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COMMON FIXED POINTS OF FISHER TYPE WEAKLY CONTRACTIVE MAPS IN b-METRIC SPACES

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Abstract. In this paper, we prove the existence and uniqueness of common fixed points for two pairs of selfmaps satisfying a Fisher type weakly contractive condition in which one pair is compatible, b-continous and the another one is weakly compatible in complete b-metric spaces. Further, we prove the same with different hypotheses on two pairs of selfmaps which satisfy b-(E.A)-property. We draw some corollaries from our results and provide examples in support of our results.

Keywords: common fixed points; b-metric space; weakly compatible maps; b-(E.A)-property; compatible maps.

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1. Introduction and preliminaries

The development of fixed point theory is based on the generalization of contraction conditions in one direction or/and generalization of ambient spaces of the operator under consideration on the other. Banach contraction principle plays an important role in solving nonlinear equations, and it is one of the most useful results in fixed point theory. In the direction of generalization of contraction conditions, in 1997, Alber and Guerre-Delabriere [3] introduced weakly contractive

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maps which are extensions of contraction maps and obtained fixed point results in the setting of Hilbert spaces. Rhoades [30] extended this concept to metric spaces. In 2008, Dutta and Choudhury [19] introduced (ψ,φ)-weakly contractive maps and proved the existence of fixed points in complete metric spaces. In 2009, Doric [18] extended it to a pair of maps. For more literature in this direction, we refer Choudhury, Konar, Rhoades and Metiya [15], Babu, Nageswara Rao and Alemayehu [7], Sastry, Babu and Kidane [32], Babu and Sailaja [8] and Zhang and Song [35]. The main idea of b-metric was initiated from the works of Bourbaki [14] and Bakhtin [11]. The concept of b-metric space or metric type space was introduced by Czerwik [16] as a generalization of metric space.

Afterwards, many authors studied fixed point theorems for single-valued and multi-valued mappings in b-metric spaces, we refer [6, 12, 13, 17, 25, 28, 29, 31, 34]. In 2002, Aamari and Moutawakil [1] introduced the notion of property (E.A). Different authors apply this concept to prove the existence of common fixed points, we refer [2, 9, 10, 27].

Throughout this paper we denote $\mathbb{R}^+ = [0,\infty)$ and
$\Psi = \{\psi/\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, nondecreasing with $\psi(t) = 0$ if and only if $t = 0\}$.

Definition 1.1. [16] Let X be a non-empty set. A function $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if the following conditions are satisfied: for any $x,y,z \in X$

(i) $0 \leq d(x,y)$ and $d(x,y) = 0$ if and only if $x = y$,

(ii) $d(x,y) = d(y,x)$,

(iii) there exists $s \geq 1$ such that $d(x,z) \leq s[d(x,y)+d(y,z)]$.

In this case, the pair (X,d) is called a b-metric space with coefficient s.

Every metric space is a b-metric space with $s = 1$. In general, every b-metric space is not a metric space.

Definition 1.2. [13] Let (X,d) be a b-metric space and $\{x_n\}$ a sequence in X.

(i) A sequence $\{x_n\}$ in X is called b-convergent if there exists $x \in X$ such that $d(x_n,x) \to 0$ as $n \to \infty$. In this case, we write $\lim\limits_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\}$ in X is called b-Cauchy if $d(x_n,x_m) \to 0$ as $n,m \to \infty$.

(iii) A b-metric space (X,d) is said to be a complete b-metric space if every b-Cauchy sequence in X is b-convergent.

(*iv*) A set $B \subset X$ is said to be b-closed if for any sequence $\{x_n\}$ in B such that $\{x_n\}$ is b-convergent to $z \in X$ then $z \in B$.

In general, a b-metric is not necessarily continuous.

Example 1.3. [22] Let $X = \mathbb{N} \cup \{\infty\}$. We define a mapping $d : X \times X \to \mathbb{R}^+$ as follows:

$$d(m,n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m,n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m,n \text{ is odd and the other is odd or } \infty, \\ 2 & \text{otherwise.} \end{cases}$$

Then (X,d) is a b-metric space with coefficient $s = \frac{5}{2}$.

Definition 1.4. [13] Let (X,d_X) and (Y,d_Y) be two b-metric spaces. A function $f : X \to Y$ is a b-continuous at a point $x \in X$, if it is b-sequentially continuous at x. i.e., whenever $\{x_n\}$ is b-convergent to x, fx_n is b-convergent to fx.

Definition 1.5. [23] A pair (A,B) of selfmaps on a metric space (X,d) is said to be compatible if $\lim\limits_{n\to\infty} d(BAx_n,ABx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 1.6. [1] A pair (A,B) of selfmaps on a metric space (X,d) is said to be satisfy (E.A)-property if there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 1.7. [27] A pair (A,B) of selfmaps on a b-metric space (X,d) is said to be satisfy b-(E.A)-property if there exists a sequence $\{x_n\}$ in X such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Bx_n = z$ for some $z \in X$.

Definition 1.8. [24] A pair (A,B) of selfmaps on a set X is said to be weakly compatible if $ABx = BAx$ whenever $Ax = Bx$ for any $x \in X$.

The following lemmas are useful in proving our main results.

Lemma 1.9. [21] Let (X,d) be a b-metric space with coefficient $s \geq 1$. Suppose that $\{x_n\}$ is a sequence in X such that $d(x_n,x_{n+1}) \leq kd(x_{n-1},x_n)$ for all $n \in \mathbb{N}$, where $k \in [0,1)$ is a constant. Then $\{x_n\}$ is a b-Cauchy sequence in X.

Lemma 1.10. [2] Let (X,d) be a b-metric space with coefficient $s \geq 1$. Suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x and y respectively, then we have

$\frac{1}{s^2}d(x,y) \leq \liminf_{n\to\infty} d(x_n,y_n) \leq \limsup_{n\to\infty} d(x_n,y_n) \leq s^2 d(x,y).$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n,y_n) = 0$. Moreover for each $z \in X$ we have

$\frac{1}{s}d(x,z) \leq \liminf_{n\to\infty} d(x_n,z) \leq \limsup_{n\to\infty} d(x_n,z) \leq sd(x,z).$

Lemma 1.11. Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing continuous map with $\psi(0) = 0$. Then for any sequence $\{t_n\} \subset \mathbb{R}^+$, we have $\liminf_{n\to\infty} \psi(t_n) = \psi(\liminf_{n\to\infty} t_n)$ and $\limsup_{n\to\infty} \psi(t_n) = \psi(\limsup_{n\to\infty} t_n)$.

The following theorem is due to Fisher [20].

Theorem 1.12. [20] Let T be a mapping of the complete metric space X into itself satisfying the inequality

$$[d(Tx,Ty)]^2 \leq a(d(x,Tx)d(y,Ty)) + b(d(x,Ty)d(y,Tx)) \tag{1.1}$$

for all $x, y \in X, 0 \leq a < 1, 0 \leq b$ then T has a fixed point in X.

Definition 1.13. Let (X,d) be a metric space, $T : X \to X$. If T satisfies (1.1) then we call T is a 'Fisher type contraction map' on X.

In 1980, Pachpatte [26] extended the result of Fisher [20] in the following way.

Theorem 1.14. [26] Let T be a mapping of the complete metric space X into itself satisfying the inequality

$$[d(Tx,Ty)]^2 \leq a[d(x,Tx)d(y,Ty) + d(x,Ty)d(y,Tx)] + b[d(x,Tx)d(y,Tx) + d(x,Ty)d(y,Ty)]$$

for all $x, y \in X$ where $a, b \geq 0$ and $a + 2b < 1$. Then T has a fixed point in X.

The following theorem is due to Sharma and Sahu [33].

Theorem 1.15. [33] Let T be a mapping of the complete metric space X into itself satisfying the inequality:

$$\begin{aligned}[d(Tx,Ty)]^2 \leq{} & \alpha_1[d(x,Tx)d(y,Ty) + d(x,Ty)d(y,Tx)] + \alpha_2[d(x,Tx)d(y,Tx) + d(x,Ty)d(y,Ty)] \\ & + \alpha_3[(d(y,Tx))^2 + (d(y,Ty))^2]\end{aligned}$$

for all $x, y \in X$ where $\alpha_1, \alpha_2, \alpha_3 \geq 0$ and $\alpha_1 + 2\alpha_2 + \alpha_3 < 1$.

Then T has a fixed point in X.

Recently, Ali and Arshad [4] and Alqahtani, Fulga, Karapinar and Özturk [5] proved the following theorems in the context of b-metric spaces.

Theorem 1.16. [4] Let (X,d) be a complete b-metric space with the coefficient $s \geq 1$ and suppose that selfmaps $f, g, h : X \to X$ satisfy the condition:

$[d(fx,gy)]^2 \leq \alpha_1[d(hx,fx)d(hy,gy)] + \alpha_2[d(hx,gy)d(hy,gy)(1 + d(fx,hy))]$

$$+\alpha_3[d(hx,hy)d(hy,gy)(1+d(fx,hy))]+\alpha_4[d(hx,gy)d(hy,gy)]$$
$$+\alpha_5[(d(hy,fx))^2+(d(hy,gy))^2]$$

for all $x,y\in X$ where $\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5\geq 0$ are nonnegative reals with

$s\alpha_1+(s^2+s)\alpha_2+s\alpha_3+(s^2+s)\alpha_4+\alpha_5<1$.

If $f(X)\cup g(X)\subseteq h(X)$ and $h(X)$ is a complete subspace of X, then the maps f,g and h have a coincidence point u in X. Moreover, if (f,h) and (g,h) are weakly compatible, then f,g and h have a unique common fixed point in X.

Theorem 1.17. [5] Let (X,d) be a complete b-metric space with the coefficient $s\geq 1$ and suppose the selfmaps $f,g,h:X\to X$ satisfy the condition

$$[d(fx,gy)]^2\leq\psi(F(x,y)),\text{ for all } x,y\in X,$$

where $\psi:\mathbb{R}^+\to\mathbb{R}^+$ increasing, there exist $k_0\in\mathbb{N}$, $a\in(0,1)$ and a convergent nonnegative series $\sum_{k=1}^{\infty}v_k$ such that $s^{k+1}\psi^{k+1}(t)\leq as^k\psi^k(t)+v_k$, for $k\geq k_0$ and any $t\geq 0$ and

$F(x,y)=\max\{d(fx,gy)d(hx,fx),d(fx,gy)d(hy,gy),d(hy,fx)d(hx,gy),\frac{1}{2s}d(hy,gy)d(hx,gy)\}$.

If $f(X)\cup g(X)\subseteq h(X)$ and $h(X)$ is a closed subspace of X, then the maps f,g and h have a coincidence point u in X. Moreover, if (f,h) and (g,h) are weakly compatible, then f,g and h have a unique common fixed point in X.

In Section 2, we prove the existence and uniqueness of common fixed points for two pairs of selfmaps satisfying a Fisher type weakly contractive condition in which one pair is compatible, b-continous and the another one is weakly compatible in complete b-metric spaces. Further, we prove the same with different hypotheses on two pairs of selfmaps which satisfy b-(E.A)-property.

2. Main results

Let A,B,S and T be mappings from a b-metric space (X,d) into itself and satisfying

$$A(X)\subseteq T(X)\text{ and } B(X)\subseteq S(X). \tag{2.1}$$

Now by (2.1), for any $x_0\in X$, there exists $x_1\in X$ such that $y_0=Ax_0=Tx_1$. In the same way for this x_1, we can choose a point $x_2\in X$ such that $y_1=Bx_1=Sx_2$ and so on. In general, we define

$y_{2n}=Ax_{2n}=Tx_{2n+1}$ and

$$y_{2n+1}=Bx_{2n+1}=Sx_{2n+2}\text{ for } n=0,1,2,\ldots. \tag{2.2}$$

Proposition 2.1. Let (X,d) be a b-metric space wuth coefficient $s \geq 1$. Assume that A,B,S and T are selfmappings of X which satisfy the following condition: there exist $\psi, \varphi \in \Psi$ such that

$$\psi(s^8[d(Ax,By)]^2) \leq \psi(M(x,y)) - \varphi(M(x,y)) \tag{2.3}$$

where

$$M(x,y) = \max\{d(Ax,By)d(Sx,Ax), d(Ax,By)d(Ty,By), d(Sx,Ty)d(Ty,By), d(Ax,Ty)d(Sx,By), \tfrac{1}{2s}d(Ty,By)d(Sx,By)\},$$

for all $x,y \in X$. Then we have the following:

(i) If $A(X) \subseteq T(X)$ and the pair (B,T) is weakly compatible and if z is a common fixed point of A and S then z is a common fixed point of A,B,S and T and it is unique.

(ii) If $B(X) \subseteq S(X)$ and the pair (A,S) is weakly compatible and if z is a common fixed point of B and T then z is a common fixed point of A,B,S and T and it is unique.

Proof. First, we assume that (i) holds.

Let z be a common fixed point of A and S. Then $Az = Sz = z$.

Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $Tu = z$.

Therefore $Az = Sz = Tu = z$.

We now prove that $Az = Bu$.

Suppose that $Az \neq Bu$.

We now consider

$$\psi(s^8[d(Az,Bu)]^2) \leq \psi(M(z,u)) - \varphi(M(z,u)) \tag{2.4}$$

where

$$\begin{aligned} M(z,u) &= \max\{d(Az,Bu)d(Sz,Az), d(Az,Bu)d(Tu,Bu), d(Sz,Tu)d(Tu,Bu), d(Az,Tu)d(Sz,Bu), \tfrac{1}{2s}d(Tu,Bu)d(Sz,Bu)\}, \\ &= \max\{0, [d(Az,Bu)]^2, 0, \tfrac{[d(Az,Bu)]^2}{2s}\} \\ &= [d(Az,Bu)]^2. \end{aligned}$$

From the inequality (2.4), we have

$$\psi(s^8[d(Az,Bu)]^2 \leq \psi([d(Az,Bu)]^2) - \varphi([d(Az,Bu)]^2) < \psi([d(Az,Bu)]^2).$$

Since ψ is monotonically increasing, we have

$$s^8[d(Az,Bu)]^2 \leq [d(Az,Bu)]^2.$$

Since $(s^8 - 1) > 0$, we have $d(Az,Bu) \leq 0$, which is a contradiction.

Hence $Az = Bu$. Therefore $Az = Bu = Sz = Tu = z$.

Since the pair (B,T) is weakly compatible and $Bu = Tu$, we have $BTu = TBu$. i.e., $Bz = Tz$.

Now we show that $Bz = z$.

If $Bz \neq z$, then we have

$$\begin{aligned}\psi(s^8[d(Bz,z)]^2) &= \psi(s^8[d(z,Bz)]^2)\\ &= \psi(s^8[d(Az,Bz)]^2)\\ &\leq \psi(M(z,z)) - \varphi(M(z,z)) \qquad (2.5)\end{aligned}$$

where

$$\begin{aligned}M(z,z) = \max\{&d(Az,Bz)d(Sz,Az), d(Az,Bz)d(Tz,Bz), d(Sz,Tz)d(Tz,Bz),\\ &d(Az,Tz)d(Sz,Bz), \tfrac{1}{2s}d(Tz,Bz)d(Sz,Bz)\},\\ = \max\{&0, [d(Az,Bz)]^2, 0, \tfrac{[d(Az,Bz)]^2}{2s}\}\\ = [d(&Az,Bz)]^2 = [d(Bz,z)]^2.\end{aligned}$$

From the inequality (2.5), we have

$$\psi(s^8[d(Bz,z)]^2) \leq \psi([d(Bz,z)]^2) - \varphi([d(Bz,z)]^2) < \psi([d(Bz,z)]^2).$$

Since ψ is monotonically increasing, we have

$$s^8[d(Bz,z)]^2 \leq [d(Bz,z)]^2.$$

Since $(s^8 - 1) > 0$, we have $d(Bz,z) \leq 0$,

which is a contradiction.

Hence $Bz = z$. Therefore $Az = Bz = Sz = Tz = z$.

Therefore z is a common fixed point of A, B, S and T.

In a similar way, under the assumption (ii), the conclusion of the proposition follows.

Uniqueness follows from the inequality (2.3). □

Remark 2.2. Selfmaps A, B, S and T of a metric space X that satisfy (2.3) is said to be Fisher type weakly contractive maps on X.

Proposition 2.3. Let A, B, S and T be selfmaps of a b-metric space (X, d) and satisfy (2.1) and Fisher type weakly contractive maps. Then for any $x_0 \in X$, the sequence $\{y_n\}$ defined by (2.2) is Cauchy in X.

Proof. Let $x_0 \in X$ and let $\{y_n\}$ be defined by (2.2).

Assume that $y_n = y_{n+1}$ for some n.

Case (i). n even.

We write $n = 2m$ for some $m \in \mathbb{N}$. Suppose that $d(y_{n+1}, y_{n+2}) > 0$.

Now we consider

$$\begin{aligned}
\psi(s^8[d(y_{n+1}, y_{n+2})]^2) &= \psi(s^8[d(y_{2m+1}, y_{2m+2})]^2) \\
&= \psi(s^8[d(y_{2m+2}, y_{2m+1})]^2) \\
&= \psi(s^8[d(Ax_{2m+2}, Bx_{2m+1})]^2) \\
&\leq \psi(M(x_{2m+2}, x_{2m+1})) - \varphi(M(x_{2m+2}, x_{2m+1})) \qquad (2.6)
\end{aligned}$$

where

$$\begin{aligned}
M(x_{2m+2}, x_{2m+1}) = \max\{ & d(Ax_{2m+2}, Bx_{2m+1})d(Sx_{2m+2}, Ax_{2m+2}), \\
& d(Ax_{2m+2}, Bx_{2m+1})d(Tx_{2m+1}, Bx_{2m+1}), \\
& d(Sx_{2m+2}, Tx_{2m+1})d(Tx_{2m+1}, Bx_{2m+1}), \\
& d(Ax_{2m+2}, Tx_{2m+1})d(Sx_{2m+2}, Bx_{2m+1}), \\
& \tfrac{1}{2s}d(Tx_{2m+1}, Bx_{2m+1})d(Sx_{2m+2}, Bx_{2m+1})\}. \\
= \max\{ & d(y_{2m+2}, y_{2m+1})d(y_{2m+1}, y_{2m+2}), d(y_{2m+2}, y_{2m+1})d(y_{2m}, y_{2m+1}), \\
& d(y_{2m+1}, y_{2m})d(y_{2m}, y_{2m+1}), d(y_{2m+2}, y_{2m})d(y_{2m+1}, y_{2m+1}), \\
& \tfrac{1}{2s}d(y_{2m}, y_{2m+1})d(y_{2m+1}, y_{2m+1})\} \\
= \max\{ & [d(y_{2m+1}, y_{2m+2})]^2, 0, 0, 0, 0\} \\
= {} & [d(y_{n+1}, y_{n+2})]^2.
\end{aligned}$$

From the inequality (2.6), we have

$$\begin{aligned}
\psi(s^8[d(y_{n+1}, y_{n+2})]^2) &\leq \psi([d(y_{n+1}, y_{n+2})]^2) - \varphi([d(y_{n+1}, y_{n+2})]^2) \\
&< \psi([d(y_{n+1}, y_{n+2})]^2).
\end{aligned}$$

Since ψ is monotonically increasing, we have

$s^8[d(y_{n+1}, y_{n+2})]^2 \leq [d(y_{n+1}, y_{n+2})]^2$.

Since $(s^8 - 1) > 0$, we have $d(y_{n+1}, y_{n+2}) \leq 0$,

which is a contradiction.

Therefore $y_{n+2} = y_{n+1} = y_n$.

In general, we have $y_{n+k} = y_n$ for $k = 0, 1, 2, \ldots$.

Case (ii). n odd.

We write $n = 2m+1$ for some $m \in \mathbb{N}$.

We now consider

$$\begin{aligned} \psi(s^8[d(y_{n+1},y_{n+2})]^2) &= \psi(s^8[d(y_{2m+2},y_{2m+3})]^2) \\ &= \psi(s^8[d(Ax_{2m+2},Bx_{2m+3})]^2) \\ &\leq \psi(M(x_{2m+2},x_{2m+3})) - \varphi(M(x_{2m+2},x_{2m+3})) \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} M(x_{2m+2},x_{2m+3}) = \max\{ & d(Ax_{2m+2},Bx_{2m+3})d(Sx_{2m+2},Ax_{2m+2}), \\ & d(Ax_{2m+2},Bx_{2m+3})d(Tx_{2m+3},Bx_{2m+3}), \\ & d(Sx_{2m+2},Tx_{2m+3})d(Tx_{2m+3},Bx_{2m+3}), \\ & d(Ax_{2m+2},Tx_{2m+3})d(Sx_{2m+2},Bx_{2m+3}), \\ & \tfrac{1}{2s}d(Tx_{2m+3},Bx_{2m+3})d(Sx_{2m+2},Bx_{2m+3})\}. \\ = \max\{ & d(y_{2m+2},y_{2m+3})d(y_{2m+1},y_{2m+2}), d(y_{2m+2},y_{2m+3})d(y_{2m+2},y_{2m+3}), \\ & d(y_{2m+1},y_{2m+2})d(y_{2m+2},y_{2m+3}), d(y_{2m+2},y_{2m+2})d(y_{2m+1},y_{2m+3}), \\ & \tfrac{1}{2s}d(y_{2m+2},y_{2m+3})d(y_{2m+1},y_{2m+3})\} \\ = \max\{ & 0, [d(y_{2m+2},y_{2m+3})]^2, 0, 0, 0, \tfrac{[d(y_{2m+2},y_{2m+3})]^2}{2}\} \\ = [& d(y_{n+1},y_{n+2})]^2. \end{aligned}$$

From the inequality (2.7), we have

$$\begin{aligned} \psi(s^8[d(y_{n+1},y_{n+2})]^2) &\leq \psi([d(y_{n+1},y_{n+2})]^2) - \varphi([d(y_{n+1},y_{n+2})]^2) \\ &< \psi([d(y_{n+1},y_{n+2})]^2). \end{aligned}$$

Since ψ is monotonically increasing, we have

$s^8[d(y_{n+1},y_{n+2})]^2 \leq [d(y_{n+1},y_{n+2})]^2$.

Since $(s^8-1) > 0$, we have $d(y_{n+1},y_{n+2}) \leq 0$,

it is a contradiction.

Therefore $y_{n+2} = y_{n+1} = y_n$.

In general, we have $y_{n+k} = y_n$ for $k = 0,1,2,\ldots$.

From Case (i) and Case (ii), we have $y_{n+k} = y_n$ for all $k = 0,1,2,\ldots$.

Hence $\{y_{n+k}\}$ is a constant sequence and hence $\{y_n\}$ is Cauchy.

Now we assume that $y_{n-1} \neq y_n$ for all $n \in \mathbb{N}$.

If n is odd, then $n = 2m+1$ for some $m \in \mathbb{N}$.

We now consider

$$\begin{aligned}\psi(s^8[d(y_n,y_{n+1})]^2) &= \psi(s^8[d(y_{2m+1},y_{2m+2})]^2)\\ &= \psi(s^8[d(y_{2m+2},y_{2m+1})]^2)\\ &= \psi(s^8[d(Ax_{2m+2},Bx_{2m+1})]^2)\\ &\leq \psi(M(x_{2m+2},x_{2m+1})) - \varphi(M(x_{2m+2},x_{2m+1})) \qquad (2.8)\end{aligned}$$

where

$$\begin{aligned}M(x_{2m+2},x_{2m+1}) = \max\{&d(Ax_{2m+2},Bx_{2m+1})d(Sx_{2m+2},Ax_{2m+2}),\\ &d(Ax_{2m+2},Bx_{2m+1})d(Tx_{2m+1},Bx_{2m+1}),\\ &d(Sx_{2m+2},Tx_{2m+1})d(Tx_{2m+1},Bx_{2m+1}),\\ &d(Ax_{2m+2},Tx_{2m+1})d(Sx_{2m+2},Bx_{2m+1}),\\ &\tfrac{1}{2s}d(Tx_{2m+1},Bx_{2m+1})d(Sx_{2m+2},Bx_{2m+1})\}\\ = \max\{&d(y_{2m+2},y_{2m+1})d(y_{2m+1},y_{2m+2}), d(y_{2m+2},y_{2m+1})d(y_{2m},y_{2m+1}),\\ &d(y_{2m+1},y_{2m})d(y_{2m},y_{2m+1}), d(y_{2m+2},y_{2m})d(y_{2m+1},y_{2m+1}),\\ &\tfrac{1}{2s}d(y_{2m},y_{2m+1})d(y_{2m+1},y_{2m+1})\}\\ = \max\{&[d(y_{2m+1},y_{2m+2})]^2, d(y_{2m},y_{2m+1})d(y_{2m+2},y_{2m+1}), [d(y_{2m},y_{2m+1})]^2, 0, 0\}\\ = \max\{&[d(y_{2m+1},y_{2m+2})]^2, d(y_{2m},y_{2m+1})d(y_{2m+1},y_{2m+2}), [d(y_{2m},y_{2m+1})]^2\}\\ = \max\{&[d(y_{2m+1},y_{2m+2})]^2, [d(y_{2m},y_{2m+1})]^2\}.\end{aligned}$$

Suppose $M(x_{2m+2},x_{2m+1}) = [d(y_{2m+1},y_{2m+2})]^2$.

From the inequality (2.8), we have

$$\begin{aligned}\psi(s^8[d(y_n,y_{n+1})]^2) &\leq \psi([d(y_{2m+1},y_{2m+2})]^2) - \varphi([d(y_{2m+1},y_{2m+2})]^2)\\ &< \psi([d(y_n,y_{n+1})]^2).\end{aligned}$$

Since ψ is monotonically increasing, we have

$s^8[d(y_n,y_{n+1})]^2 \leq [d(y_n,y_{n+1})]^2$.

Since $(s^8-1) > 0$, we have $d(y_n,y_{n+1}) \leq 0$,

which is a contradiction.

Therefore $M(x_{2m+2},x_{2m+1}) = [d(y_{2m},y_{2m+1})]^2$.

From the inequality (2.8), we have

$$\psi(s^8[d(y_n,y_{n+1})]^2) \leq \psi([d(y_{2m},y_{2m+1})]^2) - \varphi([d(y_{2m},y_{2m+1})]^2)$$
$$< \psi([d(y_{n-1},y_n)]^2).$$

Since ψ is monotonically increasing, we have

$$s^8[d(y_n,y_{n+1})]^2 \leq [d(y_{n-1},y_n)]^2 \text{ when } n \text{ is odd.} \tag{2.9}$$

Also it is easy to see that (2.9) is valid when n is even.

Hence we have

$[d(y_n,y_{n+1})]^2 \leq \frac{1}{s^8}[d(y_{n-1},y_n)]^2$ for all $n \in \mathbb{N}$.

From Lemma 1.9, we have the sequence $\{y_n\}$ is a b-Cauchy sequence in X. □

The following is the main result of this paper.

Theorem 2.4. Let A,B,S and T be selfmaps on a complete b-metric space (X,d) and satisfy (2.1) and Fisher type weakly contractive maps. If either

(i) the pair (A,S) compatible, A (or) S is b-continuous and the pair (B,T) is weakly compatible

or

(ii) the pair (B,T) compatible, B (or) T is b-continuous and the pair (A,S) is weakly compatible

then A,B,S and T have a unique common fixed point in X.

Proof. By Proposition 2.3, the sequence $\{y_n\}$ is Cauchy in X. Since X is b-complete, there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$.

Thus, $\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z$

and $\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z.$ (2.10)

Assume that (i) holds.

Since S is b-continuous, it follows that

$\lim_{n\to\infty} SSx_{2n+2} = Sz,\ \lim_{n\to\infty} SAx_{2n} = Sz.$

By the triangle inequality, we have

$d(ASx_{2n},Sz) \leq s[d(ASx_{2n},SAx_{2n}) + d(SAx_{2n},Sz)].$

Since the pair (A,S) is compatible, $\lim_{n\to\infty} d(ASx_{2n},SAx_{2n}) = 0.$

Taking limit superior as $n \to \infty$, we have

$$\limsup_{n\to\infty} d(ASx_{2n},Sz) \leq s[\limsup_{n\to\infty} d(ASx_{2n},SAx_{2n}) + \limsup_{n\to\infty} d(SAx_{2n},Sz)] = 0.$$

Therefore $\lim_{n\to\infty} ASx_{2n} = Sz$.

We now prove that $Sz = z$. Suppose that $Sz \neq z$.

From the inequality (2.3), we have

$$\psi(s^8[d(ASx_{2n+2}, Bx_{2n+1})]^2) \leq \psi(M(Sx_{2n+2}, x_{2n+1})) - \varphi(M(Sx_{2n+2}, x_{2n+1})) \tag{2.11}$$

where

$$\begin{aligned} M(Sx_{2n+2}, x_{2n+1}) = \max\{ & d(ASx_{2n+2}, Bx_{2n+1})d(SSx_{2n+2}, ASx_{2n+2}), \\ & d(ASx_{2n+2}, Bx_{2n+1})d(Tx_{2n+1}, Bx_{2n+1}), \\ & d(SSx_{2n+2}, Tx_{2n+1})d(Tx_{2n+1}, Bx_{2n+1}), \\ & d(ASx_{2n+2}, Tx_{2n+1})d(SSx_{2n+2}, Bx_{2n+1}), \\ & \tfrac{1}{2s} d(Tx_{2n+1}, Bx_{2n+1})d(SSx_{2n+2}, Bx_{2n+1})\}. \end{aligned}$$

By taking limit superior as $n \to \infty$ on $M(Sx_{2n+2}, x_{2n+1})$ and using Lemma 1.10, we obtain

$$\begin{aligned} \limsup_{n\to\infty} M(Sx_{2n+2}, x_{2n+1}) \leq \max\{ & s^4 d(Sz,z)d(Sz,z), s^4 d(Sz,z)d(z,z), s^4 d(Sz,z)d(z,z), \\ & s^4 d(Sz,z)d(Sz,z), \tfrac{s^4}{2s} d(z,z)d(Sz,z)\} \\ = {} & s^4[d(Sz,z)]^2. \end{aligned}$$

Therefore $\frac{1}{s^4}[d(Sz,z)]^2 \leq \liminf_{n\to\infty} M(Sx_{2n+2}, x_{2n+1}) \leq \limsup_{n\to\infty} M(Sx_{2n+2}, x_{2n+1}) \leq s^4[d(Sz,z)]^2$.

Taking limit superior as $n \to \infty$ in the inequality (2.11) and using Lemma 1.10, we get

$$\begin{aligned} \psi(s^8 \tfrac{1}{s^4}[d(Sz,z)]^2) &\leq \psi(s^8 \limsup_{n\to\infty}[d(ASx_{2n+2}, Bx_{2n+1})]^2) \\ &= \limsup_{n\to\infty} \psi(s^8[d(ASx_{2n+2}, Bx_{2n+1})]^2) \\ &\leq \limsup_{n\to\infty}[\psi(M(Sx_{2n+2}, x_{2n+1})) - \varphi(M(Sx_{2n+2}, x_{2n+1}))] \\ &\leq \psi(\limsup_{n\to\infty} M(Sx_{2n+2}, x_{2n+1})) - \liminf_{n\to\infty} \varphi(M(Sx_{2n+2}, x_{2n+1})) \\ &\leq \psi(\limsup_{n\to\infty} M(Sx_{2n+2}, x_{2n+1})) - \varphi(\liminf_{n\to\infty} M(Sx_{2n+2}, x_{2n+1})) \\ &< \psi(\limsup_{n\to\infty} M(Sx_{2n+2}, x_{2n+1})) \\ &\leq \psi(s^4[d(Sz,z)]^2). \end{aligned}$$

Therefore $\psi(s^4[d(Sz,z)]^2) < \psi(s^4[d(Sz,z)]^2)$,

which is a contradiction.

Therefore $Sz = z$.

We now show that $Az = z$. Suppose that $Az \neq z$.

$$\psi(s^8[d(Az, Bx_{2n+1})]^2) \leq \psi(M(z, x_{2n+1})) - \varphi(M(z, x_{2n+1})) \tag{2.12}$$

where

$$\begin{aligned}M(z,x_{2n+1}) = \max\{&d(Az,Bx_{2n+1})d(Sz,Az), d(Az,Bx_{2n+1})d(Tx_{2n+1},Bx_{2n+1}),\\ &d(Sz,Tx_{2n+1})d(Tx_{2n+1},Bx_{2n+1}), d(Az,Tx_{2n+1})d(Sz,Bx_{2n+1}),\\ &\tfrac{1}{2s}d(Tx_{2n+1},Bx_{2n+1})d(Sz,Bx_{2n+1})\}.\end{aligned}$$

On letting limit superior as $n\to\infty$ on $M(z,x_{2n+1})$ and using Lemma 1.10, we obtain

$$\begin{aligned}\limsup_{n\to\infty} M(z,x_{2n+1}) &\le \max\{sd(Az,z)d(z,Az), s^3d(Az,z)d(z,z), s^3d(z,z)d(z,z), s^2d(Az,z)d(z,z),\\ &\qquad \tfrac{1}{2s}s^3d(z,z)d(z,z)\}\\ &= s[d(Az,z)]^2.\end{aligned}$$

Therefore $\frac{1}{s}[d(Az,z)]^2 \le \liminf_{n\to\infty} M(z,x_{2n+1}) \le \limsup_{n\to\infty} M(z,x_{2n+1}) \le s[d(Sz,z)]^2$.

Taking limit superior as $n\to\infty$ in the inequality (2.12) and using Lemma 1.10, we get

$$\begin{aligned}\psi(s^8\tfrac{1}{s}[d(Az,z)]^2) &\le \psi(s^8\limsup_{n\to\infty}[d(Az,Bx_{2n+1})]^2)\\ &= \limsup_{n\to\infty}\psi(s^8[d(Az,Bx_{2n+1})]^2)\\ &\le \limsup_{n\to\infty}[\psi(M(z,x_{2n+1}))-\varphi(M(z,x_{2n+1}))]\\ &\le \psi(\limsup_{n\to\infty} M(z,x_{2n+1}))-\liminf_{n\to\infty}\varphi(M(z,x_{2n+1}))\\ &\le \psi(\limsup_{n\to\infty} M(z,x_{2n+1}))-\varphi(\liminf_{n\to\infty} M(z,x_{2n+1}))\\ &< \psi(\limsup_{n\to\infty} M(z,x_{2n+1}))\\ &\le \psi(s[d(Az,z)]^2).\end{aligned}$$

Since ψ is monotonically increasing, we have

$s^7[d(Az,z)]^2 \le s[d(Az,z)]^2$.

Since $(s^6-1)>0$, we have

$[d(Az,z)]^2 \le 0$ which implies that $d(Az,z)\le 0$ and implies that $Az=z$.

Therefore $Az=Sz=z$. Hence z is a common fixed point of A and S.

Now by Proposition 2.1, we have z is a unique common fixed point of A,B,S and T.

Assume that A is b-continuous, it follows that

$\lim_{n\to\infty} AAx_{2n} = Az,\ \lim_{n\to\infty} ASx_{2n+2} = Az.$

By the b-triangle inequality, we have

$d(SAx_{2n},Az) \le s[d(SAx_{2n},ASx_{2n}) + d(ASx_{2n},Az)]$.

Since the pair (A,S) is compatible, $\lim_{n\to\infty} d(ASx_{2n},SAx_{2n}) = 0$.

Taking limit superior as $n\to\infty$, we have

$$\limsup_{n\to\infty} d(SAx_{2n},Az) \le s[\limsup_{n\to\infty} d(SAx_{2n},ASx_{2n}) + \limsup_{n\to\infty} d(ASx_{2n},Az)] = 0.$$

Therefore $\lim_{n\to\infty} SAx_{2n} = Az$.

Now we prove that $Az = z$. Suppose that $Az \neq z$.

From the inequality (2.3), we have

$$\psi(s^8[d(AAx_{2n}, Bx_{2n+1})]^2) \leq \psi(M(Ax_{2n}, x_{2n+1})) - \varphi(M(Ax_{2n}, x_{2n+1})) \tag{2.13}$$

where

$$\begin{aligned} M(Ax_{2n}, x_{2n+1}) = \max\{ & d(AAx_{2n}, Bx_{2n+1})d(SAx_{2n}, AAx_{2n}), \\ & d(AAx_{2n}, Bx_{2n+1})d(Tx_{2n+1}, Bx_{2n+1}), \\ & d(SAx_{2n}, Tx_{2n+1})d(Tx_{2n+1}, Bx_{2n+1}), \\ & d(AAx_{2n}, Tx_{2n+1})d(SAx_{2n}, Bx_{2n+1}), \\ & \tfrac{1}{2s} d(Tx_{2n+1}, Bx_{2n+1})d(SAx_{2n}, Bx_{2n+1})\}. \end{aligned}$$

On letting limit superior as $n \to \infty$ on $M(Ax_{2n}, x_{2n+1})$ and using Lemma 1.10, we obtain

$$\begin{aligned} \limsup_{n\to\infty} M(Ax_{2n}, x_{2n+1}) &\leq \max\{s^4 d(Az,z)d(Az,z), s^4 d(Az,z)d(z,z), s^4 d(Az,z)d(z,z), \\ &\qquad s^4 d(Az,z)d(Az,z), \tfrac{s^4}{2s} d(z,z)d(Az,z)\} \\ &= s^4[d(Az,z)]^2. \end{aligned}$$

Therefore $\frac{1}{s^4}[d(Az,z)]^2 \leq \liminf_{n\to\infty} M(Ax_{2n}, x_{2n+1}) \leq \limsup_{n\to\infty} M(Ax_{2n}, x_{2n+1}) \leq s^4[d(Az,z)]^2$.

Taking limit superior as $n \to \infty$ in the inequality (2.13) and using Lemma 1.10, we get

$$\begin{aligned} \psi(s^8 \tfrac{1}{s^4}[d(Az,z)]^2) &\leq \psi(s^8 \limsup_{n\to\infty}[d(AAx_{2n}, Bx_{2n+1})]^2) \\ &= \limsup_{n\to\infty} \psi(s^8[d(AAx_{2n}, Bx_{2n+1})]^2) \\ &\leq \limsup_{n\to\infty}[\psi(M(Ax_{2n}, x_{2n+1})) - \varphi(M(Ax_{2n}, x_{2n+1}))] \\ &\leq \psi(\limsup_{n\to\infty} M(Ax_{2n}, x_{2n+1})) - \liminf_{n\to\infty} \varphi(M(Ax_{2n}, x_{2n+1})) \\ &\leq \psi(\limsup_{n\to\infty} M(Ax_{2n}, x_{2n+1})) - \varphi(\liminf_{n\to\infty} M(Ax_{2n}, x_{2n+1})) \\ &< \psi(\limsup_{n\to\infty} M(Ax_{2n}, x_{2n+1})) \\ &\leq \psi(s^4[d(Az,z)]^2). \end{aligned}$$

Therefore $\psi(s^4[d(Az,z)]^2) < \psi(s^4[d(Az,z)]^2)$,

it is a contradiction.

Therefore $Az = z$. Since $A(X) \subseteq T(X)$, there exists $u \in X$ such that $z = Tu$.

We now show that $Bu = z$. Suppose that $Bu \neq z$.

$$\psi(s^8[d(Ax_{2n}, Bu)]^2) \leq \psi(M(x_{2n}, u)) - \varphi(M(x_{2n}, u)) \tag{2.14}$$

where

$$M(x_{2n},u)=\max\{d(Ax_{2n},Bu)d(Sx_{2n},Ax_{2n}),d(Ax_{2n},Bu)d(Tu,Bu),$$
$$d(Sx_{2n},Tu)d(Tu,Bu),d(Ax_{2n},Tu)d(Sx_{2n},Bu),$$
$$\tfrac{1}{2s}d(Tu,Bu)d(Sx_{2n},Bu)\}.$$

By taking limit superior as $n\to\infty$ on $M(x_{2n},u)$ and using Lemma 1.10, we obtain

$$\begin{aligned}\limsup_{n\to\infty}M(x_{2n},u)&\leq\max\{s^3d(z,Bu)d(z,z),sd(z,Bu)d(z,Bu),sd(z,z)d(z,Bu),\\&\qquad s^2d(z,z)d(z,Bu),\tfrac{s}{2s}d(z,Bu)d(z,Bu)\}\\&=s[d(z,Bu)]^2.\end{aligned}$$

Therefore $\frac{1}{s}[d(z,Bu)]^2\leq\liminf_{n\to\infty}M(Ax_{2n},u)\leq\limsup_{n\to\infty}M(Ax_{2n},u)\leq s[d(z,Bu)]^2$.

Taking limit superior as $n\to\infty$ in the inequality (2.14) and using Lemma 1.10, we get

$$\begin{aligned}\psi(s^8\tfrac{1}{s}[d(z,Bu)]^2)&\leq\psi(s^8\limsup_{n\to\infty}[d(Ax_{2n},Bu)]^2)\\&=\limsup_{n\to\infty}\psi(s^8[d(Ax_{2n},Bu)]^2)\\&\leq\limsup_{n\to\infty}[\psi(M(x_{2n},u))-\varphi(M(x_{2n},u))]\\&\leq\psi(\limsup_{n\to\infty}M(x_{2n},u))-\liminf_{n\to\infty}\varphi(M(x_{2n},u))\\&\leq\psi(\limsup_{n\to\infty}M(x_{2n},u))-\varphi(\liminf_{n\to\infty}M(x_{2n},u))\\&<\psi(\limsup_{n\to\infty}M(x_{2n},u))\\&\leq\psi(s[d(z,Bu)]^2).\end{aligned}$$

Since ψ is monotonically increasing, we have

$s^7[d(z,Bu)]^2\leq s[d(z,Bu)]^2$.

Since $(s^6-1)>0$, we have

$[d(z,Bu)]^2\leq 0$ which implies that $Bu=z$.

Therefore $Bu=Tu=z$. Since the pair (B,T) is weakly compatible and $Bu=Tu$, we have

$BTu=TBu$. i.e., $Bz=Tz$.

We now show that $Bz=z$. Suppose that $Bz\neq z$.

$$\psi(s^8[d(Ax_{2n},Bz)]^2)\leq\psi(M(x_{2n},z))-\varphi(M(x_{2n},z))\tag{2.15}$$

where

$$M(x_{2n},z)=\max\{d(Ax_{2n},Bz)d(Sx_{2n},Ax_{2n}),d(Ax_{2n},Bz)d(Tz,Bz),$$
$$d(Sx_{2n},Tz)d(Tz,Bz),d(Ax_{2n},Tz)d(Sx_{2n},Bz),\tfrac{1}{2s}d(Tz,Bz)d(Sx_{2n},Bz)\}.$$

By taking limit superior as $n\to\infty$ on $M(x_{2n},z)$ and using Lemma 1.10, we obtain

$$\limsup_{n\to\infty}M(x_{2n},z)\leq\max\{s^3d(z,Bz)d(z,z),sd(z,Bz)d(z,Bz),sd(z,z)d(z,Bz),$$

$$s^2 d(z,z)d(z,Bz), \tfrac{s}{2s}d(z,Bz)d(z,Bz)\}$$
$$= s[d(z,Bz)]^2.$$

Therefore $\frac{1}{s}[d(z,Bz)]^2 \le \liminf_{n\to\infty} M(Ax_{2n},z) \le \limsup_{n\to\infty} M(Ax_{2n},z) \le s[d(z,Bz)]^2$.

Taking limit superior as $n\to\infty$ in the inequality (2.15) and using Lemma 1.10, we get

$$\begin{aligned}
\psi(s^8\tfrac{1}{s}[d(z,Bz)]^2) &\le \psi(s^8 \limsup_{n\to\infty}[d(Ax_{2n},Bz)]^2)\\
&= \limsup_{n\to\infty} \psi(s^8[d(Ax_{2n},Bz)]^2)\\
&\le \limsup_{n\to\infty}[\psi(M(x_{2n},z)) - \varphi(M(x_{2n},z))]\\
&\le \psi(\limsup_{n\to\infty} M(x_{2n},z)) - \liminf_{n\to\infty} \varphi(M(x_{2n},z))\\
&\le \psi(\limsup_{n\to\infty} M(x_{2n},z)) - \varphi(\liminf_{n\to\infty} M(x_{2n},z))\\
&< \psi(\limsup_{n\to\infty} M(x_{2n},z))\\
&\le \psi(s[d(z,Bz)]^2).
\end{aligned}$$

Since ψ is monotonically increasing, we have

$s^7[d(z,Bz)]^2 \le s[d(z,Bz)]^2$.

Since $(s^6-1) > 0$, we have

$[d(z,Bz)]^2 \le 0$.

Which implies that $d(Bz,z) \le 0$.

Hence $Bz = z$.

Therefore $Bz = Tz = z$.

Hence z is a common fixed point of A and S.

Now by Proposition 2.1, we have z is a unique common fixed point of A,B,S and T.

In a similar way, under the assumption (ii), the conclusion of the theorem holds. □

Theorem 2.5. Let (X,d) be a b-metric space with coefficient $s\ge 1$. Let $A,B,S,T : X\to X$ be selfmaps of X and satisfy (2.1) and Fisher type weakly contractive maps. Suppose that one of the pairs (A,S) and (B,T) satisfies the b-(E.A)-property and that one of the subspace $A(X),B(X),S(X)$ and $T(X)$ is b-closed in X. Then the pairs (A,S) and (B,T) have a point of coincidence in X. Moreover, if the pairs (A,S) and (B,T) are weakly compatible, then A,B,S and T have a unique common fixed point in X.

Proof. We first assume that the pair (A,S) satisfies the b-(E.A)-property. So there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = q \text{ for some } q \in X. \tag{2.16}$$

Since $A(X) \subseteq T(X)$, there exists a sequence $\{y_n\}$ in X such that $Ax_n = Ty_n$, and hence

$$\lim_{n\to\infty} Ty_n = q. \tag{2.17}$$

Now we show that $\lim_{n\to\infty} By_n = q$.

From the inequality (2.3), we have

$$\psi(s^8[d(Ax_n,By_n)]^2) \le \psi(M(x_n,y_n)) - \varphi(M(x_n,y_n)) \tag{2.18}$$

where

$$M(x_n,y_n) = \max\{d(Ax_n,By_n)d(Sx_n,Ax_n), d(Ax_n,By_n)d(Ty_n,By_n),$$
$$d(Sx_n,Ty_n)d(Ty_n,By_n), d(Ax_n,Ty_n)d(Sx_n,By_n), \tfrac{1}{2s}d(Ty_n,By_n)d(Sx_n,By_n)\}.$$

By taking limit superior as $n \to \infty$ on $M(x_n,y_n)$, and using (2.16) and (2.17), we obtain

$$\begin{aligned}\limsup_{n\to\infty} M(x_n,y_n) &= \max\{0, \limsup_{n\to\infty}[d(Ax_n,By_n)]^2, 0, 0, \tfrac{1}{2s}\limsup_{n\to\infty}[d(Ax_n,By_n)]^2\} \\ &= \limsup_{n\to\infty}[d(Ax_n,By_n)]^2. \end{aligned}\tag{2.19}$$

Similarly, we obtain

$$\liminf_{n\to\infty} M(x_n,y_n) = \liminf_{n\to\infty}[d(Ax_n,By_n)]^2. \tag{2.20}$$

On taking limit superior as $n \to \infty$ in (2.18), and using (2.19) and (2.20), we get

$$\begin{aligned}\psi(s^8 \limsup_{n\to\infty}[d(Ax_n,By_n)]^2) &= \limsup_{n\to\infty} \psi(s^8[d(Ax_n,By_n)]^2) \\ &\le \limsup_{n\to\infty} \psi(M(x_n,y_n)) - \liminf_{n\to\infty} \varphi(M(x_n,y_n)) \\ &= \psi(\limsup_{n\to\infty} M(x_n,y_n)) - \varphi(\liminf_{n\to\infty} M(x_n,y_n)) \\ &= \psi(\limsup_{n\to\infty}[d(Ax_n,By_n)]^2) - \varphi(\liminf_{n\to\infty}[d(Ax_n,By_n)]^2) \\ &< \psi(\limsup_{n\to\infty}[d(Ax_n,By_n)]^2).\end{aligned}$$

Since ψ is monotonically increasing, we have

$$s^8 \limsup_{n\to\infty}[d(Ax_n,By_n)]^2 \le \limsup_{n\to\infty}[d(Ax_n,By_n)]^2.$$

Since $(s^8 - 1) > 0$, we have

$\limsup_{n\to\infty}[d(Ax_n,By_n)]^2 \le 0$ implies that $\lim_{n\to\infty}[d(Ax_n,By_n)]^2 = 0$.

Therefore $\lim_{n\to\infty} d(Ax_n,By_n) = 0$. (2.21)

By the b-triangle inequality, we have

$$d(q,By_n) \le s[d(q,Ax_n) + d(Ax_n,By_n)]. \tag{2.22}$$

On takin limits as $n \to \infty$ in (2.22), and using (2.16) and (2.21), we get

$$\lim_{n\to\infty} d(q,By_n) \le s[\lim_{n\to\infty} d(q,Ax_n) + \lim_{n\to\infty} d(Ax_n,By_n)] = 0.$$

Therefore $\lim_{n\to\infty} d(q,By_n)=0.$

Case (i). Assume that $T(X)$ is a b-closed subset of X.

In this case $q\in T(X)$, we can choose $r\in X$ such that $Tr=q$.

We now prove that $Br=q$. Suppose that $d(Br,q)>0$.

From the inequality (2.3), we have

$$\psi(s^8[d(Ax_n,Br)]^2)\leq \psi(M(x_n,r))-\varphi(M(x_n,r)) \tag{2.23}$$

where

$$M(x_n,r)=\max\{d(Ax_n,Br)d(Sx_n,Ax_n),d(Ax_n,Br)d(Tr,Br),\\ d(Sx_n,Tr)d(Tr,Br),d(Ax_n,Tr)d(Sx_n,Br),\tfrac{1}{2s}d(Tr,Br)d(Sx_n,Br)\}.$$

On letting limit superior as $n\to\infty$ in (2.23), and using Lemma 1.10, we obtain

$$\begin{aligned}\limsup_{n\to\infty} M(x_n,r)&\leq \max\{0,s[d(q,Br)]^2,0,0,\tfrac{1}{2s}s[d(q,Br)]^2\}\\ &=s[d(q,Br)]^2.\end{aligned}$$

Therefore $\frac{1}{s}[d(q,Br)]^2\leq \liminf_{n\to\infty} M(x_n,r)\leq \limsup_{n\to\infty} M(x_n,r)\leq s[d(q,Br)]^2.$

Taking limit superior as $n\to\infty$ in (2.23) and using Lemma 1.10, we have

$$\begin{aligned}\psi(s^8\tfrac{1}{s}[d(q,Br)]^2)&=\psi(s^8\limsup_{n\to\infty}[d(Ax_n,Br)]^2)\\ &=\limsup_{n\to\infty}\psi(s^8[d(Ax_n,Br)]^2)\\ &\leq \limsup_{n\to\infty}[\psi(M(x_n,r))-\varphi(M(x_n,r))]\\ &=\limsup_{n\to\infty}\psi(M(x_n,r))-\liminf_{n\to\infty}\varphi(M(x_n,r))\\ &=\psi(\limsup_{n\to\infty} M(x_n,r))-\varphi(\liminf_{n\to\infty} M(x_n,r))\\ &<\psi(\limsup_{n\to\infty} M(x_n,r))\\ &\leq s[d(q,Br)]^2.\end{aligned}$$

Since ψ is monotonically increasing, we have

$s^7[d(q,Br)]^2\leq s[d(q,Br)]^2.$

Since $(s^6-1)>0$, we have

$[d(q,Br)]^2\leq 0$ which implies that $d(q,Br)\leq 0$,

it is a contradiction.

Therefore $Br=q$. Hence $Br=Tr=q$, so that q is a coincidence point of B and T.

Since $B(X)\subseteq S(X)$, we have $q\in S(X)$, there exists $z\in X$ such that $Sz=q=Br$.

Now we show that $Az=q$. Suppose $Az\neq q$.

From the inequality (2.3), we have

$$\psi(s^8[d(Az,q)]^2) = \psi(s^8[d(Az,Br)]^2) \leq \psi(M(z,r)) - \varphi(M(z,r)) \tag{2.24}$$

where

$$\begin{aligned} M(z,r) = \max\{&d(Az,Br)d(Sz,Az), d(Az,Br)d(Tr,Br), d(Sz,Tr)d(Tr,Br), \\ &d(Az,Tr)d(Sz,Br), \tfrac{1}{2s}d(Tr,Br)d(Sz,Br)\} \\ = \max\{&[d(Az,q)]^2, 0, 0, 0, 0\} \\ = [d&(Az,q)]^2. \end{aligned}$$

From the inequality (2.5.9), we have

$$\psi(s^8[d(Az,q)]^2) \leq \psi([d(Az,q)]^2) - \varphi([d(Az,q)]^2) < \psi([d(Az,q)]^2).$$

Since ψ is monotonically increasing, we have

$$s^8[d(Az,q)]^2 \leq [d(Az,q)]^2.$$

Since $(s^8 - 1) > 0$, we have

$[d(Az,q)]^2 \leq 0$ which implies that $d(Az,q) \leq 0$,

which is a contradiction.

Therefore $Az = Sz = q$ so that z is a coincidence point of A and S.

Since the pairs (A,S) and (B,T) are weakly compatible, we have $Aq = Sq$ and $Bq = Tq$.

Therefore q is also a coincidence point of the pairs (A,S) and (B,T).

We now show that q is a common fixed point of A, B, S and T.

Suppose $Aq \neq q$.

From the inequality (2.3), we have

$$\psi(s^8[d(Aq,q)]^2) = \psi(s^8[d(Aq,Br)]^2) \leq \psi(M(q,r)) - \varphi(M(q,r)) \tag{2.25}$$

where

$$\begin{aligned} M(q,r) = \max\{&d(Aq,Br)d(Sq,Aq), d(Aq,Br)d(Tr,Br), d(Sq,Tr)d(Tr,Br), \\ &d(Aq,Tr)d(Sq,Br), \tfrac{1}{2s}d(Tr,Br)d(Sq,Br)\} \\ = \max\{&0, 0, 0, [d(Aq,q)]^2, 0\} \\ = [d&(Aq,q)]^2. \end{aligned}$$

From the inequality (2.25), we have

$$\psi(s^8[d(Aq,q)]^2) \leq \psi([d(Aq,q)]^2) - \varphi([d(Aq,q)]^2) < \psi([d(Aq,q)]^2).$$

Since ψ is monotonically increasing, we have

$s^8[d(Aq,q)]^2 \leq [d(Aq,q)]^2$.

Since $(s^8-1)>0$, we have

$[d(Aq,q)]^2 \leq 0$ which implies that $d(Aq,q) \leq 0$,

which is a contradiction.

Therefore $Aq = Sq = q$ so that q is a common fixed point of A and S.

By Proposition 2.1, we get that q is a unique common fixed point of A,B,S and T.

Case (ii). Suppose $A(X)$ is b-closed.

In this case, we have $q \in A(X)$ and since $A(X) \subseteq T(X)$, we choose $r \in X$ such that $q = Tr$.

The proof follows as in Case (i).

Case (iii). Suppose $S(X)$ is b-closed.

We follow the argument similar as Case (i) and we get conclusion.

Case (iv). Suppose $B(X)$ is b-closed.

As in Case (ii), we get the conclusion.

For the case of (B,T) satisfies the b-(E.A)-property, we follow the argument similar to the case (A,S) satisfies the b-(E.A)-property. □

3. COROLLARIES AND EXAMPLES

In this section we draw some corollaries from our main results and we provide examples in support of our results.

Corollary 3.1. Let (X,d) be a b-metric space and f and g be selfmaps of X. Assume that there exist $\psi, \varphi \in \Psi$ such that

$$\psi(s^8[d(fx,fy)]^2) \leq \psi(M(x,y)) - \varphi(M(x,y)) \tag{3.1}$$

where

$$M(x,y) = \max\{d(fx,fy)d(gx,fx), d(fx,fy)d(gy,fy), d(gx,gy)d(gy,fy), d(fx,gy)d(gx,fy), \tfrac{1}{2s}d(gy,fy)d(gx,fy)\},$$

for all $x,y \in X$. If $f(X) \subseteq g(X)$, the pair (f,g) is compatible and f or g is b-continuous then f and g have a unique common fixed point in X.

Proof. The proof follows by choosing $A = B = f$ and $T = S = g$ in Theorem 2.4. □

Corollary 3.2. Let (X,d) be a b-metric space with coefficient $s \geq 1$. Let $f,g : X \to X$ be selfmaps of X and satisfy $f(X) \subseteq g(X)$ and the inequality (3.1). Suppose that the pair (f,g) satisfies the b-(E.A)-property and that one of the subspace $f(X)$ and $g(X)$ is b-closed in X. Then the pairs (f,g) have a point of coincidence in X. Moreover, if the pair (f,g) is weakly compatible, then f and g have a unique common fixed point in X.

Proof. By taking $A = B = f$ and $S = T = g$ in Theorem 2.5, the conclusion of the corollary follows. □

The following is an example in support of Theorem 2.4.

Example 3.3. Let $X = \mathbb{R}^+$ and let $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 4 & \text{if } x,y \in (0,1), \\ \frac{9}{2} + \frac{1}{x+y} & \text{if } x,y \in [1,\infty), \\ \frac{12}{5} & \text{otherwise.} \end{cases}$$

Then clearly (X,d) is a complete b-metric space with coefficient $s = \frac{25}{24}$.

Here we observe that when $x = \frac{10}{9}, z = 1 \in [1,\infty)$ and $y \in (0,1)$, we have

$d(x,z) = \frac{9}{2} + \frac{1}{x+z} = \frac{189}{38}$ and $d(x,y) + d(y,z) = \frac{12}{5} + \frac{12}{5} = \frac{24}{5}$ so that

$d(x,z) \neq d(x,y) + d(y,z)$.

Hence, d is a b-metric with $s = \frac{25}{24} (> 1)$ but not a metric.

We define $A,B,S,T : X \to X$ by

$A(x) = 1$ if $x \in [0,\infty)$, $B(x) = \begin{cases} x & \text{if } x \in [0,1) \\ \frac{1}{x} & \text{if } x \in [1,\infty), \end{cases}$ $\quad S(x) = \begin{cases} \frac{x(5-x)}{4} & \text{if } x \in [0,1) \\ \frac{1+x}{2} & \text{if } x \in [1,\infty), \end{cases}$

and $T(x) = \begin{cases} x(2-x) & \text{if } x \in [0,1) \\ 2x-1 & \text{if } x \in [1,\infty). \end{cases}$

Clearly $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. A is b-continuous.

We choose a sequence $\{x_n\}$ with $\{x_n\} = 1 + \frac{1}{2n}, n \geq 1$, we have

$ASx_n = A(1 + \frac{1}{4n}) = 1$ and $SAx_n = S1 = 1$.

Therefore $\lim\limits_{n\to\infty} d(ASx_n, SAx_n) = 0$ so that the pair (A,S) is compatible and clearly the pair (B,T) is weakly compatible.

We define $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = \frac{5}{4}t, \varphi(t) = \frac{1}{4}t$.

Case (i). $x,y \in (0,1)$.

$d(Ax,By) = \frac{12}{5}, d(Sx,Ty) = 4, d(Ty,By) = 4, d(Sx,Ax) = \frac{12}{5}, d(Ax,Ty) = \frac{12}{5}, d(Sx,By) = 4.$

$$\begin{aligned} M(x,y) = \max\{&d(Ax,By)d(Sx,Ax), d(Ax,By)d(Ty,By), d(Sx,Ty)d(Ty,By),\\ &d(Ax,Ty)d(Sx,By), \tfrac{1}{2s}d(Ty,By)d(Sx,By)\}\\ = \max\{&\tfrac{144}{25}, \tfrac{48}{5}, 16, \tfrac{48}{5}, \tfrac{192}{25}\} = 16. \end{aligned}$$

Now we consider

$$\begin{aligned} \psi(s^8[d(Ax,By)]^2) &= \psi(\tfrac{6103515625}{764411904})\\ &\leq \psi(16) - \varphi(16)\\ &= \psi(M(x,y)) - \varphi(M(x,y)). \end{aligned}$$

Case (ii). $x,y \in [1,\infty)$.

$d(Ax,By) = \frac{12}{5}, d(Sx,Ty) = \frac{9}{2} + \frac{1}{x+y}, d(Ty,By) = \frac{9}{2} + \frac{1}{x+y}, d(Sx,Ax) = \frac{9}{2} + \frac{1}{x+y},$

$d(Ax,Ty) = \frac{9}{2} + \frac{1}{x+y}, d(Sx,By) = \frac{12}{5}.$

$$\begin{aligned} M(x,y) = \max\{&d(Ax,By)d(Sx,Ax), d(Ax,By)d(Ty,By), d(Sx,Ty)d(Ty,By),\\ &d(Ax,Ty)d(Sx,By), \tfrac{1}{2s}d(Ty,By)d(Sx,By)\}\\ = \max\{&(\tfrac{12}{5})(\tfrac{9}{2} + \tfrac{1}{x+y}), (\tfrac{12}{5})(\tfrac{9}{2} + \tfrac{1}{x+y}), [\tfrac{9}{2} + \tfrac{1}{x+y}]^2, (\tfrac{12}{5})(\tfrac{9}{2} + \tfrac{1}{x+y}), (\tfrac{144}{25})(\tfrac{9}{2} + \tfrac{1}{x+y})\}\\ = [&\tfrac{9}{2} + \tfrac{1}{x+y}]^2. \end{aligned}$$

We now consider

$$\begin{aligned} \psi(s^8[d(Ax,By)]^2) &= \psi(\tfrac{6103515625}{764411904})\\ &\leq \psi([\tfrac{9}{2} + \tfrac{1}{x+y}]^2) - \varphi([\tfrac{9}{2} + \tfrac{1}{x+y}]^2)\\ &= \psi(M(x,y)) - \varphi(M(x,y)). \end{aligned}$$

Case (iii). $x \in (0,1), y \in [1,\infty)$.

$d(Ax,By) = \frac{12}{5}, d(Sx,Ty) = \frac{12}{5}, d(Ty,By) = \frac{12}{5}, d(Sx,Ax) = \frac{12}{5},$

$d(Ax,Ty) = \frac{9}{2} + \frac{1}{x+y}, d(Sx,By) = 4.$

$$\begin{aligned} M(x,y) = \max\{&d(Ax,By)d(Sx,Ax), d(Ax,By)d(Ty,By), d(Sx,Ty)d(Ty,By),\\ &d(Ax,Ty)d(Sx,By), \tfrac{1}{2s}d(Ty,By)d(Sx,By)\}\\ = \max\{&\tfrac{144}{25}, \tfrac{144}{25}, \tfrac{144}{25}, 18 + \tfrac{4}{x+y}, \tfrac{576}{125}\} = 18 + \tfrac{4}{x+y}. \end{aligned}$$

Now we consider

$$\psi(s^8[d(Ax,By)]^2) = \psi(\tfrac{6103515625}{764411904})$$

$$\leq \psi(18+\frac{4}{x+y}) - \varphi(18+\frac{4}{x+y})$$
$$= \psi(M(x,y)) - \varphi(M(x,y)).$$

Case (iv). $x \in [1,\infty), y \in (0,1)$.

$d(Ax,By) = \frac{12}{5}, d(Sx,Ty) = \frac{12}{5}, d(Ty,By) = 4, d(Sx,Ax) = \frac{9}{2} + \frac{1}{x+y},$

$d(Ax,Ty) = \frac{12}{5}, d(Sx,By) = \frac{12}{5}.$

$$M(x,y) = \max\{d(Ax,By)d(Sx,Ax), d(Ax,By)d(Ty,By), d(Sx,Ty)d(Ty,By),$$
$$d(Ax,Ty)d(Sx,By), \tfrac{1}{2s}d(Ty,By)d(Sx,By)\}$$
$$= \max\{\tfrac{54}{5} + \tfrac{12}{5(x+y)}, \tfrac{48}{5}, \tfrac{48}{5}, \tfrac{144}{25}, \tfrac{576}{125}\} = \tfrac{54}{5} + \tfrac{12}{5(x+y)}.$$

We now consider

$$\psi(s^8[d(Ax,By)]^2) = \psi(\tfrac{6103515625}{764411904})$$
$$\leq \psi(\tfrac{54}{5} + \tfrac{12}{5(x+y)}) - \varphi(\tfrac{54}{5} + \tfrac{12}{5(x+y)})$$
$$= \psi(M(x,y)) - \varphi(M(x,y)).$$

From all the above four cases, A, B, S and T are Fisher type weakly contractive maps.

Therefore A, B, S and T satisfy all the hypotheses of Theorem 2.4 and 1 is the unique common fixed point of A, B, S and T.

The following is an example in support of Theorem 2.5.

Example 3.4. Let $X = [0,1]$ and let $d : X \times X \to \mathbb{R}^+$ defined by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \frac{23}{30} & \text{if } x,y \in (0,\frac{2}{3}), \\ \frac{4}{5} + \frac{x+y}{10} & \text{if } x,y \in [\frac{2}{3},1], \\ \frac{12}{25} & \text{otherwise.} \end{cases}$$

Then clearly (X,d) is a complete b-metric space with coefficient $s = \frac{52}{49}$.

Here we observe that when $x = 1, z = \frac{9}{10} \in [\frac{2}{3},1]$ and $y \in (0,\frac{2}{3})$, we have

$d(x,z) = \frac{4}{5} + \frac{x+y}{10} = \frac{99}{100}$ and $d(x,y) + d(y,z) = \frac{12}{25} + \frac{12}{25} = \frac{24}{25}$ so that

$d(x,z) \neq d(x,y) + d(y,z)$.

Hence, d is a b-metric with $s = \frac{52}{49} (> 1)$ but not a metric.

We define $A, B, S, T : X \to X$ by

$$A(x) = \tfrac{2}{3} \text{ if } x \in [0,1],\ B(x) = \begin{cases} \frac{1}{2} & \text{if } x \in [0,\frac{2}{3}) \\ 1 - \frac{x}{2} & \text{if } x \in [\frac{2}{3},1], \end{cases} \quad S(x) = \begin{cases} \frac{1}{4} & \text{if } x \in [0,\frac{2}{3}) \\ \frac{4}{3} - x & \text{if } x \in [\frac{2}{3},1], \end{cases}$$

and $T(x) = \begin{cases} \frac{1}{3} & \text{if } x \in [0, \frac{2}{3}) \\ x & \text{if } x \in [\frac{2}{3}, 1]. \end{cases}$

Clearly $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. $A(X) = \{\frac{2}{3}\}$ is b-closed.

We choose a sequence $\{x_n\}$ with $\{x_n\} = \frac{2}{3} + \frac{1}{n}, n \geq 4$ with

$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{2}{3}$, hence the pair (A, S) satisfies the b-(E.A)-property.

Clearly the pairs (A, S) and (B, T) are weakly compatible.

We define $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = \frac{5}{4}t, \varphi(t) = \frac{1}{4}t$.

Case (i). $x, y \in (0, \frac{2}{3})$.

$d(Ax, By) = \frac{12}{25}, d(Sx, Ty) = \frac{23}{30}, d(Ty, By) = \frac{23}{30}, d(Sx, Ax) = \frac{12}{25}, d(Ax, Ty) = \frac{12}{25},$

$d(Sx, By) = \frac{23}{30}$.

$$\begin{aligned} M(x,y) &= \max\{d(Ax,By)d(Sx,Ax), d(Ax,By)d(Ty,By), d(Sx,Ty)d(Ty,By), \\ &\qquad d(Ax,Ty)d(Sx,By), \tfrac{1}{2s}d(Ty,By)d(Sx,By)\} \\ &= \max\{\tfrac{144}{625}, \tfrac{92}{250}, \tfrac{529}{900}, \tfrac{529}{900}, \tfrac{25921}{93600}\} = \tfrac{529}{900}. \end{aligned}$$

Now we consider

$$\begin{aligned} \psi(s^8[d(Ax,By)]^2) &= \psi(\tfrac{7698200908529664}{20770581606000625}) \\ &\leq \psi(\tfrac{529}{900}) - \varphi(\tfrac{529}{900}) \\ &= \psi(M(x,y)) - \varphi(M(x,y)). \end{aligned}$$

Case (ii). $x, y \in [\frac{2}{3}, 1]$.

$d(Ax, By) = \frac{12}{25}, d(Sx, Ty) = \frac{12}{25}, d(Ty, By) = \frac{12}{25}, d(Sx, Ax) = \frac{12}{25},$

$d(Ax, Ty) = \frac{4}{5} + \frac{x+y}{10}, d(Sx, By) = \frac{23}{30}$.

$$\begin{aligned} M(x,y) &= \max\{d(Ax,By)d(Sx,Ax), d(Ax,By)d(Ty,By), d(Sx,Ty)d(Ty,By), \\ &\qquad d(Ax,Ty)d(Sx,By), \tfrac{1}{2s}d(Ty,By)d(Sx,By)\} \\ &= \max\{\tfrac{144}{625}, \tfrac{144}{625}, \tfrac{144}{625}, \tfrac{46}{75} + \tfrac{23(x+y)}{300}, \tfrac{2392}{6125}\} = \tfrac{46}{75} + \tfrac{23(x+y)}{300}. \end{aligned}$$

We now consider

$$\begin{aligned} \psi(s^8[d(Ax,By)]^2) &= \psi(\tfrac{7698200908529664}{20770581606000625}) \\ &\leq \psi(\tfrac{46}{75} + \tfrac{23(x+y)}{300}) - \varphi(\tfrac{46}{75} + \tfrac{23(x+y)}{300}) \\ &= \psi(M(x,y)) - \varphi(M(x,y)). \end{aligned}$$

Case (iii). $x \in (0, \frac{2}{3}), y \in [\frac{2}{3}, 1]$.

$d(Ax, By) = \frac{12}{25}, d(Sx, Ty) = \frac{12}{25}, d(Ty, By) = \frac{12}{25}, d(Sx, Ax) = \frac{12}{25},$

$d(Ax,Ty) = \frac{4}{5} + \frac{x+y}{10}, d(Sx,By) = \frac{23}{30}$.

$$M(x,y) = \max\{d(Ax,By)d(Sx,Ax), d(Ax,By)d(Ty,By), d(Sx,Ty)d(Ty,By), d(Ax,Ty)d(Sx,By), \tfrac{1}{2s}d(Ty,By)d(Sx,By)\}$$
$$= \max\{\tfrac{144}{625}, \tfrac{144}{625}, \tfrac{144}{625}, \tfrac{92}{150} + \tfrac{23(x+y)}{300}, \tfrac{2392}{6125}\} = \tfrac{46}{75} + \tfrac{23(x+y)}{300}.$$

We now consider

$$\psi(s^8[d(Ax,By)]^2) = \psi(\tfrac{7698200908529664}{20770581606000625}) \leq \psi(\tfrac{46}{75} + \tfrac{23(x+y)}{300}) - \varphi(\tfrac{46}{75} + \tfrac{23(x+y)}{300}) = \psi(M(x,y)) - \varphi(M(x,y)).$$

Case (iv). $x \in [\frac{2}{3}, 1], y \in (0, \frac{2}{3})$.

$d(Ax,By) = \frac{12}{25}, d(Sx,Ty) = \frac{23}{30}, d(Ty,By) = \frac{23}{30}, d(Sx,Ax) = \frac{12}{25}$,

$d(Ax,Ty) = \frac{12}{25}, d(Sx,By) = \frac{23}{30}$.

$$M(x,y) = \max\{d(Ax,By)d(Sx,Ax), d(Ax,By)d(Ty,By), d(Sx,Ty)d(Ty,By), d(Ax,Ty)d(Sx,By), \tfrac{1}{2s}d(Ty,By)d(Sx,By)\}$$
$$= \max\{\tfrac{144}{625}, \tfrac{144}{625}, \tfrac{529}{900}, \tfrac{92}{250}, \tfrac{25921}{93600}\} = \tfrac{529}{900}.$$

We now consider

$$\psi(s^8[d(Ax,By)]^2) = \psi(\tfrac{7698200908529664}{20770581606000625}) \leq \psi(\tfrac{529}{900}) - \varphi(\tfrac{529}{900}) = \psi(M(x,y)) - \varphi(M(x,y)).$$

From all the above four cases, A, B, S and T are Fisher type weakly contractive maps.

Therefore A, B, S and T satisfy all the hypotheses of Theorem 2.5 and $\frac{2}{3}$ is the unique common fixed point.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] M. Aamri and D. El. Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl., 270(2002), 181-188.

[2] A. Aghajani, M. Abbas and J. R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered b-metric spaces, Math. Slovaca, 64(4)(2014), 941-960.

[3] Ya. I. Alber and Guerre-Delabriere, Principles of weakly contractive maps in Hilbet spaces, New results in operator theory and its applications (I. Gohberg, and Yu. Lyubich, Eds.), Oper. Theory Adv. Appl., (1997), 7-22.

[4] M. Ali and M. Arshad, *b*-metric generalization of some fixed point theorems, J. Function Spaces, 2018 (2018), Article ID 2658653, 9 pages.

[5] B. Alqahtani, A. Fulga, E. Karapinar and A. Özturk, Fisher-type fixed point results in *b*-metric spaces, Mathematics, 7(2019), Article ID 102.

[6] H. Aydi, M. F. Bota, E. Karapinar and S. Mitrović, A fixed point theorem for set-valued quasi contractions in *b*-metric spaces, Fixed Point Theory Appl., 2012 (2012), Article ID 88.

[7] G. V. R. Babu, K. Nageswara Rao and G. N. Alemayehu, Common fixed points of two pair of generalized weakly contractive maps, Adv. Studies in Contemporary Math., 20(4)(2010), 575-594.

[8] G. V. R. Babu and P. D. Sailaja, A fixed point theorem of generalized weakly contractive maps in orbitally complete metric spaces, Thai J. Math., 9(1)(2011), 1-10.

[9] G. V. R. Babu and G. N. Alemayehu, A common fixed point theorem for weakly contractive mappings satisfying property (E.A), Applied Mathematics E-Notes, 24(6)(2012), 975-981.

[10] G. V. R. Babu and T. M. Dula, Common fixed points of two pairs of selfmaps satisfying (E.A)-property in *b*-metric spaces using a new control function, Inter. J. Math. Appl., 5(1-B)(2017), 145-153.

[11] I. A. Bakhtin, The contraction mapping principle in almost metric spaces, Func. Anal. Gos. Ped. Inst. Unianowsk, 30(1989), 26-37.

[12] M. Boriceanu, Strict fixed point theorems for multivalued operators in *b*-metric spaces, Int. J. Mod. Math., 4(3)(2009), 285-301.

[13] M. Boriceanu, M. Bota and A. Petrusel, Multivalued fractals in *b*-metric spaces, Cent. Eur. J. Math., 8(2)(2010), 367-377.

[14] N. Bourbaki, Topologie Generale, Herman: Paris, France, 1974.

[15] B. S. Choudhury, P. Konar, B. E. Rhoades and N. Metiya, Fixed point therorems for generalized weakly contractive mappings, Nonlinear Anal. Theory, Methods and Appl., 74(6)(2011), 2116-2126.

[16] S. Czerwik, Contraction mappings in *b*-metric spaces, Acta Math. Inform. Univ. Ostraviensis, 1(1993), 5-11.

[17] S. Czerwik, Nonlinear set-valued contraction mappings in *b*-metric spaces, Atti del Seminario Matematico e Fisico (DellUniv. di Modena), 46(1998), 263-276.

[18] D. Doric, Common fixed point for generalized $(\psi,\varphi)-$weak contraction, Appl. Math. Lett. 22(2009), 1896-1900.

[19] P. N. Dutta and B. S. Choudhury, A generalization of contraction principle in metric spaces, Fixed Point Theory and Appl., 2008 (2008), Article ID 406368, 8 pages.

[20] B. Fisher, Fixed point and constant mappings on metric spaces, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 61(1977), 329-332.

[21] H. Huang, G. Deng and S. Radenović, Fixed point theorems for C-class functions in b-metric spaces and applications, J. Nonlinear Sci. Appl., 10(2017), 5853-5868.

[22] N. Hussain, V. Paraneh, J. R. Roshan and Z. Kadelburg, Fixed points of cycle weakly (ψ,φ,L,A,B)-contractive mappings in ordered b-metric spaces with applications, Fixed Point Theory Appl., 2013(2013), Article ID 256.

[23] G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci., 9(1986), 771-779.

[24] G. Jungck and B. E. Rhoades, Fixed points of set-valued functions without continuity, Indian J. Pure Appl. Math., 29(3)(1998), 227-238.

[25] P. Kumam and W. Sintunavarat, The existence of fixed point theorems for partial q-set valued quasi-contractions in b-metric spaces and related results, Fixed point theory Appl., 2014(2014), Article ID 226.

[26] B. G. Pachpatte, On certain fixed point mappings in metric spaces, J. Maulana Azad Coll. Technol., 13(1980), 59-63.

[27] V. Ozturk and D. Turkoglu, Common fixed point theorems for mappings satisfying (E.A)-property in b-metric spaces, J. Nonlinear Sci. Appl., 8(2015), 1127-1133.

[28] V. Ozturk and S. Radenović, Some remarks on b-(E.A)-property in b-metric spaces, Springer Plus, 5(2016), 544, 10 pages.

[29] V. Ozturk and A. H. Ansari, Common fixed point theorems for mapping satisfying (E.A)-property via C-class functions in b-metric spaces, Appl. Gen. Topol., 18(1)(2017), 45-52.

[30] B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal., 47(2001), 2683-2693.

[31] J. R. Roshan, V. Paraneh and Z. Kadelburg, Common fixed point theorems for weakly isotone increasing mappings in ordered b-metric spaces, J. Nonlinear Sci. Appl., 7(4)(2014), 229-245.

[32] K. P. R. Sastry, G. V. R. Babu and K. T. Kidane, A common fixed point of generalized $(\psi,\phi)-$weakly contractive maps where ϕ is nondecreasing (not necessarily continuous or lower semicontinuous), Advs. Fixed Point Theory, 2(2012), 203-223.

[33] P. L. Sharma and M. K. Sahu, A unique fixed point theorem in complete metric space, Acta Cienc. Indic. Math., 17(1991), 685-688.

[34] W. Shatanawi, Fixed and common fixed point for mappings satisfying some nonlinearcontractions in b-metric spaces, J. Math. Anal., 7(4)(2016), 1-12.

[35] Q. Zhang and V. Song, Fixed point theory for generalized $\varphi-$weak contractions, Appl. Math. Lett., 22(2009), 75-78.