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AUXILIARY PRINCIPLE TECHNIQUE FOR SOLVING A GENERALIZED MULTI-VALUED QUASI-VARIATIONAL-LIKE INEQUALITY PROBLEM

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Abstract. In this paper, we consider a generalized multi-valued quasi-variational-like inequality problem (GMQVLIP) in real Hilbert space. Further, we define an auxiliary problem (AP) for GMQVLIP and establish an existence result for AP. Using this result, we construct an algorithm for GMQVLIP. Furthermore, we prove the existence of solution of GMQVLIP and discuss the convergence analysis of iterative sequences generated by the algorithm. The approach used in this paper may be treated as an extension and unification of approaches for studying existence results for various important classes of quasi-variational and quasi-variational-like inequalities given by many authors in this direction.

Keywords: generalized multi-valued quasi-variational-like inequality problem; auxiliary problem; coercive mapping; strongly mixed monotone mapping; mixed Lipschitz continuous mapping.

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1. Introduction

Variational inequality theory has become very effective and powerful tool for studying a wide range of problems arising in mechanics, optimization, operation research, equilibrium problems and boundary valued problems, etc. Variational inequalities have been generalized and extended

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in different directions using novel and innovative techniques. A useful and important generalization of variational inequality is called the quasi-variational-like inequality.

There are many numerical techniques including the projection technique and its variant forms, linear approximation, descent and Newton methods for solving variational inequalities. However, there are very few techniques for solving variational-like inequalities. It is worth mentioning that the projection type technique cannot be used to suggest iterative algorithms for variational-like inequalities, since it is not possible to find the projection of the solution. To overcome this drawback, one uses usually the auxiliary principle technique introduced by Glowinski *et al.* [3]. This technique deals with finding a suitable auxiliary problem and proving that the approximate solution of auxiliary problem converges to the solution of original problem.

Recently, Huang and Deng [5], Noor [9] and Zeng *et al.* [13] extended the auxiliary principle technique to various classes of variational-like inequalities involving multi-valued mappings. Very recently, Chidume *et al.* [1], Ding [2], Huang and Fang [6], Huang *et al.* [7], Tian [11] and Yao [12] extended the auxiliary principle technique to some important classes of quasi-variational and quasi-variational-like inequalities involving single and multi-valued mappings.

Inspired by recent research works in this field, in this paper, we consider a generalized multi-valued quasi-variational-like inequality problem (GMQVLIP) in real Hilbert space. Further, we define an auxiliary problem (AP) for GMQVLIP and prove the existence result for AP. Using this result, we construct an algorithm for GMQVLIP. Furthermore, we prove the existence of solution of GMQVLIP and discuss the convergence analysis of the algorithm. The technique presented in this paper can be used to generalize and improve the results given by many authors, see for example [2,4-7,9,11,13].

2. Preliminaries and Formulation of Problem

Let H^* be the topological dual of a real Hilbert space H. Let $CB(H)$ denote the family of all nonempty, closed and bounded subsets of H and let 2^H denote the power set of H. We denote the inner product of H and the duality pairing between H and H^* by $\langle \cdot, \cdot \rangle$, and denote the induced norms of Hilbert space H and its dual spaces by $\|\cdot\|$. The Hausdorff metric $\mathscr{H}(\cdot,\cdot)$

on $CB(H)$ is defined by

$$\mathscr{H}(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x,y),\ \sup_{y\in B}\inf_{x\in A} d(x,y)\right\},\ A,B \in CB(H). \tag{2.1}$$

First, we review the following concepts and known results.

Definition 2.1[6]. A mapping $\eta : H \times H \to H$ is said to be

(i) τ-strongly monotone if there exists a constant $\tau > 0$ such that

$$\langle \eta(y,x),\, y-x\rangle \ \geq\ \tau\|y-x\|^2,\ \forall x,y \in H;$$

(ii) ξ-Lipschitz continuous if there exists a constant $\xi > 0$ such that

$$\|\eta(y,x)\| \ \leq\ \xi\|y-x\|,\ \forall x,y \in H.$$

Definition 2.2[6]. A mapping $m : H \to H$ is said to be σ-Lipschitz continuous if there exists a constant $\sigma > 0$ such that

$$\|m(x)-m(y)\| \ \leq\ \sigma\|x-y\|,\ \forall x,y \in H.$$

Definition 2.3[8]. A multi-valued mapping $T : H \to CB(H)$ is said to be k-$\mathscr{H}$-Lipschitz continuous if there exists a constant $k > 0$ such that

$$\mathscr{H}(T(x),\, T(y)) \ \leq\ k\|x-y\|,\ \forall x,y \in H.$$

Definition 2.4. Let $m : H \to H$ be a single-valued mapping and let $T,A,S : H \to CB(H)$ be multi-valued mappings. A mapping $N : H \times H \times H \to H$ is said to be

(i) α-m-strongly mixed monotone with respect to T, A and S if there exists a constant $\alpha > 0$ such that

$$\langle N(u_1,v_1,w_1)-N(u_2,v_2,w_2),\, m(x)-m(y)\rangle \ \geq\ \alpha\|x-y\|^2,$$

$$\forall x,y \in H,\ u_1 \in T(x),\ u_2 \in T(y),\ v_1 \in A(x),\ v_2 \in A(y),\ w_1 \in S(x),\ w_2 \in S(y);$$

(ii) $(\beta_1,\beta_2,\beta_3)$-mixed Lipschitz continuous if there exist constants $\beta_1,\beta_2,\beta_3 > 0$ such that

$$\|N(x_1,y_1,z_1)-N(x_2,y_2,z_2)\| \ \leq\ \beta_1\|x_1-x_2\|+\beta_2\|y_1-y_2\|+\beta_3\|z_1-z_2\|,$$

$$\forall x_1,x_2,y_1,y_2,z_1,z_2 \in H.$$

Assumption 2.1. Let the function $a : H \times H \to \mathbb{R}$ satisfy the following conditions:

(i) a is bilinear;

(ii) a is γ-continuous, that is, there exists a constant $\gamma > 0$ such that

$$a(x,y) \ \leq \ \gamma\|x\|\,\|y\|, \ \forall x,y \in H;$$

(iii) a is ν-coercive, that is, there exists a constant $\nu > 0$ such that

$$a(x,x) \ \geq \ \nu\|x\|^2, \ \forall x \in H.$$

Since a is a continuous and bilinear function on H, then by the Riesz-Frechet representation theorem, there exists a continuous linear mapping $S : H \to H^*$ such that

$$a(x,y) \ \equiv \ {}_{H^*}\langle Sx,y\rangle_H, \ \forall x,y \in H. \tag{2.2}$$

It can be shown that $\|S\| \leq \gamma$. Finally, we define Λ, a canonical isomorphism, from H^* onto H, as

$${}_{H^*}\langle f,x\rangle_H \ \equiv \ \langle \Lambda f,x\rangle, \ \forall f \in H^*, \ x \in H. \tag{2.3}$$

Then $\|\Lambda\|_{H^*} = 1 = \|\Lambda^{-1}\|_H$.

Assumption 2.2. Let the function $b : H \times H \to \mathbb{R}$ satisfy the following conditions:

(i) b is linear in the first argument;

(ii) b is μ-continuous;

(iii) $b(x,y) - b(x,z) \ \leq \ b(x,y-z), \ \forall x,y,z \in H$;

(iv) b is convex in the second argument.

Remark 2.1. From Assumption 2.2 (i)-(iv), we have

(i) $|b(x,y)| \ \leq \ \mu\|x\|\,\|y\|$;

(ii) $b(x,0) \ = \ b(0,y) \ = \ 0$;

(iii) $|b(x,y) - b(x,z)| \ \leq \ \mu\|x\|\,\|y-z\|, \ \forall x,y,z \in H$.

Assumption 2.3. Let the mapping $\eta : H \times H \to \mathbb{R}$ satisfy the following conditions:

(i) $\eta(x,y) + \eta(y,z) = \eta(x,z), \ \forall x,y,z \in H$;

(ii) for any $x,y,z,t \in H$, $x - y = z - t$ implies that $\eta(x,y) = \eta(z,t)$;

(iii) for given $x,u,v,w \in H$, the mapping $y \to \langle N(u,v,w), \eta(y,x)\rangle$ is convex and lower semi-continuous.

In many practical problems, $K(x)$ has the following form:

$$K(x) \equiv m(x) + K,\ \forall x \in H, \tag{2.4}$$

where $m : H \to H$ is a single-valued mapping and K is a nonempty, closed and convex set of H. Let $K : H \to 2^H$ be a multi-valued mapping such that for each $x \in H$, $K(x)$ is a nonempty, closed and convex subset of H.

Let $T,A,S : H \to CB(H)$ be multi-valued mappings and let $N : H \times H \times H \to H$; $\eta : H \times H \to H$ be nonlinear single-valued mappings. Let $a : H \times H \to \mathbb{R}$ be a bilinear function and let $b : H \times H \to R$ be a function which is not necessarily differentiable. We consider the following generalized multi-valued quasi-variational-like inequality problem (GMQVLIP): Find $x \in H$, $u \in T(x)$, $v \in A(x)$ and $w \in S(x)$ such that $x \in K(x)$ and

$$a(x, y-x) + \langle N(u,v,w),\ \eta(y,x)\rangle + b(x,y) - b(x,x)\ \geq\ 0,\ \forall y \in K(x). \tag{2.5}$$

For a suitable choices of mappings T,A,S,N,K,a,b,η and the space H, it is easy to see that GMQVLIP (2.5) includes a number of known classes of quasi-variational and quasi-variational-like inequalities studied by many authors as special cases, see for example [2,4-7,9,11,13] and the references therein.

3. Existence of Solution of Auxiliary Problem

We consider the following auxiliary problem for GMQVLIP (2.5).

Auxiliary Problem (AP): Given $x \in H$, $u \in T(x)$, $v \in A(x)$ and $w \in S(x)$, find $t \in K(x)$ such that

$$\langle t, y-t\rangle \geq \langle x, y-t\rangle - \rho a(x, y-t) - \rho\langle N(u,v,w), \eta(y,t)\rangle + \rho b(x,t) - \rho b(x,y),\ \forall y \in K(x), \tag{3.1}$$

where $\rho > 0$ is a constant.

Next, using the technique of Ding [2], we prove the existence of solution of AP (3.1).

Theorem 3.1. Let $T, A, S : H \to CB(H)$; $\eta : H \times H \to H$; $N : H \times H \times H \to H$ be nonlinear mappings. Let $K : H \to 2^H$ be a multi-valued mapping such that for each $x \in H$, $K(x)$ is a nonempty, closed and convex subset of H. Let the function $a : H \times H \to \mathbb{R}$ be linear and lower semicontinuous in the second argument, and let the function $b : H \times H \to \mathbb{R}$ be convex and lower semicontinuous in the second argument. Moreover, suppose that Assumption 2.3 holds. Then for any given $x \in H$, $u \in T(x)$, $v \in A(x)$ and $w \in S(x)$, the following problem:

$$\min_{y \in K(x)} J(y) \tag{3.2}$$

where $J(y) \;=\; \frac{1}{2}\langle y, y\rangle + j(y)$,

$$j(y) \;=\; \rho a(x, y-x) + \rho\langle N(u,v,w), \eta(y,x)\rangle + \rho b(x,y) - \langle x, y\rangle, \tag{3.3}$$

admits a unique solution and t is a solution of the problem (3.2) if and only if t is a solution of AP (3.1).

Proof. Since the function a is linear and lower semicontinuous and the function b is convex and lower semicontinuous in the second argument, it follows from Assumption 2.3 that $j(y)$ is convex and lower semicontinuous on $K(x)$ and $J(x)$ is strictly convex and lower semicontinuous on $K(x)$. By Theorem 2.5 of [10], j is bounded from below by hyperplane $f(y) = \langle h; y\rangle + r$, where $h \in H$ and $r \in \mathbb{R}$. Hence, we have

$$\begin{aligned} J(y) \;&=\; \frac{1}{2}\langle y, y\rangle + j(y) \;\geq\; \frac{1}{2}\|y\|^2 + \langle h, y\rangle + r \\ &=\; \frac{1}{2}\|y+h\|^2 - \frac{1}{2}\|h\|^2 + r. \end{aligned}$$

This implies that

$$J(y) \to \infty \text{ and } \|y\| \to \infty. \tag{3.4}$$

Now, let $\{y_n\} \subset K(x)$ be a minimizing sequence of J on $K(x)$, that is,

$$\lim_{n\to\infty} J(y_n) \;=\; d \text{ and } d \;=\; \inf_{y \in K(x)} J(y).$$

We claim that $\{y_n\}$ is bounded. If it is false, then there exists a subsequence $\{y_{n_k}\} \subset \{y_n\}$ such that $\|y_{n_k}\| \geq k$, $k = 1, 2, 3, \ldots$. By (3.4), we have $J(y_{n_k}) \to \infty$ which contradicts that fact

$\lim_{k\to\infty} J(y_{n_k}) = d < \infty$. Therefore, there exists a constant $r_1 > 0$ such that

$$\{y_n\} \subset K(x) \cap B(0, r_1) \ = \ \{y \in K(x) : \|y\| \leq r_1\}.$$

By Weierstrass theorem (see [10]), there exists $t \in K(x)$ such that

$$J(t) \ = \ \min_{y\in K(x)} J(y).$$

Since J is strictly convex, we know that t is the unique solution of the problem (3.2).

Now, suppose that t is a unique solution of the problem (3.2). We show that t is also a solution of AP (3.1). For any $y \in K(x)$ and $q \in [0,1]$, we have

$$\begin{aligned} J(t) &= \frac{1}{2}\langle t, t\rangle + j(t) \\ &\leq J(t + q(y-t)) \\ &= \frac{1}{2}\langle t + q(y-t), t + q(y-t)\rangle + j(t + q(y-t)) \\ &\leq \frac{1}{2}\langle t, t\rangle + \frac{q^2}{2}\langle y-t, y-t\rangle + q\langle t, y-t\rangle + j(t) + q(j(y) - j(t)). \end{aligned}$$

This implies that

$$\frac{q}{2}\langle y-t, y-t\rangle + \langle t, y-t\rangle + j(y) - j(t) \ \geq \ 0. \tag{3.5}$$

Letting $q \to 0$ in the above inequality (3.5), we obtain

$$\begin{aligned} &\langle t, y-t\rangle + \rho a(x, y-x) + \rho\langle N(u,v,w), \eta(y,x)\rangle + \rho b(x,y) - \langle x, y\rangle - \rho a(x, t-x) \\ &\qquad - \rho\langle N(u,v,w), \eta(t,x)\rangle - \rho b(x,t) + \langle x,t\rangle \ \geq \ 0. \end{aligned} \tag{3.6}$$

Since a is linear in the second argument, it follows from Assumption 2.3 (i) and (3.6) that

$$\langle t, y-t\rangle \geq \langle x, y-t\rangle - \rho a(x, y-t) - \rho\langle N(u,v,w), \eta(y,t)\rangle + \rho b(x,t) - \rho b(x,y), \ \forall y \in K(x).$$

This shows that t is a solution of AP (3.1).

Conversely, suppose that t is a solution of AP (3.1), it follows from (3.1) that

$$\begin{aligned}
\frac{1}{2}[\langle y,y\rangle-\langle t,t\rangle] = {} & \langle t, y-t\rangle+\frac{1}{2}\langle y-t, y-t\rangle \\
\geq {} & \langle t, y-t\rangle \\
\geq {} & \langle x, y-t\rangle-\rho a(x,y-t)-\rho\langle N(u,v,w), \eta(y,t)\rangle+\rho b(x,t)-\rho b(x,y) \\
= {} & \langle x, y\rangle-\langle x, t\rangle-\rho a(x,y-x)-\rho\langle N(u,v,w), \eta(y,x)\rangle+\rho a(x,t-x) \\
& +\rho\langle N(u,v,w), \eta(t,x)\rangle+\rho b(x,t)-\rho b(x,y),\ \forall y\in K(x).
\end{aligned}$$

This implies that $J(y)\geq J(t), \forall y\in K(x)$ and so t is a solution of the problem (3.2) and the proof is complete.

4. ALGORITHM, EXISTENCE OF SOLUTION AND CONVERGENCE ANALYSIS

Based on Theorem 3.1, we construct the following algorithm for GMQVLIP (2.5).

Algorithm 4.1: Let $x_0\in H$, $u_0\in T(x_0)$, $v_0\in A(x_0)$, $w_0\in S(x_0)$, where $T(x_0), A(x_0), S(x_0)\in CB(H)$. By Nadler's technique [8] and induction process, we have sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$ such that $x_{n+1}\in K(x_n)$,

$$\begin{aligned}
\langle x_{n+1}, y-x_{n+1}\rangle \geq {} & \langle x_n, y-x_{n+1}\rangle-\rho a(x_{n+1},y-x_{n+1})-\rho\langle N(u_n,v_n,w_n), \eta(y, x_{n+1})\rangle \\
& +\rho b(x_n, x_{n+1})-\rho b(x_n, y),\ \forall y\in K(x_n),\ n=0,1,2,\ldots, \qquad (4.1)
\end{aligned}$$

$$u_{n+1}\in T(x_{n+1}),\ \|u_{n+1}-u_n\| \leq (1+(1+n)^{-1})\, \mathscr{H}(T(x_{n+1}),T(x_n)), \qquad (4.2)$$

$$v_{n+1}\in A(x_{n+1}),\ \|v_{n+1}-v_n\| \leq (1+(1+n)^{-1})\, \mathscr{H}(A(x_{n+1}),A(x_n)), \qquad (4.3)$$

$$w_{n+1}\in S(x_{n+1}),\ \|w_{n+1}-w_n\| \leq (1+(1+n)^{-1})\, \mathscr{H}(S(x_{n+1}),S(x_n)). \qquad (4.4)$$

Remark 4.1[4]. If $a(x,y)\equiv 0$ for all $x,y\in H$, then Algorithm 4.1 reduces to the corresponding algorithm due to Huang and Fang [6].

Next, we prove the existence of solution of GMQVLIP (2.5) and discuss the convergence analysis of Algorithm 4.1.

Theorem 4.1. Let $T,A,S : H \to CB(H)$ be multi-valued mappings such that T,A and S are k_1-$\mathscr{H}$-Lipschitz continuous, k_2-$\mathscr{H}$-Lipschitz continuous and k_3-$\mathscr{H}$-Lipschitz continuous, respectively. Let the mapping $N : H \times H \times H \to H$ be α-m-strongly mixed monotone with respect to T, A and S, and $(\beta_1, \beta_2, \beta_3)$-mixed Lipschitz continuous. Let the mapping $m : H \to H$ be σ-Lipschitz continuous, and let the mapping $\eta : H \times H \to H$ be ξ-Lipschitz continuous. Let $K : H \to 2^H$ be a multi-valued mapping such that $K(x)$ has the form of (2.4). Suppose that Assumptions 2.1, 2.2 and 2.3 hold and $\rho > 0$ satisfies the following condition:

$$\theta \ := \ \sigma + \rho(\mu + (1+\xi)L_N) + \sqrt{2}\,\sqrt{(1+\sigma^2) - 2\rho(\nu+\alpha) + \rho^2(\gamma^2 + L_N^2)} \ < \ 1, \tag{4.5}$$

where $L_N := (\beta_1 k_1 + \beta_2 k_2 + \beta_3 k_3)$. Then the iterative sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ generated by Algorithm 4.1 converge strongly to x, u, v and w, respectively, and (x,u,v,w) is a solution of GMQVLIP (2.5).

Proof. It follows from (4.1) that

$$\begin{aligned}\langle x_{n+1}, y - x_{n+1}\rangle \geq \langle x_n, y - x_{n+1}\rangle - \rho a(x_n, y - x_{n+1}) - \rho\langle N(u_n, v_n, w_n), \eta(y, x_{n+1})\rangle \\ + \rho b(x_n, x_{n+1}) - \rho b(x_n, y),\ \forall y \in K(x_n),\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\langle x_{n+2}, y - x_{n+2}\rangle \geq \langle x_{n+1}, y - x_{n+2}\rangle - \rho a(x_{n+1}, y - x_{n+2}) - \rho\langle N(u_{n+1}, v_{n+1}, w_{n+1}), \eta(y, x_{n+2})\rangle \\ + \rho b(x_{n+1}, x_{n+2}) - \rho b(x_{n+1}, y),\ \forall y \in K(x_{n+1}).\end{aligned} \tag{4.7}$$

Adding $\langle -m(x_n), y - x_{n+1}\rangle$ to both sides of (4.6) and then taking $y = x_{n+2} - m(x_{n+1}) + m(x_n) \in K(x_n)$, we obtain

$$\begin{aligned}&\langle x_{n+1} - m(x_n), x_{n+2} - x_{n+1} - m(x_{n+1}) + m(x_n)\rangle \\ &\geq \langle x_n - m(x_n), x_{n+2} - x_{n+1} - m(x_{n+1}) + m(x_n)\rangle - \rho a(x_n, x_{n+2} - x_{n+1} - m(x_{n+1}) + m(x_n)) \\ &\quad - \rho\langle N(u_n, v_n, w_n), \eta(x_{n+2} - m(x_{n+1}) + m(x_n), x_{n+1})\rangle \\ &\quad + \rho b(x_n, x_{n+1}) - \rho b(x_n, x_{n+2} - m(x_{n+1}) + m(x_n)).\end{aligned} \tag{4.8}$$

Adding $\langle -m(x_{n+1}), y-x_{n+2}\rangle$ to both sides of (4.7) and then taking $y=x_{n+1}-m(x_n)+m(x_{n+1})\in K(x_{n+1})$, we obtain

$$\begin{aligned}
&\langle x_{n+2}-m(x_{n+1}),\, x_{n+1}-x_{n+2}-m(x_n)+m(x_{n+1})\rangle\\
&\geq \langle x_{n+1}-m(x_{n+1}),\, x_{n+1}-x_{n+2}-m(x_n)+m(x_{n+1})\rangle-\rho a(x_{n+1},\, x_{n+1}-x_{n+2}-m(x_n)+m(x_{n+1}))\\
&\quad -\rho\langle N(u_{n+1},v_{n+1},w_{n+1}),\eta(x_{n+1}-m(x_n)+m(x_{n+1}),\, x_{n+2})\rangle\\
&\quad +\rho b(x_{n+1},x_{n+2})-\rho b(x_{n+1},\, x_{n+1}-m(x_n)+m(x_{n+1})). \qquad (4.9)
\end{aligned}$$

By Assumption 2.3, it follows that

$$\begin{aligned}
\eta(x_{n+1}-m(x_n)+m(x_{n+1}),\, x_{n+2}) &= -\eta(x_{n+2}-m(x_{n+1})+m(x_n),\, x_{n+1})\\
&= \eta(x_{n+1}-x_{n+2},\, -m(x_n)-m(x_{n+1})). \qquad (4.10)
\end{aligned}$$

From (4.8)-(4.10) and Assumptions 2.1 and 2.2, it follows that

$$\begin{aligned}
&\|x_{n+1}-x_{n+2}-(m(x_n)-m(x_{n+1}))\|^2\\
&\leq \langle x_n-x_{n+1}-(m(x_n)-m(x_{n+1})),\, x_{n+1}-x_{n+2}-(m(x_n)-m(x_{n+1}))\rangle\\
&\quad -\rho a(x_n,x_{n+1}-x_{n+2}-(m(x_n)-m(x_{n+1})))\\
&\quad +\rho a(x_{n+1},x_{n+1}-x_{n+2}-(m(x_n)-m(x_{n+1})))\\
&\quad -\rho\langle N(u_n,v_n,w_n),\eta(x_{n+1}-x_{n+2},m(x_n)-m(x_{n+1}))\rangle\\
&\quad +\rho\langle N(u_{n+1},v_{n+1},w_{n+1}),\eta(x_{n+1}-x_{n+2},m(x_n)-m(x_{n+1}))\rangle\\
&\quad +\rho[b(-x_n,x_{n+1})-b(-x_n,x_{n+2}-m(x_{n+1})+m(x_n))\\
&\quad +b(x_{n+1},x_{n+1}-m(x_n)+m(x_{n+1}))-b(x_{n+1},x_{n+2})]\\
&\leq \langle x_n-x_{n+1}-(m(x_n)-m(x_{n+1})),\, x_{n+1}-x_{n+2}-(m(x_n)-m(x_{n+1}))\rangle\\
&\quad -\rho a(x_n-x_{n+1},x_{n+1}-x_{n+2}-(m(x_n)-m(x_{n+1})))\\
&\quad -\rho\langle N(u_n,v_n,w_n)-N(u_{n+1},v_{n+1},w_{n+1}),\eta(x_{n+1}-x_{n+2},m(x_n)-m(x_{n+1}))\rangle\\
&\quad +\rho b(x_{n+1}-x_n,\, x_{n+1}-x_{n+2}-(m(x_n)-m(x_{n+1})))\\
&\leq \langle x_n-x_{n+1}-\rho\Lambda S(x_n-x_{n+1})-[m(x_n)-m(x_{n+1})\\
&\quad -\rho(N(u_n,v_n,w_n)-N(u_{n+1},v_{n+1},w_{n+1}))],x_{n+1}-x_{n+2}-(m(x_n)-m(x_{n+1}))\rangle\\
&\quad -\rho\langle N(u_n,v_n,w_n)-N(u_{n+1},v_{n+1},w_{n+1}),\, x_{n+1}-x_{n+2}-(m(x_n)-m(x_{n+1}))
\end{aligned}$$

$$+\eta(x_{n+1}-x_{n+2},m(x_n)-m(x_{n+1}))\rangle$$

$$+\rho\mu\|x_{n+1}-x_n\|\,\|x_{n+1}-x_{n+2}-(m(x_n)-m(x_{n+1}))\|$$

$$\leq \{\|x_n-x_{n+1}-\rho\Lambda S(x_n-x_{n+1})-[m(x_n)-m(x_{n+1})-\rho(N(u_n,v_n,w_n)-N(u_{n+1},v_{n+1},w_{n+1}))]\|$$

$$+\rho\mu\|x_{n+1}-x_n\|\}\|x_{n+1}-x_{n+2}-(m(x_n)-m(x_{n+1}))\|+\rho\|N(u_n,v_n,w_n)-N(u_{n+1},v_{n+1},w_{n+1})\|$$

$$\{\|x_{n+1}-x_{n+2}-(m(x_n)-m(x_{n+1}))\|+\|\eta(x_{n+1}-x_{n+2},m(x_n)-m(x_{n+1}))\|\}$$

$$\leq \{\|x_n-x_{n+1}-\rho\Lambda S(x_n-x_{n+1})-[m(x_n)-m(x_{n+1})-\rho(N(u_n,v_n,w_n)-N(u_{n+1},v_{n+1},w_{n+1}))]\|$$

$$+\rho(1+\xi)\|N(u_n,v_n,w_n)-N(u_{n+1},v_{n+1},w_{n+1})\|+\rho\mu\|x_{n+1}-x_n\|\}\|x_{n+1}-x_{n+2}-(m(x_n)-m(x_{n+1}))\|,$$

where η is ξ-Lipschitz continuous.

Hence, we have

$$\begin{aligned}\|x_{n+1}-x_{n+2}\| \leq\; & \|m(x_n)-m(x_{n+1})\|+\|x_n-x_{n+1}-\rho\Lambda S(x_n-x_{n+1})\\ & -[m(x_n)-m(x_{n+1})-\rho(N(u_n,v_n,w_n)-N(u_{n+1},v_{n+1},w_{n+1}))]\|\\ & +\rho(1+\xi)\|N(u_n,v_n,w_n)-N(u_{n+1},v_{n+1},w_{n+1})\|+\rho\mu\|x_n-x_{n+1}\|. \quad (4.11)\end{aligned}$$

Using $\|x-y\|^2\leq 2(\|x\|^2+\|y\|^2)$, we have

$$\|x_n-x_{n+1}-\rho\Lambda S(x_n-x_{n+1})-[m(x_n)-m(x_{n+1})-\rho(N(u_n,v_n,w_n)-N(u_{n+1},v_{n+1},w_{n+1}))]\|^2$$

$$\leq 2\{\|x_n-x_{n+1}-\rho\Lambda S(x_n-x_{n+1})\|^2+\|m(x_n)-m(x_{n+1})-\rho(N(u_n,v_n,w_n)-N(u_{n+1},v_{n+1},w_{n+1}))\|^2\}. \quad (4.12)$$

Using Assumption 2.1, (2.2) and (2.3), we estimate

$$\begin{aligned}&\|x_n-x_{n+1}-\rho\Lambda S(x_n-x_{n+1})\|^2\\ &= \|x_n-x_{n+1}\|^2-2\rho\langle\Lambda S(x_n-x_{n+1}),x_n-x_{n+1}\rangle+\rho^2\|\Lambda S(x_n-x_{n+1})\|^2\\ &= \|x_n-x_{n+1}\|^2-2\rho a(x_n-x_{n+1},x_n-x_{n+1})+\rho^2\|S(x_n-x_{n+1})\|^2\\ &\leq (1-2\rho\nu+\rho^2\gamma^2)\|x_n-x_{n+1}\|^2. \quad (4.13)\end{aligned}$$

Since N is α-m-strongly mixed monotone with respect to T, A and S, and $(\beta_1,\beta_2,\beta_3)$-mixed Lipschitz continuous; T, A and S are k_1-$\mathscr{H}$-Lipschitz continuous, k_2-$\mathscr{H}$-Lipschitz continuous and k_3-$\mathscr{H}$-Lipschitz continuous, respectively; m is σ-Lipschitz continuous, then we have

$$\|N(u_n,v_n,w_n)-N(u_{n+1},v_{n+1},w_{n+1})\| \leq \beta_1\|u_n-u_{n+1}\|+\beta_2\|v_n-v_{n+1}\|+\beta_3\|w_n-w_{n+1}\|$$

$$\leq (1+(1+n)^{-1})\{\beta_1\mathscr{H}(T(x_n),T(x_{n+1}))+\beta_2\mathscr{H}(A(x_n),A(x_{n+1}))+\beta_3\mathscr{H}(S(x_n),S(x_{n+1}))\}$$
$$\leq (1+(1+n)^{-1})(\beta_1 k_1+\beta_2 k_2+\beta_3 k_3)\|x_n-x_{n+1}\|, \tag{4.14}$$

and

$$\|m(x_n)-m(x_{n+1})-\rho(N(u_n,v_n,w_n)-N(u_{n+1},v_{n+1},w_{n+1}))\|^2$$
$$\leq \|m(x_n)-m(x_{n+1})\|^2-2\rho\langle N(u_n,v_n,w_n)-N(u_{n+1},v_{n+1},w_{n+1}),\, m(x_n)-m(x_{n+1})\rangle$$
$$+\rho^2\|N(u_n,v_n,w_n)-N(u_{n+1},v_{n+1},w_{n+1})\|^2$$
$$\leq \Big(\sigma^2-2\rho\alpha+\rho^2(1+(1+n)^{-1})^2(\beta_1 k_1+\beta_2 k_2+\beta_3 k_3)^2\Big)\|x_n-x_{n+1}\|^2. \tag{4.15}$$

It follows from (4.11)-(4.15) that

$$\|x_{n+1}-x_{n+2}\| \ \leq\ \theta_n\|x_n-x_{n+1}\|, \tag{4.16}$$

where

$$\theta_n \ :=\ \{\sigma+\sqrt{2}\,\sqrt{(1+\sigma^2)-2\rho(\nu+\alpha)+\rho^2(\gamma^2+l_n^2L_N^2)}+\rho(1+\xi)l_nL_N+\rho\mu\}. \tag{4.17}$$

where $l_n := (1+(1+n)^{-1})$ and $L_N := (\beta_1 k_1+\beta_2 k_2+\beta_3 k_3)$.

Letting $n\to\infty$, then $\theta_n\to\theta$, where

$$\theta \ :=\ \{\sigma+\rho(\mu+(1+\xi)L_N)+\sqrt{2}\,\sqrt{(1+\sigma^2)-2\rho(\nu+\alpha)+\rho^2(\gamma^2+L_N^2)}\,\}. \tag{4.18}$$

where $L_N := (\beta_1 k_1+\beta_2 k_2+\beta_3 k_3)$.

It follows from (4.5) that $\theta\in(0,1)$. Hence $\theta<1$ for n sufficiently large. Therefore, (4.16) implies that $\{x_n\}$ is a Cauchy sequence in H and hence we suppose that $x_n\to x\in H$. Since T, A and S are $\mathscr{H}$-Lipschitz continuous, then from (4.2)-(4.4), we have

$$\|u_{n+1}-u_n\| \ \leq\ (1+(1+n)^{-1})\mathscr{H}(T(x_{n+1}),\,T(x_n))$$
$$\leq\ (1+(1+n)^{-1})\,k_1\|x_{n+1}-x_n\|, \tag{4.19}$$
$$\|v_{n+1}-v_n\| \ \leq\ (1+(1+n)^{-1})\,k_2\|x_{n+1}-x_n\|, \tag{4.20}$$

and

$$\|w_{n+1}-w_n\| \ \leq\ (1+(1+n)^{-1})\,k_3\|x_{n+1}-x_n\|. \tag{4.21}$$

Thus, from (4.19)-(4.21), it shows that $\{u_n\}$, $\{v_n\}$ and $\{w_n\}$ are also Cauchy sequences in H. Let $u_n \to u \in H$, $v_n \to v \in H$ and $w_n \to w \in H$ as $n \to \infty$.

Further, we have

$$\begin{aligned} d(u, T(x)) &\leq \|u_n - u\| + d(u_n, T(x)) \\ &\leq \|u_n - u\| + \mathscr{H}(T(x_n), T(x)) \\ &\leq \|u_n - u\| + k_1 \|x_n - x\| \\ &\to 0 \text{ as } n \to \infty, \end{aligned}$$

where $d(u,T(x)) = \inf\{\|u-z\| : z \in T(x)\}$. This implies that $u \in T(x)$. Similarly, we can prove that $v \in A(x)$ and $w \in S(x)$.

By Theorem 3.1, we know that there exists a unique $t \in K(x)$ such that

$$\langle t, y-t\rangle \geq \langle x, y-t\rangle - \rho a(x, y-t) - \rho\langle N(u,v,w), \eta(y,t)\rangle + \rho b(x,t) - \rho b(x,y),\ \forall y \in K(x). \tag{4.22}$$

We show that $x = t$. By applying (4.6), (4.22) and similar arguments as proving (4.11), we can prove that

$$\begin{aligned} \|x_{n+1} - t\| &\leq \|m(x_n) - m(x)\| + \|x_n - x - \rho\Lambda S(x_n - x)\| \\ &\quad + \|m(x_n) - m(x) - \rho(N(u_n, v_n, w_n) - N(u,v,w))\| \\ &\quad + \rho(1+\xi)\|N(u_n, v_n, w_n) - N(u,v,w)\| + \rho\mu\|x_n - x\|. \end{aligned} \tag{4.23}$$

Since N, m and S are continuous, then (4.23) implies that $x_n \to t$ as $n \to \infty$. Since $x_n \to x$, we must have $x = t$. It follows from (4.22) that (x,u,v,w) is a solution of GMQVLIP (2.5). This completes the proof.

Remark 4.2. Since the GMQVLIP (2.5) includes many known classes of quas-variational and quasi-variational-like inequalities as special cases, Theorem 3.1 and Theorem 4.1 improve and generalize the known results given in [2,5-7,9,11,13].

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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References

[1] C.E. Chidume, K.R. Kazmi and H. Zegeye, General auxiliary problem and algorithm for a generalized multi-valued variational-like inequality problem in reflexive Banach spaces, Appl. Anal. 82 (12) (2003), 1099-1109.

[2] X.P. Ding, Algorithm of solutions for mixed implicit quasi-variational inequalities with fuzzy mappings, Comput. Math. Appl. 38 (1999), 231-241.

[3] R. Glowinski, J.L. Lions and R. Tremolieres, Numerical Analysis of Variational Inequalities, North-Holland, Amsterdam, 1981.

[4] X. He, On Φ-strongly accretive mappings and some set-valued variational problems, J. Math. Anal. Appl. 277 (2003), 504-511.

[5] N.J. Huang and C.X. Deng, Auxiliary principle and iterative algorithms for generalized set-valued strongly nonlinear mixed variational-like inequalities, J. Math. Anal. Appl. 256 (2001), 345-359.

[6] N.J. Huang and Y.P. Fang, Auxiliary principle technique for solving generalized set-valued nonlinear quasi-variational-like inequalities, Math. Inequal. Appl. 6 (2) (2003), 339-350.

[7] N.J. Huang, M.R. Bai, Y.J. Cho and S.M. Kang, Generalized nonlinear mixed quasi-variational inequalities, Comput. Math. Appl. 40 (2-3) (2000), 205-215.

[8] S.B. Nadler, Jr., Multivalued contraction mappings, Pacific J. Math. 30 (1969), 475-487.

[9] M.A. Noor, Auxiliary principle for generalized mixed variational-like inequalities, J. Math. Anal. Appl. 215 (1997), 78-85.

[10] D. Pascali and S. Sburlan, Nonlinear Mappings of Monotone Type, Sijthoff and Noordhoof, The Netherlands, 1978.

[11] G. Tian, Generalized quasi-variational-like inequality problem, Math. Oper. Res. 18 (1993), 752-764.

[12] J.C. Yao, The generalized quasi-variational inequality problem with applications, J. Math. Anal. Appl. 185 (1991), 139-160.

[13] L.C. Zeng, S. Schaible and J.C. Yao, Iterative algorithm for generalized set-valued strongly nonlinear mixed variational-like inequalities, J. Optim. Theory Appl. 124 (3) (2005), 725-737.