1 > 0$ and $\delta \in (0,t_1)$ small enough, we have

$$\begin{aligned} G_5 \leq & \left\| \int_0^{t_1-\delta} (t_1-s)^{\sigma-1}\left[\mathscr{V}_{\sigma}(t_2-s)-\mathscr{V}_{\sigma}(t_1-s)\right] f_2\big(s,x(s)\big)\, ds \right\| \\ & + \left\| \int_{t_1-\delta}^{t_1} (t_1-s)^{\sigma-1}\left[\mathscr{V}_{\sigma}(t_2-s)-\mathscr{V}_{\sigma}(t_1-s)\right] f_3\big(s,x(s)\big)\, ds \right\| \\ \leq & \frac{\widetilde{M}(t_1^{\sigma}-\delta^{\sigma})}{\sigma} \sup_{s\in[0,t_1-\delta]} \left\|\mathscr{V}_{\sigma}(t_2-s)-\mathscr{V}_{\sigma}(t_1-s)\right\| \\ & + \frac{3M\widetilde{M}\delta^{\sigma}}{\Gamma(\sigma+1)}. \end{aligned}$$

Hence $G_5 \to 0$ as $t_2-t_1 \to 0$ and $\delta \to 0$ because of (H1). Let $C \subset D$ be an arbitrary chain. For any $W_2 \in A_3(C)$, there is $x \in C$ such that $W_2(t) = (A_3x)(t)$ for all $t \in J$. So, for any $t_1,t_2 \in J$ with $t_1 < t_2$, by the definition of the operator A_3, the inequality

$$\left\|W_2(t_2)-W_2(t_1)\right\| = \left\|(A_3x)(t_2)-(A_3x)(t_1)\right\| \leq G_4+G_5+G_6$$

implies that

$$\left\|W_2(t_2)-W_2(t_1)\right\| \to 0$$

as $t_2-t_1 \to 0$. This further implies that $A_3(C)$ is equi-continuous on J. On the other hand, by (H3), we have

$$\left\|W_2(t)\right\| = \left\|(A_3x)(t)\right\| \leq \int_0^t (t-s)^{\sigma-1}\left\|\mathscr{V}_{\sigma}(t-s)f_3\big(s,x(s)\big)\right\| ds \leq \frac{M\widetilde{M}b^{\sigma}}{\Gamma(\sigma+1)}, \quad \forall t \in J.$$

It follows that $\|W_2\|_C \leq \frac{M\widetilde{M}b^{\sigma}}{\Gamma(\sigma+1)}$. Hence $A_3(C)$ is uniformly bounded in E. By the Arzela-Ascoli theorem, $A_3 : D \to D$ is partially compact. □

Theorem 22. *Let the hypotheses (H1)-(H4) hold. Then the fractional evolution system 15 has a tripled mild solution on J.*

Proof. Define three operators A_1, A_2 and A_3 as in 4.2, 4.3 and 4.4. By Lemmas 19,20 and 21, we deduce that $A_1 : D \to D$ is Φ-orbitally continuous, nondecreasing and a partially nonlinear $\mathfrak{D}$-contraction as well as $A_2, A_3 : D \to D$ is Φ-orbitally continuous, nondecreasing and partially compact.

To apply Theorem 11, it remains to prove that there is an element $\bar{x} \in D$ satisfying

$$\bar{x} \leq A_1\bar{x} + A_2 y + A_3 z$$

for all $y, z \in D$. By (H4), there is an elements $\bar{x} \in D$ satisfying

$$\begin{cases} {}^{C}D_t^{\sigma}\bar{x}(t) + A\bar{x}(t) \leq f_1(t,\bar{x}(t)) + f_2(t,y(t)) + f_3(t,z(t)),, \quad t \in J, \\ \bar{x}(0) \leq \tau_0 \in X, \end{cases}$$

for all $y, z \in D$. Let $F(t) = D^{\sigma}\bar{x}(t) + A\bar{x}(t)$, $t \in J$. Then we have

$$\begin{aligned} \bar{x}(t) &= \mathscr{U}_{\sigma}(t)\bar{x}(0) + \int_0^t (t-s)^{\sigma-1}\mathscr{V}_{\sigma}(t-s)F(s)\,ds \\ &\leq \mathscr{U}_{\sigma}(t)\tau_0 + \int_0^t (t-s)^{\sigma-1}\mathscr{V}_{\sigma}(t-s)\big[f_1\big(t,\bar{x}(t)\big) + f_2\big(t,y(t) + f_3\big(t,z(t)\big)\big]\,ds \\ &= (A_1\bar{x})(t) + (A_2 y)(t) + (A_3 z)(t) \end{aligned}$$

for all $y, z \in D$ and $t \in J$. Hence all the conditions of Theorem 11 are satisfied. By Theorem 11, the system 4.5 has a tripled fixed point in $\widehat{E}$. Therefore, the fractional evolution system 4.1 has a tripled mild solution in $\widehat{E}$. □

By Theorem 22, we can obtain the following corollaries easily.

Corollary 23. *Let the hypotheses (H1), (H3) and (H4) hold. In addition, the following condition is satisfied:*

(H2)' The function $f_1 : J \times X \to X$ is continuous in x for all $t \in J$ and there is a constant $\beta \in (0, \frac{\Gamma(\sigma+1)}{Mb^{\sigma}})$ such that

$$0 \leq f_1(t,u) - f_1(t,v) \leq \beta(u-v), \quad \forall u \geq v, t \in J.$$

Then the fractional evolution system 4.1 has a tripled mild solution on J.

Corollary 24. *Let the hypotheses (H1), (H3) and (H4) hold. In addition, the following condition is satisfied:*

(H2)" The function $f_1 : J \times X \to X$ is continuous in x for all $t \in J$ and there is a constant $\gamma \in (0, \frac{\Gamma(\sigma+1)}{Mb^{\sigma}})$ such that

$$0 \leq f_1(t,u) - f_1(t,v) \leq \frac{\gamma\|u-v\|}{1+\|u-v\|}, \quad \forall u \geq v, t \in J.$$

Then the fractional evolution system 4.1 has a tripled mild solution on J.

5. Applications

In this section, we apply the obtained abstract results to the following fractional hybrid dynamic system:

$$\begin{cases} {}^{C}D_t^{\frac{1}{3}} x(w,t) + \frac{\partial x(w,t)}{\partial w} = f_1(w,t,x(w,t)) + f_2(w,t,y(w,t)) + f_3(w,t,z(w,t)), & (w,t) \in I \times I, \\ {}^{C}D_t^{\frac{1}{3}} y(w,t) + \frac{\partial y(w,t)}{\partial w} = f_1(w,t,y(w,t)) + f_2(w,t,x(w,t)) + f_3(w,t,y(w,t)), & (w,t) \in I \times I, \\ {}^{C}D_t^{\frac{1}{3}} z(w,t) + \frac{\partial z(w,t)}{\partial w} = f_1(w,t,z(w,t)) + f_2(w,t,y(w,t)) + f_3(w,t,x(w,t)) & (w,t) \in I \times I, \\ x(0,t) = x(1,t) = 0, \qquad y(0,t) = y(1,t) = 0, \qquad z(0,t) = z(1,t) = 0, & t \in I, \\ x(w,0) = y(w,0) = z(w,0) = \tau_0(w), \quad w \in (0,1), \end{cases} \tag{5.1}$$

where $I = [0,1]$, $f_2, f_3 : I \times I \times \mathbb{R} \to \mathbb{R}$ is defined by

$$f_2(z,t,u) = f_3(z,t,u) = \begin{cases} 3, & u \leq 0, \\ 3 + \frac{5u}{1+3u}, & 0 < u < 3, \\ 9, & u \geq 3. \end{cases}$$

It is clear that $f_2, f_3 : I \times I \times \mathbb{R} \to \mathbb{R}$ is continuous, nondecreasing and

$$\left| f_2(z,t,u) \right| \leq 9$$

$$\left| f_3(z,t,u) \right| \leq 9.$$

This implies that the condition (H3) holds.

Theorem 25. *Suppose that the following conditions are satisfied:*

(P1) The function $f_1 : I \times I \times \mathbb{R} \to \mathbb{R}$is continuous and there exist a constant $\rho \in (0, \frac{\sqrt{\pi}}{3})$and a $\mathscr{D}$-function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(r) < r$ for $r > 0$ such that

$$0 \leq f_1\big(w,t,u(w,t)\big) - f_1\big(w,t,v(w,t)\big) \leq \rho\phi\left(\left|u(w,t) - v(w,t)\right|\right)$$

for all $(w,t) \in I \times I$ *and* $u, v \in C(I \times I, \mathbb{R})$ *with* $u \geq v$.

(P2) There exists a function $\hat{x} \in C(I \times I, \mathbb{R})$ *such that*

$$\begin{cases} {}^{\mathrm{C}}D_t^{\frac{1}{3}} \hat{x}(w,t) + \frac{\partial \hat{x}(w,t)}{\partial w} \leq f_1(w,t,\hat{x}(w,t)) + f_2(w,t,y(w,t)) + f_3(w,t,z(w,t)), \quad (w,t) \in I \times I, \\ \hat{x}(w,0) \leq \tau_0(w), \quad w \in (0,1), \end{cases}$$

for any $y, z \in C(I \times I, \mathbb{R})$.

Then the fractional hybrid dynamic system 5.1 has a tripled mild solution.

Proof. Let $X = C(I, \mathbb{R})$. Then X is a Φ-orbitally complete normed linear space with the norm $\|x(t)\|_C = \max_{t \in I} |x(t)|$. Define a positive cone in X by $\overline{K} = \{x \in X : x \geq 0\}$. Then K is a closed convex cone in X, which is normal. Define an operator $A : D(A) \subset X \to X$ by

$$Au = u', \quad u \in D(A) := \big\{u \in X : u' \in X, u(0) = u(1) = 0\big\}.$$

It is well known that A generates a C_0-semigroup $S(t)$ $(t \geq 0)$ given by

$$S(t)u(w) = u(t+w), \quad t \geq 0, u \in X.$$

Then $S(t)$ $(t \geq 0)$ is an equi-continuous C_0-semigroup, but it is not compact, and $\sup_{t \in I} \|S(t)\| \leq 1$. This implies that the condition (H1) holds. Let

$$\begin{aligned} x(t)(w) &= x(w,t), \\ y(t)(w) &= y(w,t), \\ z(t)(w) &= z(w,t) \\ f_1\big(t,x(t)\big)(w) &= f_1\big(w,t,x(w,t)\big), \\ f_2\big(t,x(t)\big)(w) &= f_2\big(w,t,x(w,t)\big) \\ f_3\big(t,x(t)\big)(w) &= f_3\big(w,t,x(w,t)\big). \end{aligned}$$

Then the fractional hybrid dynamic system 5.1 can be rewritten into the abstract fractional evolution system (4.1). By the assumptions (P1) and (P2), the conditions (H2) and (H4) hold. Hence by Theorem 22, the abstract fractional evolution system 4.1 has a tripled mild solution, which is also the tripled mild solution of the fractional hybrid dynamic system 5.1. □

Similarly, using Corollaries 23 and 24, we can obtain the following theorems.

Theorem 26. *Assume that the condition (P2) and the following condition are satisfied:*

(P3) The function $f_1 : I \times I \times \mathbb{R} \to \mathbb{R}$is continuous and there exist a constant $\beta \in (0, \frac{\sqrt{\pi}}{2})$ such that

$$0 \leq f_1\big(w,t,u(w,t)\big) - f_1\big(w,t,v(w,t)\big) \leq \beta\big(u(w,t) - v(w,t)\big)$$

for all $(w,t) \in I \times I$ and $u, v \in C(I \times I, \mathbb{R})$ with $u \geq v$.

Then the fractional hybrid dynamic system (5.1) has a tripled mild solution.

Theorem 27. *Let the condition (P2) and the following condition hold:*

(P4) The function $f_1 : I \times I \times \mathbb{R} \to \mathbb{R}$is continuous and there exist a constant $\gamma \in (0, \frac{\sqrt{\pi}}{2})$ such that

$$0 \leq f_1\big(w,t,u(w,t)\big) - f_1\big(w,t,v(w,t)\big) \leq \frac{\gamma \|u-v\|_C}{1 + \|u-v\|_C}$$

for all $(w,t) \in I \times I$ and $u, v \in C(I \times I, \mathbb{R})$ with $u \geq v$.

Then the fractional hybrid dynamic system 5.1 has a tripled mild solution.

6. ACKNOWLEDGEMENTS

The authors are thankful to the Editors for their deep observations and pertinent suggestions, which greatly helped us to improve the paper significantly.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] Agarwal, R, El-Gebeily, M, ORegan, D: Generalized contractions in partially ordered metric spaces. Appl. Anal. 87 (2008), 109-116.

[2] Borkut M., Berinde V., Tripled fixed point theorems for contractive type mappings in partially ordered metric spaces, Nonlinear Anal. 74 (15) (2011), 4889–4897.

[3] Dhage, B: A nonlinear alternative with applications to nonlinear perturbed differential equations. Nonlinear Stud. 13 (2006), 343-354.

[4] Dhage, B: Hybrid fixed point theory in partially ordered normed linear spaces and applications to fractional integral equations. J. Differ. Equ. Appl. 2 (2013), 155-184.

[5] Dhage, B: Partially continuous mappings in partially ordered normed linear spaces and applications to functional integral equations. Tamkang J. Math. 45 (2014), 397-426.

[6] Granas, A, Dugundji, J: Fixed Point Theory. Springer, New York (2003).

[7] Gupta A., Yadava R.N., Shrivastava R., Tripled PBVPS of Nonlinear Second Order Differential Equations, Bangmood Int. J. Math. Comp. Sci. 2 (1) (2016), 95-106.

[8] Gupta A., Maitra J. K., Tripled Coincidence Point Theorem in Fuzzy Metric Spaces, Konuralp J. Math. 5 (1) (2017), 68-76.

[9] Gupta A., Manaro S., A New Type of Tripled Fixed Point Theorem in Partially Ordered Complete Metric Space, Adv. Anal. 2 (2) (2017), 63-70.

[10] Gupta A., Ulam- Hyers Stability Theorem by Tripled Fixed Point Theorem, Fasciculi Math. 56 (2016), 77-97.

[11] Harjani, J, Sadarangani, K: Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations. Nonlinear Anal. TMA 72 (2010), 1188-1197.

[12] Kilbas, A, Srivastava, H, Trujillo, J: Theory and Applications of Fractional Differential Equations. North Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam (2006).

[13] Lakshmikantham, V, Ciric, L: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. TMA 70 (2009), 4341-4349.

[14] Liang, J, Yang, H: Controllability of fractional integro-differential evolution equations with nonlocal conditions. Appl. Math. Comput. 254 (2015), 20-29.

[15] Luong, N, Thuan, N: Coupled fixed points in partially ordered metric spaces and application. Nonlinear Anal. TMA 74 (2011), 983-992.

[16] Nieto, J, Rodriguez-Lpez, R: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order 22 (2005), 223-239.

[17] Nieto, J, Rodriguez-Lpez, R: Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. Acta Math. Sin. Engl. Ser. 23(12) (2007), 2205-2212.

[18] ORegan, D, Petrusel, A: Fixed point theorems for generalized contractions in ordered metric spaces. J. Math. Anal. Appl. 341 (2008), 1241-1252.

[19] Ran, A, Reurings, M: A fixed point theorem in partially ordered sets and some applications to metric equations. Proc. Am. Math. Soc. 132 (2003), 1435-1443.

[20] Samet, B: Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces. Nonlinear Anal. TMA 72 (2010), 4508-4517.

[21] Zhou, Y, Jiao, F: Nonlocal Cauchy problem for fractional evolution equations. Nonlinear Anal., Real World Appl. 11 (2010), 4465-4475.