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SOME EQUIVALENTS OF THE HAHN-BANACH THEOREM

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Abstract. It is well-known that the Brouwer fixed point theorem in 1912, the weak Sperner combinatorial lemma in 1928, and the Knaster-Kuratowski-Mazurkiewitz (KKM) intersection theorem in 1929 are mutually equivalent, and have a large number of equivalent statements in the realm of the KKM theory. In 1991, Granas and Lassonde introduced the convex-valued KKM theorem with numerous applications in different areas of mathematics. Recently, Horvath and Ben-El-Mechaiekh obtained some equivalents of the convex-valued KKM theorem. In the present article, we add up some more consequences of them including equivalents of the Hahn-Banach theorem. Consequently, we have a large scaled logical system covering all consequences and applications of the KKM theory where equivalents of the Hahn-Banach theorem belong to a partial realm.

Keywords: KKM theory; convex-valued KKM theorem; Brouwer fixed point theorem; Hahn-Banach theorem.

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1. Introduction

In 1929, Knaster, Kuratowski, and Mazurkiewicz (simply, KKM) obtained an intersection theorem which is known to be equivalent to the Brouwer fixed point theorem in 1912 and the

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weak Sperner combinatorial lemma in 1928. The KKM theory is first named by ourselves in 1991 as the study of applications of generalizations or equivalents of the KKM theorem. Nowadays the realm of the theory is very broad; see Park [16].

Independently to the above progress, the Hahn-Banach theorem originated from Hahn [8] and Banach [1] is of basic importance in the analysis of problems concerning the existence of continuous linear functionals. Later there have been appeared a large number of articles on the Hahn-Banach theorem related to its generalization, variants, proofs, and applications; see MathSciNet, Google Scholar, and ResearchGate. Note that only a few of them are closely related to the KKM theory.

Edwards in 1974 showed that a number of variational inequalities that had hitherto been proved by using the Brouwer fixed point theorem or the KKM theorem can in fact be deduced more simply from the Hahn-Banach theorem. Motivated by this fact, Simons in 1978-2008 studied various results of the Hahn-Banach type or the Brouwer type (equivalently, the KKM type). He actually obtained the Hahn-Banach type proofs of some KKM type theorems. In addition, some authors claim that even the Hahn-Banach theorem is of the Brouwer type or the KKM type; see Hirano, Komiya, and Takahashi [9], and Granas and Lassonde [6, 7].

In 1991, Granas and Lassonde [6] introduced the convex-valued KKM theorem which has numerous applications in different areas of mathematics. Many applications of the theorem to known results are systemically given on systems of inequalities, variational inequalities, minimax equalities, theorems of Markhoff-Kakutani, Mazur-Orlicz and Hahn-Banach, variational problems, maximal monotone operators, and others in convex analysis. Moreover, in 1995, Granas and Lassonde [7] complemented and elucidated the preceding approach within the context of complete metric spaces. Their aim of [7] is to provide simple proofs of several known results, stated in super-reflexive Banach spaces, concerning minimization of quasi-convex functions, variational inequalities, game theory, systems of inequalities, and maximal monotone operators, by using their intersection principle which is elementary.

In 2014, Horvath [10] showed that the convex-valued KKM theorem is equivalent to Klee's 1951 theorem, the elementary Alexandroff-Pasynkov theorem, the elementary Ky Fan theorem and the Sion-von Neumann minimax theorem, as well as a few other classical results with an extra convexity assumption. Moreover, Ben-El-Mechaiekh [2] recognized that the convex-valued KKM principle is equivalent to the Hahn-Banach theorem, the Markov-Kakutani fixed point theorem, and the Sion-von Neumann minimax principle. Furthermore, in our previous article [15], we showed that the distinction between the Hahn-Banach type and the KKM type theorems is not strict, and that actually the two types are very close in a broad sense. Actually, we showed that the KKM theorem implies the Hahn-Banach theorem in [17]. Based on these articles, we recollect some history of the study on the relation between the Hahn-Banach type and the KKM type, and conclude that all consequences and applications of the Hahn-Banach theorem belong to a partial realm of the KKM theory.

Our aim in this survey is to collect some equivalent statements to the Hahn-Banach theorem and the convex-valued KKM theorem within the realm of the KKM theory. This article is a supplement of our previous works [15, 17].

In this article, numbers attached to Lemmas, Theorems, Corollaries and Definition are the ones in their original sources.

2. THE HAHN-BANACH THEOREM

It begins with dominated extension theorems proved by H. Hahn in 1926 [8] and S. Banach in 1929 [1], respectively, by making use of the ideas of F. Riesz and E. Helly. Their theorems were later generalized to linear operators taking their values in normed spaces with the binary intersection property and ordered vector spaces.

Since then the Hahn-Banach theorem is of basic importance in the analysis of problems concerning the existence of continuous linear functionals. Its original proof of Banach, as well as the proofs given in most textbooks, relies on the axiom of choice. We borrow the following version from Dunford and Schwartz [3]:

Theorem. (Hahn-Banach) *Let the real function p on the real linear space X satisfying*

$$p(x+y) \leq p(x)+p(y);\ p(\alpha x) = \alpha p(x);\ \alpha \geq 0;\ x,y \in X.$$

Let f be a real linear functional on a subspace Y of X with

$$f(x) \leq p(x);\ \ x \in Y.$$

Then there is a real linear functional F on X for which

$$F(x) = f(x);\ \ x \in Y;\ \ F(x) \leq p(x),\ \ x \in X.$$

Here the proof is based on Zorn's lemma. Several consequences and comments on the theorem are given in Dunford-Schwartz ([3], Part I after p.62 and pp.85–88). Nowadays there are many proofs of the Hahn-Banach theorem without using the axiom of choice or Zorn's lemma. For the recent literature on the Hahn-Banach theorem, see [15] and [17].

3. Works of Kakutani and Takahashi

Early in 1938, Kakutani [12] gave a proof of the Hahn-Banach theorem by using the Markov-Kakutani fixed point theorem, which follows from the Tychonoff fixed point theorem.

Theorem. (Markov 1936) *Let B be a non-vacuous, convex, bicompact subset of a locally convex linear topological space E, and let* Γ *be an abelian family of continuous affine transformation* $\varphi(x)$ *of B into itself; then there is a point* $x \in B$ *such that we have* $\varphi(x) = x$ *for any* $\varphi \in \Gamma$.

Kakutani [11] proved this theorem without assuming the local convexity and showed [12] that

Markov-Kakutani theorem $\Longrightarrow$ Hahn-Banach theorem.

In fact, Takahashi [18] indicated the following:

Hahn-Banach theorem

$\Longrightarrow$ a minimax theorem of Fan in 1953

$\Longrightarrow$ Fan's convex inequalities theorem in 1957

$\Longrightarrow$ Markov-Kakutani theorem

$\Longrightarrow$ Hahn-Banach theorem (Kakutani [12]).

Therefore, already Takahashi [20] indicated the following result of Werner [21]:

Hahn-Banach theorem $\Longrightarrow$ Markov-Kakutani fixed point theorem.

Moreover, Hirano-Komiya-Takahashi in 1982 [9] used the Markov-Kakutani theorem to prove the following generalized form of the Hahn-Banach theorem:

Lemma 3. ([9]) *If p is sublinear on a linear space E and $x_0 \in E$, then there is an $f \in E^*$ such that $f(x) \leq p(x)$ for all $x \in E$ and $f(x_0) = p(x_0)$.*

4. Works of Granas and Lassonde [6] in 1991

The *geometric principle* in [6] is as follows:

Theorem 1. ([6]) *Let D be a non-empty subset of a topological vector space E, and $G : D \multimap E$ a KKM map (that is, $\mathrm{co}A \subset G(A)$ for all finite $A \subset D$) with closed convex values. Then the family $\{G(x) : x \in D\}$ has the finite intersection property.*

Note that this is a particular case of the 1961 KKM lemma of Ky Fan [5], and appeared already in works of Valentine in 1964 and Asakawa in 1986 with some applications; see [6].

In [6], it is shown that

Geometric lemma (reformulation of Klee's lemma in 1951)

$\Longleftrightarrow$ Geometric principle (Valentine in 1964 and Asakawa in 1986)

$\Longleftrightarrow$ Corollary 1.1

$\Longleftrightarrow$ Theorem 2 (Analytic form)

In [6], the following theorem of Banach (version de base du Théoréme de Hahn-Banach) is recalled:

Lemme 1. (Banach) *If $p : E \to \mathbb{R}$ is sublinear, then there exists $f \in E^*$ such that $f(x) \leq p(x)$ for all $x \in E$.*

From this they deduced Lemme 2 (the same to Lemma 3 of Hirano-Komiya-Takahashi in 1982 [9]), which generalizes the Hahn-Banach theorem.

Consequently, they deduced

Markov-Kakutani theorem

$\Longrightarrow$ Lemme 1 (Banach)

$\Longrightarrow$ Lemme 2

$\Longrightarrow$ Hahn-Banach theorem.

Moreover, Granas and Lassonde [6] deduced

Lemme 2 + Corollaire 4.1 (Fan in 1957 [4] and others)

$\Longrightarrow$ Théoréme 9 (Mazur-Orlicz)

$\Longrightarrow$ Corollaire 9.1

$\Longrightarrow$ Hahn-Banach theorem.

Later Ben-El-Mechaiekh [2] showed that

Markov-Kakutani theorem $\Longleftrightarrow$

Geometric principle (the convex KKM principle in [2]).

Recall that Hirano-Komiya-Takahashi in 1982 [9] showed

Markov-Kakutani theorem $\Longrightarrow$ Lemme 2,

and that Granas-Lassonde in 1991 [6] showed

Geometric principle $\Longrightarrow$ Corollaire 4.1.

Therefore, we have

KKM theorem $\Longrightarrow$ Geometric principle (or Markov-Kakutani theorem)

$\Longrightarrow$ Lemme 2 + Corollaire 4.1 (Fan in 1957 [4] and others)

$\Longrightarrow$ Théorème 9 (Mazur-Orlicz)

$\Longrightarrow$ Corollaire 9.1

$\Longrightarrow$ Hahn-Banach theorem.

Consequently, "*the Hahn-Banach theorem can be derived from the KKM theorem*"; see Park [17].

Moreover, Takahashi [20] indicated the following:

Hahn-Banach theorem

$\Longrightarrow$ a minimax theorem of Fan in 1953

$\Longrightarrow$ Lemme 2 + Fan's convex inequalities theorem in 1957 ([6], Corollaire 4.1)

$\Longrightarrow$ Markov-Kakutani theorem

$\Longrightarrow$ Hahn-Banach theorem (Kakutani [12]).

Consequently, we have

Hahn-Banach theorem

$\Longleftrightarrow$ Markov-Kakutani theorem

$\Longrightarrow$ Lemma 2 + Fan's convex inequalities theorem in 1957

$\Longrightarrow$ Markov-Kakutani theorem

$\Longleftrightarrow$ Hahn-Banach theorem (Kakutani [12]).

Recall that the following in [6]:

Théorème 9. ([6], Mazur-Orlicz) *Let E be a t.v.s., E^* its algebraic dual, $p : E \to \mathbb{R}$ a sublinear functional, T an abstract set, and $x : T \to E$ and $\beta : T \to \mathbb{R}$ two functions. Then the following two properties are equivalent:*

(A) *there exists $f \in E^*$ such that $f(x) \le p(x)$ for every $x \in E$, and $\beta(t) \le f(x(t))$ for every $t \in T$.*

(B) *for every $\{t_1, \dots, t_n\} \subset T$ and $(\lambda_1, \dots, \lambda_n) \in \Delta^{n-1}$, a standard simplex, we have*

$$\sum_{i=1}^{n} \lambda_i \beta(t_i) \le p(\sum_{i=1}^{n} \lambda_i x(t_i)).$$

Corollaire 9.1. ([6]) *Let $p : E \to \mathbb{R}$ be a sub-linear functional, $C \subset E$ a convex subset of a real vector space E, and $g : C \to \mathbb{R}$ a concave function such that $g(y) \le p(y)$ for all $y \in C$. Then there exists $f \in E^*$ such that $f(x) \le p(x)$ for all $x \in E$ and $f(y) \ge g(y)$ for every $y \in C$.*

Then their version of the Hahn-Banach theorem is an evident particular case of Corollaire 9.1. Recall that *the Hahn-Banach theorem can be derived from the KKM theorem*; see Park [15, 17].

5. Works of Horvath [10] in 2014

In 2014, Horvath [10] published an article related to [6]. It is shown in [10] that the Elementary KKM theorem is equivalent to Klee's theorem, the Elementary Alexandroff-Pasynkov theorem, the Elementary Ky Fan theorem and the Sion-von Neumann minimax theorem, as well as a few other classical results with an extra convexity assumption.

It is well-known that the KKM theorem, the Alexandroff-Pasynkov theorem, and the Ky Fan theorem are mutually equivalent. Horvath [10] showed that "elementary" versions of these theorems are equivalent to Klee's theorem and the Sion-von Neumann minimax theorem.

Theorem 2.1. (Klee's theorem, 1951) *If $A_0,\dots,A_n$ is a family of closed convex sets such that, for all $j \in \{0,\dots,n\}$, $\bigcap_{i\neq j} A_i \neq \emptyset$ and $\bigcup_{i=0}^n A_i$ is convex, then $\bigcap_{i=0}^n A_i \neq \emptyset$.*

Theorem 2.2. (Elementary KKM) *A family $F_i,\ i \in \{0,\dots,n\}$, of closed and convex subsets of the n-dimensional simplex $\Delta_n = conv\{e_i : i = \{0,\dots,n\}\}$ such that, for all nonempty subsets $J \subset \{0,\dots,n\}$, $\Delta_J \subset \bigcup_{j\in J} F_j$, has a nonempty intersection.*

Theorem 2.3. (Elementary Alexandroff-Pasynkoff theorem, 1957) *Let $\{M_0,\dots,M_n\}$ be a cover of Δ_n by closed convex sets. If, for all $i \in \{0,\dots,n\}$, $\mathrm{conv}\{e_j : j \neq i\} \subset M_i$, then $\bigcap_{i=0}^n M_i \neq \emptyset$.*

The first chain of implications in [10] is as follows:

Klee's theorem $\Longrightarrow$ Elementary KKM

$\Longrightarrow$ Elementary Alexandroff-Pasynkoff

$\Longrightarrow$ Klee's theorem.

Elementary Ky Fan's theorem. *Let X be a convex subset of a topological vector space and let $\Omega : X \multimap X$ be a multifunction with closed values such that*

(1) *for all $x \in X,\ x \in \Omega x$,*

(2) *for all $y \in X,\ X \setminus \Omega^{-1} y$ is convex.*

Then $x \mapsto \mathrm{conv}(\Omega x)$ *has the finite intersection property.*

Elementary Ky Fan's inequality. *Let* X *be a compact convex set and let* $f : X \times X \to \mathbb{R}$ *be a function such that*

(1) *for all* $x \in X,\ f(x,x) \le 0$,

(2) *for all* $x \in X,\ y \mapsto f(x,y)$ *is l.s.c. and quasi-convex,*

(3) *for all* $y \in X,\ x \mapsto f(x,y)$ *is quasi-concave.*

Then, there exists $y_0 \in X$ *such that, for all* $x \in X,\ f(x,y_0) \le 0$.

The second chain of implications:

Elementary KKM theorem

$\Longrightarrow$ Geometric KKM principle

$\Longrightarrow$ Elementary Ky Fan's theorem

$\Longrightarrow$ Elementary Ky Fan's inequality

$\Longrightarrow$ von Neumann's minimax theorem

$\Longrightarrow$ Klee's theorem

$\Longrightarrow$ Elementary KKM theorem

The third chain of implications:

von Neumann's minimax theorem

$\Longrightarrow$ Klee's theorem

$\Longrightarrow$ Sion's theorem

$\Longrightarrow$ von Neumann's minimax theorem.

All three chains in [2] contain Klee's theorem and hence, all results in them are equivalent each other.

6. BEN-EL-MECHAIEKH'S ARTICLE [2] IN 2015

Ben-El-Mechaiekh [2] stated : "A number of landmark existence theorems of nonlinear functional analysis follow in a simple and direct way from the basic separation of convex closed sets in finite dimension via elementary versions of the Knaster-Kuratowski-Mazurkiewicz principle —- which we extend to arbitrary topological vector spaces —- and a coincidence property for so-called von Neumann relations. The method avoids the use of deeper results of topological essence such as the Brouwer fixed point theorem or the Sperner's lemma and underlines the crucial role played by convexity. It turns out that the convex KKM principle is equivalent to the Hahn-Banach theorem, the Markov-Kakutani fixed point theorem, and the Sion-von Neumann minimax principle."

Note that the geometric principle of Granas and Lassonde [6] is called the convex KKM principle by Ben-El-Mechaiekh [2].

Ben-El-Mechaiekh [2] derived the following:

Definition 8. ([2]) A *von Neumann relation* is a subset A of a cartesian product $X \times Y$, where X and Y are subsets of topological vector spaces, satisfying:

(i) for every $x \in X$, the section $A(x)$ is convex and non-empty;

(ii) for every $y \in Y$, the section $A^{-1}(y)$ is open in X and $X \setminus A^{-1}(y)$ is convex.

Denote by $N(X,Y)$ the class of von Neumann relations in $X \times Y$ and by $N^{-1}(X,Y) := \{A : X \multimap Y \mid A^{-1} \in N(Y,X)\}$.

Theorem 9. ([2], Fixed Point for N-maps) *Let E be a t.v.s., $\emptyset \neq Y \subset X \subset E$ with X convex, and let $A \in N(X,Y)$. If there exist a compact subset K of X and a compact convex subset D of Y such that for every $x \in X \setminus K$, $A(x) \cap D \neq \emptyset$, then A has a fixed point, i.e., $(\hat{x},\hat{x}) \in A$ for some $\hat{x} \in X$.*

Ben-El-Mechaiekh [2] noted that Theorem 9 is equivalent to the Klee intersection theorem and the convex KKM theorem (with certain compactness condition).

7. EQUIVALENTS OF THE HAHN-BANACH THEOREM

In this section, we summarize the known equivalent formulations of the Hahn-Banach theorem or the convex-valued KKM principle in the chronological order as follows:

Hahn-Banach theorem (1926, 1929)

von Neumann's minimax theorem (1927)

Banach's theorem (1929)

Markov-Kakutani theorem (1938)

Klee's intersection theorem (1951) [reformulated to "Le lemme géométrique" [6])

Fan's minimax theorem (1953)

Fan's convex inequalities theorem (1957)

Sion's minimax equality (1958)

Lemma 3 of Hirano-Komiya-Takahashi (1982)

A generalized Hahn-Banach theorem of Hirano-Komiya-Takahashi (1982)

The convex-valued KKM principle

Geometric lemma of Granas and Lassonde (1991) [reformulation of Klee's lemma (1951)]

Geometric principle of Granas and Lassonde (1991) [Valentine (1964) and Asakawa (1986)]

Corollary 1.1 of Granas and Lassonde (1991)

Theorem 2 (Analytic form) of Granas and Lassonde (1991)

Theorem 9 of Mazur-Orlicz (see Granas and Lassonde (1991))

Corollary 9.1 of Granas and Lassonde (1991)

Elementary Alexandroff-Pasynkoff theorem of Horvath (2014)

Elementary Ky Fan's theorem of Horvath (2014)

Elementary Ky Fan's inequality of Horvath (2014)

Fixed point theorem for N-maps by Ben-El-Mechaiekh (2015)

8. A LARGE SCALED LOGICAL SYSTEM RELATED THE KKM THEORY

It is well-known that the Brouwer fixed point theorem, the weak Sperner combinatorial lemma, and the Knaster-Kuratowski-Mazurkiewicz (KKM) theorem are mutually equivalent and have nearly one hundred equivalent formulations and several thousand applications.

Recently, Nyman and Su [13] choose a particular form of Fan's 1952 Lemma and called it "Fan's $N+1$ lemma". They showed that the $N+1$ lemma is equivalent to Borsuk-Ulam theorem, LSB theorem, and Tucker's lemma, and directly implies the weak Sperner lemma. Therefore "Fan's $N+1$ lemma" leads equivalents of the KKM theorem and their applications.

In this section, we consider an imaginary realm consisting of consequences of the $N+1$ lemma. In our recent article [18] we gave a panoramic view of this realm.

As we have seen, in this article, that the KKM theorem implies the Hahn-Banach theorem and its equivalents, a large scaled logical system related to the KKM theory can be adequately expressed as follows:

$$\text{Fan's 1952 Lemma}$$
$$\Longleftrightarrow \{\text{Borsuk-Ulam theorem, LSB theorem, and Tucker's lemma}\}$$
$$\Longrightarrow \{\text{equivalents of the Brouwer fixed point theorem}\}$$
$$\Longrightarrow \{\text{equivalents of the Hahn-Banach theorem.}\}$$

We can add thousands of consequences of these results.

9. Historical remarks

In this section, we add some historical remarks.

(1) In our previous work in 2010 [14], we stated that "But there are no evidence for that any of (XVIII)-(XX) and (XXVI) implies the Brouwer fixed point theorem" in Remark 3 at the end of Section 3.

Note that (XVIII)-(XX) and (XXVI) are all generalized abstract form of the von Neumann - Sion minimax theorem, and reduce to Sion's minimax theorem in 1958. This is equivalent to Elementary KKM theorem in view of Horvath [10] and hence not equivalent to the KKM theorem nor the Brouwer theorem.

For a long period, the author could not recognize this fact by Horvath.

(2) Some authors previously studied the relation of results of the Hahn-Banach theorem and the Brouwer fixed point theorem (equivalently, the KKM theorem).

In our previous work [15], we presented a large number of examples of the Hahn-Banach type or of the KKM type, and show that the Hahn-Banach theorem is of the KKM type. Therefore, the distinction between these two types is not strict, and the two types are actually same in a broad sense.

In [15], we incorrectly stated that Klee's intersection theorem in 1951 and Sion's minimax equality in 1958 are equivalent to the Brouwer fixed point theorem.

(3) In our recent work [15], we recalled some history of the study on the relation of results of the Hahn-Banach theorem and the KKM theorem, and showed that the Hahn-Banach theorem can be derived from the KKM theorem and not conversely. The present article shows that the Hahn-Banach theorem is equivalent to a particular form of the KKM theorem (or the geometric principle of Granas-Lassonde [6]).

(4) In our previous work [19], we introduced the contents of two articles of Granas and Lassonde [6, 7] on applications of some elementary principles of convex analysis. In the first article, they presented a geometric approach in the theory of minimax inequalities, which has numerous applications in different areas of mathematics. In the second article, they complement and elucidate the preceding approach within the context of complete metric spaces. In this paper, we give abstract convex space versions of the basic results of [6, 7], and, as the supplements of overviews on recently developed KKM theory, we introduce applications appeared in [6, 7]. Consequently, many of known results in the traditional convex analysis can be deduced from the KKM theory.

(5) Since 1983, it is known that the KKM theorem holds for open-valued KKM maps, we can consider the open-valued version of the geometric principle of Granas-Lassonde [6] as follows:

Theorem. *Let D be a non-empty subset of a topological vector space E, and $G : D \multimap E$ a KKM map (that is, $\mathrm{co}A \subset G(A)$ for all finite $A \subset D$) with open convex values. Then the family $\{G(x) : x \in D\}$ has the finite intersection property.*

It would be interesting to study consequences of this theorem.

Conflict of Interests

The authors declare that there is no conflict of interests.

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