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COMMON FIXED POINTS VIA C_k-CLASS FUNCTIONS IN S-METRIC SPACES

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Abstract. In this paper, we define C_k-class functions in three variables and prove the existence and uniqueness of common fixed points of three self mappings in S-metric spaces using C_k-class functions involving generalized altering distance function in five variables and by using property (E. A.) under weakly compatible property. Our results extend and generalize the results of Sedghi, Shobe and Aliouche [17]. Supporting examples are provided to illustrate our results.

Keywords: S-metric space; C-class functions; C_k-class functions; common fixed point; property (E. A.).

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1. Introduction

In 1986, Jungck[10] initiated the study of the existence of common fixed points of compatible mappings. On the other hand in 1998, Jungck and Rhoades[11] introduced the notation of weakly compatible mappings which are weaker than compatible mappings. In 2002, Aamri and

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Moutawakil [1] defined a new property for a pair of mappings called property (E. A.). In 2007, Pathak, Rodriguez-Lopez and Verma [15] illustrated that weakly compatibility and property (E. A.) are independent to each other.

In 2014, Ansari [2] introduced C-class functions and generalized contraction conditions by using C-class functions and proved fixed point theorems of generalized contractions involving C-class functions. For more works on contraction conditions involving C-class functions, we refer [3], [4], [7], [13].

2. PRELIMINARIES

Definition 2.1. [2] A mapping $F:[0,\infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and satisfies the following axioms:

(1) $F(s,t) \leq s$
(2) $F(s,t) = s$ implies that either $s = 0$ or $t = 0$ for all $s,t \in [0,\infty)$.

We denote the set of all C-class functions by $\mathscr{C}$.

We now define C_k-class functions as follows.

Definition 2.2. Let $F:[0,\infty)^2 \to \mathbb{R}$ be a function and $k \geq 1$. If F is continuous and

(1) $F(s,t) \leq ks$
(2) $F(s,t) = ks$ implies that either $s = 0$ or $t = 0$ for all $s,t \in [0,\infty)$,

then we say that F is a C_k-class function.

Every C-class function is a C_k-class function for any $k \geq 1$. But its converse need not be true.

Example 2.3. A mapping $F:[0,\infty)^2 \to \mathbb{R}$ is a function such that $F(s,t) = 2s$ for all $s,t \in [0,\infty)$. Then clearly, F is a C_k-class function for any $k \geq 2$. But it is not a C-class function.

Now we extend C_k-class function to three variables as follows.

Definition 2.4. Let $F:[0,\infty)^3 \to \mathbb{R}$. Let $k \geq 1$. If F if continuous and satisfies the following conditions :

(1) $F(s,s,t) \leq ks$

(2) $F(s,r,t) = k\max\{s,r\}$ implies that either $s = 0$ or $r = 0$ or $t = 0$ for all $s,\ r,\ t \in [0,\infty)$,

then we call F is a C_k-class function in three variables.

Here onwards, we denote the class of all C_k-class functions in three variables by $\mathscr{C}_k$. The following are examples of $\mathscr{C}_k$.

Example 2.5. We define $F : [0,\infty)^3 \to \mathbb{R}$ as follows: for any $s,r,t \in [0,\infty)$,

(1) $F(s,s,t) = ms$ where $0 < m < 1$.
(2) $F(s,t) = s\beta(s)$ where $\beta : [0,\infty) \to [0,1)$ is a continuous function.
(3) $F(s,r,t) = m(s+r)$ where $0 \leq m < \frac{1}{2}$.
(4) $F(s,r,t) = \phi(t)se^{-r}$ where $\phi : [0,\infty) \to \mathbb{R}$ is a bounded function.

Then all of the above functions $F \in \mathscr{C}_k$.

In 2012, Sedghi, Shobe and Aliouche[17] introduced a new concept of S-metric spaces and studied some properties of these spaces.

Definition 2.6. [17] Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions: for each $x,y,z,a \in X$

(S1) $S(x,y,z) \geq 0$,
(S2) $S(x,y,z) = 0$ if and only if $x = y = z$ and
(S3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called an S-metric space.

Some examples of S-metric spaces are the following:

Example 2.7. [17] Let $X = \mathbb{R}^n$ and $||.||$ be a norm on X. Then $S(x,y,z) = ||x-z|| + ||y-z||$ for all $x,y,z \in \mathbb{R}$ is an S-metric on X.

Example 2.8. [5] Let $X = \mathbb{R}$, the set of all real numbers. Then $S(x,y,z) = \max\{|x-z|, |y-z|\}$ for all $x,y,z \in X$ is an S-metric on X.

Example 2.9. [17] Let $X = \mathbb{R}^n$, and $||.||$ be a norm on X then $S(x,y,z) = ||y+z-2x|| + ||y-z||$ for all $x,y,z \in X$ is an S-metric on X.

Example 2.10. [17] Let X be a nonempty set and d be a metric on X. Then $S(x,y,z) = d(x,z) + d(y,z)$ for all $x,y,z \in X$ is an S-metric on X.

Example 2.11. Let $X = [0,1]$ and we define $S : X^3 \to [0,\infty)$ by

$$S(x,y,z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x,y,z\} & \text{otherwise.} \end{cases}$$

Then S is an S-metric on X.

Lemma 2.12. [17] In an S-metric space, we have $S(x,x,y) = S(y,y,x)$.

Definition 2.13. [17] Let (X,S) be an S-metric space. We define the following:

(i) a sequence $\{x_n\}$ in X converges to a point $x \in X$ if and only if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0,\ S(x_n,x_n,x) < \varepsilon$ and we denote it by $\lim_{n\to\infty} x_n = x$.

(ii) a sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \varepsilon$ for all $n,m \geq n_0$.

(iii) an S-metric space (X,S) is said to be complete if each Cauchy sequence in X is convergent.

Lemma 2.14. [17] Let (X,S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then x is unique.

Lemma 2.15. [17] Let (X,S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n,x_n,y_n) = S(x,x,y)$.

Lemma 2.16. [6] Let (X,S) be an S-metric space. If a sequence $\{x_n\}$ in X converges to x and $S(x_n,x_n,y_n) \to 0$ then $y_n \to x$.

Definition 2.17. [6] Let (X,S) be an S-metric space and f,T be two self maps on X. Then a point $x \in X$ is called a common fixed point of f and T if $x = fx = Tx$. The pair (f,T) is said to be

(i) commuting on X if $fTx = Tfx$ for all $x \in X$.

(ii) S-weakly commuting on X if $S(fTx,fTx,Tfx) \leq S(fx,fx,Tx)$ for every $x \in X$.

(iii) S-compatible if $\lim_{n\to\infty} S(fTx_n, fTx_n, Tfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

(iv) S-weakly compatible if they commute at their coincidence point. That is, for $x \in X$, if $fx = Tx$ holds then $fTx = Tfx$.

(v) property (E. A.), if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in X$.

Remark 2.18. (1) Every commuting pair of maps in an S-metric space X is S-weakly commuting, but its converse need not be true.

(2) Every S-weakly commuting pair of maps is S-compatible, but its converse need not be true.

(3) Every S-compatible pair of maps is S-weakly compatible, but its converse need not be true.

(4) Property (E. A.) and S-weakly compatible pair of maps are independent to each other.

For more details we refer [6].

The following is the Banach contraction principle in S-metric spaces.

Theorem 2.19. [17] Let (X,S) be an S-metric space. A map $F : X \to X$ is said to be a contraction i.e., there exists a constant $0 \le L < 1$ such that

$$S(Fx, Fx, Fy) \le LS(x, x, y) \tag{2.1}$$

for all $x, y \in X$. Then F has a unique fixed point u in X.

For more works on the existence of fixed points of mappings satisfying certain contractive conditions on S-metric spaces, we refer [5], [9], [14], [16], [19].

We now define a generalized altering distance function in five variables as follows.

Definition 2.20. A mapping $\psi : [0,\infty)^5 \to [0,\infty)$ is said to be a generalized altering distance function in five variables , if

(i) ψ is continuous in each of its variables

(ii) ψ is non-decreasing in each of its variables

(iii) $\psi(t_1,t_2,t_3,t_4,t_5) = 0$ if and only if $t_i = 0$ for $i = 1,\ 2,\ 3,\ 4,\ 5$

(iv) If $\eta(t) = \psi(t,t,t,t,t)$ then $\eta(t) \le t$ for $t > 0$.

We denote the set of all generalized altering distance functions in five variables by Ψ.

In this paper, in Section 3, we prove the existence and uniqueness of common fixed points of three self mappings on S-metric space via C_k-class functions involving generalized altering distance function in five variables satisfying property (E. A.) under weakly compatible property. Also, we establish a common fixed point theorem by replacing property (E. A.) with asymptotically regular property. Corollaries and examples are provided in Section 4.

3. Main results

Let (X,S) be an S- metric space. Given $x_0 \in X$ and self mappings f, G and T on X, if there exists a sequence $\{x_n\} \in X$ such that $Gx_{2n} = fx_{2n+1}$ and $Tx_{2n+1} = fx_{2n+2}$ then $O(G,T;f,x_0) = \{fx_{n+1} : n = 0,1,2,.....\}$ is called a (G,T)-orbit at x_0 with respect to f.

Definition 3.1. Let (X,S) be an S-metric space. Let $x_0 \in X$. X is said to be orbitally complete at x_0 if every Cauchy sequence in $O(G,T;f,x_0)$ converges in X.

Definition 3.2. Let $G,\ T : X \to X$. The pair (G,T) is said to be asymptotically regular at a point x_0 in X with respect to f if there exists a sequence $\{x_n\}$ in X such that $Gx_{2n} = fx_{2n+1}$ and $Tx_{2n+1} = fx_{2n+2}$ where $S(fx_n, fx_n, fx_{n+1}) \to 0$ as $n \to \infty$.

Lemma 3.3. *Let (X,S) be an S-metric space. Let $\{x_n\}$ be a sequence in X converging to x in X. Then* $\lim\limits_{n\to\infty} S(x_n,x_n,y_n) = \lim\limits_{n\to\infty} S(x,x,y_n)$.

Proof. Let $\{x_n\}$ be a sequence in X that converges to $x \in X$. We consider
$S(x_n,x_n,y_n) \leq 2S(x_n,x_n,x) + S(x,x,y_n)$. Then

$$\lim_{n\to\infty} S(x_n,x_n,y_n) \leq \lim_{n\to\infty} S(x,x,y_n). \tag{3.1}$$

Now we consider
$S(x,x,y_n) \leq 2S(x,x,x_n) + S(x_n,x_n,y_n)$
and hence

$$\lim_{n\to\infty} S(x,x,y_n) \leq \lim_{n\to\infty} S(x_n,x_n,y_n). \tag{3.2}$$

From (3.1) and (3.2) we have, $\lim\limits_{n\to\infty} S(x_n,x_n,y_n) = \lim\limits_{n\to\infty} S(x,x,y_n)$. □

Lemma 3.4. *Let (X,S) be an S-metric space and $\{y_n\}$ be a sequence in X such that $\lim_{n\to\infty} S(y_n,y_n,y_{n+1})=0$. Then the sequence $\{y_n\}$ is Cauchy if and only if $\{y_{2n}\}$ is Cauchy.*

Proof. Let $\varepsilon>0$ be given. Since $\{y_n\}$ is a sequence such that $\lim_{n\to\infty} S(y_n,y_n,y_{n+1})=0$, then there exists $n_1\in\mathbb{N}$ such that

$$S(y_n,y_n,y_{n+1})<\frac{\varepsilon}{3} \tag{3.3}$$

for all $n\geq n_1$. First suppose that $\{y_n\}$ is a Cauchy sequence in X. Then there exists $n_0\in\mathbb{N}$ such that

$$S(y_n,y_n,y_m)<\frac{\varepsilon}{3} \tag{3.4}$$

for all $n>m\geq n_0$. Let $2n>2m>l$ where $l=\max\{n_0,n_1\}$. We now consider

$$\begin{aligned} S(y_{2n},y_{2n},y_{2m}) &\leq 2S(y_{2n-1},y_{2n-1},y_{2n})+S(y_{2n-1},y_{2n-1},y_{2m})\\ &< 2\tfrac{\varepsilon}{3}+\tfrac{\varepsilon}{3}=\varepsilon. \end{aligned}$$

Therefore $\{y_{2n}\}$ is Cauchy.

Conversely, suppose that $\{y_{2n}\}$ is Cauchy. Then there exists $n_0\in\mathbb{N}$ such that

$$S(y_{2n},y_{2n},y_{2m})<\frac{\varepsilon}{3} \tag{3.5}$$

for all $2n>2m\geq n_0$. We now prove $S(y_n,y_n,y_m)<\varepsilon$ for all $n>m\geq n_0$.

Case (i): If n and m are even then by (3.3), we are through.

Case (ii): If n is even and m is odd then $n=2n_1$ and $m=2m_1+1$, where $m_1,n_1\in\mathbb{N}$. In this case,

$$\begin{aligned} S(y_n,y_n,y_m) &= S(y_{2n_1},y_{2n_1},y_{2m_1+1})\\ &\leq 2S(y_{2n_1},y_{2n_1},y_{2m_1})+S(y_{2m_1},y_{2m_1},y_{2m_1+1})\\ &< 2\tfrac{\varepsilon}{3}+\tfrac{\varepsilon}{3}=\varepsilon. \end{aligned}$$

Case (iii): If n,m are odd then $n=2n_1+1$ and $m=2m_1+1$ where $n_1,m_1\in\mathbb{N}$. Now

$$\begin{aligned} S(y_n,y_n,y_m) &= S(y_{2n_1+1},y_{2n_1+1},y_{2m_1+1})\\ &\leq 2S(y_{2n_1+1},y_{2n_1+1},y_{2n_1})+S(y_{2n_1},y_{2n_1},y_{2m_1+1})\\ &< 2\tfrac{\varepsilon}{3}+\tfrac{\varepsilon}{3}=\varepsilon. \end{aligned}$$

Case (iv): If n is odd and m is even then $n=2n_1+1$ and $m=2m_1$, where $n_1,m_1\in\mathbb{N}$. In this case,

$$\begin{aligned} S(y_n,y_n,y_m) &= S(y_{2n_1+1},y_{2n_1+1},y_{2m_1}) \\ &\leq 2S(y_{2n_1+1},y_{2n_1+1},y_{2n_1}) + S(y_{2n_1},y_{2n_1},y_{2m_1}) \\ &< 2\tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Hence $\{y_n\}$ is Cauchy. □

Lemma 3.5. *(*[5], [8]*)* Let (X,S) be an S-metric space and $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = 0. \tag{3.6}$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $m_k > n_k > k$ such that

$$S(x_{m_k},x_{m_k},x_{n_k}) \geq \varepsilon \tag{3.7}$$

$$S(x_{m_k-1},x_{m_k-1},x_{n_k}) < \varepsilon. \tag{3.8}$$

Then we have the following:

(i) $\lim_{k\to\infty} S(x_{m_k},x_{m_k},x_{n_k}) = \varepsilon$ (ii) $\lim_{k\to\infty} S(x_{m_k-1},x_{m_k-1},x_{n_k}) = \varepsilon$
(iii) $\lim_{k\to\infty} S(x_{m_k},x_{m_k},x_{n_k-1}) = \varepsilon$ (iv) $\lim_{k\to\infty} S(x_{m_k-1},x_{m_k-1},x_{n_k-1}) = \varepsilon$.

Lemma 3.6. *Let (X,S) be an S-metric space. Let f,G and T be self mappings on X satisfying the following condition: there exists $k \geq 1$ such that*

$$kS(Gx,Gy,Tz) \leq F(\psi(t_1,t_2,t_3,t_4,t_5),\psi(t_1,t_2,t_3,t_4,t_5),\varphi(t_1,t_2,t_3,t_4,t_5)) \tag{3.9}$$

for all $x,y \in X$, where $\psi,\varphi \in \Psi$ and $F \in \mathscr{C}_k$ with $t_1 = S(fx,fy,fz)$, $t_2 = \frac{S(fx,fx,Gx)+S(fy,fy,Gy)}{2}$, $t_3 = \frac{S(fx,fx,Tz)+S(fy,fy,Tz)}{2}$, $t_4 = \frac{S(fz,fz,Gx)+S(fz,fz,Gy)}{2}$ and $t_5 = S(fz,fz,Tz)$.

If (f,G) satisfies property (E. A.) and $f(X)$ is closed then f and G have a coincidence point in X.

Proof. We assume that the pair (f,G) satisfies property (E. A.). So there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Gx_n = z$ for some $z \in X$. We now prove that $\lim_{n\to\infty} Tx_n = z$. By taking $x = x_n, y = x_n, z = x_n$ in (3.9), we have

$$kS(Gx_n,Gx_n,Tx_n) \leq F(\psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5^n),\psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5^n),\varphi(t_1^n,t_2^n,t_3^n,t_4^n,t_5^n)) \tag{3.10}$$

where $t_1^n = S(fx_n, fx_n, fx_n) = 0$; $t_2^n = \frac{S(fx_n,fx_n,Gx_n)+S(fx_n,fx_n,Gx_n)}{2} = S(fx_n, fx_n, Gx_n)$;
$t_3^n = S(fx_n, fx_n, Tx_n)$; $t_4^n = \frac{S(fx_n,fx_n,Gx_n)+S(fx_n,fx_n,Gx_n)}{2} = S(fx_n, fx_n, Gx_n)$; $t_5^n = S(fx_n, fx_n, Tx_n)$.

Here we observe that

$\lim_{n\to\infty} t_2^n = S(z,z,z) = 0$; $\lim_{n\to\infty} t_3^n = \lim_{n\to\infty} S(fx_n, fx_n, Tx_n) = \lim_{n\to\infty} S(z,z,Tx_n)$; $\lim_{n\to\infty} t_4^n = 0$;
$\lim_{n\to\infty} t_5^n = \lim_{n\to\infty} S(z,z,Tx_n)$.

On taking limits as $n\to\infty$ in (3.10) and using Lemma 3.3, we have

$$\begin{aligned}
k\lim_{n\to\infty} S(z,z,Tx_n) &\le \lim_{n\to\infty} F(\psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5^n), \psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5^n), \varphi(t_1^n,t_2^n,t_3^n,t_4^n,t_5^n))\\
&= F(\lim_{n\to\infty}\psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5^n), \lim_{n\to\infty}\psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5^n), \lim_{n\to\infty}\varphi(t_1^n,t_2^n,t_3^n,t_4^n,t_5^n))\\
&= F(\psi(0,0,\lim_{n\to\infty} S(z,z,Tx_n),0,\lim_{n\to\infty} S(z,z,Tx_n)),\\
&\quad \psi(0,0,\lim_{n\to\infty} S(z,z,Tx_n),0,\lim_{n\to\infty} S(z,z,Tx_n)),\\
&\quad \varphi(0,0,\lim_{n\to\infty} S(z,z,Tx_n),0,\lim_{n\to\infty} S(z,z,Tx_n)))\\
&\le k\psi(0,0,\lim_{n\to\infty} S(z,z,Tx_n),0,\lim_{n\to\infty} S(z,z,Tx_n))\\
&\le k\eta(\lim_{n\to\infty} S(z,z,Tx_n)) \le k\lim_{n\to\infty} S(z,z,Tx_n).
\end{aligned}$$

Hence

$$\begin{aligned}
F(\psi(0,0,\lim_{n\to\infty} S(z,z,Tx_n),0,\lim_{n\to\infty} S(z,z,Tx_n)), \psi(0,0,\lim_{n\to\infty} S(z,z,Tx_n),0,\lim_{n\to\infty} S(z,z,Tx_n)),\\
\varphi(0,0,\lim_{n\to\infty} S(z,z,Tx_n),0,\lim_{n\to\infty} S(z,z,Tx_n)))\\
= k\psi(0,0,\lim_{n\to\infty} S(z,z,Tx_n),0,\lim_{n\to\infty} S(z,z,Tx_n)).
\end{aligned}$$

By using the second property of F, we have

either $\psi(0,0,\lim_{n\to\infty} S(z,z,Tx_n),0,\lim_{n\to\infty} S(z,z,Tx_n)) = 0$
or $\varphi(0,0,\lim_{n\to\infty} S(z,z,Tx_n),0,\lim_{n\to\infty} S(z,z,Tx_n))$,

so that $\lim_{n\to\infty} S(z,z,Tx_n) = 0$. Hence $\lim_{n\to\infty} Tx_n = z$. Thus $\lim_{n\to\infty} Gx_n = z = \lim_{n\to\infty} Tx_n$. Therefore there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} Gx_n = \lim_{n\to\infty} Tx_n = z. \tag{3.11}$$

Since $f(X)$ is closed, we have $z \in f(X)$ so that $z = fu$ for some $u \in X$.

Now taking $x = u,\ y = u,\ z = x_n$ in (3.9), we have

$$kS(Gu,Gu,Tx_n) \le F(\psi(t_1^n,t_2,t_3^n,t_4^n,t_5^n), \psi(t_1^n,t_2,t_3^n,t_4^n,t_5^n), \varphi(t_1^n,t_2,t_3^n,t_4^n,t_5^n)) \tag{3.12}$$

where $t_1^n = S(fu,fu,fx_n)$; $t_2 = \frac{S(fu,fu,Gu)+S(fu,fu,Gu)}{2} = S(fu,fu,Gu) = S(z,z,Gu)$;
$t_3^n = \frac{S(fu,fu,Tx_n)+S(fu,fu,Tx_n)}{2} = S(z,z,Tx_n)$; $t_4^n = \frac{S(fx_n,fx_n,Gu)+S(fx_n,fx_n,Gu)}{2} = S(fx_n,fx_n,Gu)$;

$t_5^n = S(fx_n, fx_n, Tx_n)$.

Here we observe that

$\lim_{n\to\infty} t_1^n = S(z,z,z) = 0$; $\lim_{n\to\infty} t_3^n = 0$; $\lim_{n\to\infty} t_4^n = S(z,z,Gu)$; $\lim_{n\to\infty} t_5^n = 0$.

On letting $n \to \infty$ in (3.12), we have

$$\begin{aligned} kS(Gu,Gu,z) &\le \lim_{n\to\infty} F(\psi(t_1^n,t_2,t_3^n,t_4^n,t_5^n), \psi(t_1^n,t_2,t_3^n,t_4^n,t_5^n), \varphi(t_1^n,t_2,t_3^n,t_4^n,t_5^n)) \\ &= F(\lim_{n\to\infty} \psi(t_1^n,t_2,t_3^n,t_4^n,t_5^n), \lim_{n\to\infty} \psi(t_1^n,t_2,t_3^n,t_4^n,t_5^n), \lim_{n\to\infty} \varphi(t_1^n,t_2,t_3^n,t_4^n,t_5^n)) \\ &= F(\psi(0,S(z,z,Gu),0,S(z,z,Gu),0), \psi(0,S(z,z,Gu),0,S(z,z,Gu),0), \\ &\quad \varphi(0,S(z,z,Gu),0,S(z,z,Gu),0)) \\ &\le k\psi(0,S(z,z,Gu),0,S(z,z,Gu),0) \\ &\le k\eta(S(z,z,Gu)) \le kS(z,z,Gu). \end{aligned}$$

That is

$$\begin{aligned} &F(\psi(0,S(z,z,Gu),0,S(z,z,Gu),0), \psi(0,S(z,z,Gu),0,S(z,z,Gu),0), \\ &\qquad \varphi(0,S(z,z,Gu),0,S(z,z,Gu),0)) \\ &\qquad = k\psi(0,S(z,z,Gu),0,S(z,z,Gu),0)\ S(z,z,Gu) \end{aligned}$$

which implies that either $\psi(0,S(z,z,Gu),0,S(z,z,Gu),0) = 0$

or $\varphi(0,S(z,z,Gu),0,S(z,z,Gu),0) = 0$

so that $S(z,z,Gu) = 0$. Hence $z = Gu$. Therefore $fu = Gu = z$. □

Theorem 3.7. *Let (X,S) be an S-metric space. Let f,G and T be self mappings on X satisfying* (3.9). *Suppose that*

(a) *either the pair (f,G) or (f,T) satisfies property (E. A.);*

(b) *$f(X)$ is a closed subspace of X;*

(c) *the pair (f,G) or (f,T) is weakly compatible.*

Then f,G and T have a unique common fixed point in X.

Proof. If the pair (f,G) satisfies property (E. A.) and $f(X)$ is closed then by Lemma 3.6, f and G have a coincidence point u(say) in X. That is $fu = Gu = z$. By the weak compatibility of the pair (f,G), we have $fGu = Gfu$ which implies that $fz = Gz$.

Now taking $x = y = z$ in (3.9) and using $fz = Gz$, it follows that

$$kS(Gz,Gz,Tz) \le F(\psi(t_1,t_2,t_3,t_4,t_5), \psi(t_1,t_2,t_3,t_4,t_5), \varphi(t_1,t_2,t_3,t_4,t_5)) \tag{3.13}$$

where $t_1 = S(fz,fz,fz) = 0$; $t_2 = \frac{S(fz,fz,Gz)+S(fz,fz,Gz)}{2} = 0$;
$t_3 = \frac{S(fz,fz,Tz)+S(fz,fz,Tz)}{2} = S(fz,fz,Tz) = S(Gz,Gz,Tz)$; $t_4 = 0$; $t_5 = S(fz,fz,Tz) = S(Gz,Gz,Tz)$.

From (3.13), we have

$$\begin{aligned} kS(Gz,Gz,Tz) &\le F(\psi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz)),\psi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz)),\\ &\qquad \varphi(0,0,S(Gz,Gz,Tz),0,S(fz,fz,Tz)))\\ &\le k\psi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz))\\ &\le k\eta(S(Gz,Gz,Tz)) \le kS(Gz,Gz,Tz). \end{aligned}$$

Hence

$$\begin{aligned} &F(\psi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz)),\psi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz)),\\ &\qquad \varphi(0,0,S(Gz,Gz,Tz),0,S(fz,fz,Tz))) = k\psi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz)) \end{aligned}$$

which implies that either $\psi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz)) = 0$

or $\varphi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz)) = 0$

so that $S(Gz,Gz,Tz) = 0$. Hence $Gz = Tz$. Therefore $fz = Gz = Tz$.

Again for $x = x_n,\ y = x_n$ and $z = z$ in (3.9), we have

$$kS(Gx_n,Gx_n,Tz) \le F(\psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5),\psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5),\varphi(t_1^n,t_2^n,t_3^n,t_4^n,t_5)) \tag{3.14}$$

where $t_1^n = S(fx_n,fx_n,fz)$; $t_2^n = S(fx_n,fx_n,Gx_n)$; $t_3^n = S(fx_n,fx_n,Tz)$; $t_4^n = S(fz,fz,Gx_n)$;
$t_5 = S(fz,fz,Tz) = 0$.

Here we observe that

$\lim_{n\to\infty} t_1^n = S(z,z,Tz)$; $\lim_{n\to\infty} t_2^n = S(z,z,z) = 0$. $\lim_{n\to\infty} t_3^n = S(z,z,Tz)$; $\lim_{n\to\infty} t_4^n = S(z,z,Tz)$.

On letting $n \to \infty$ in (3.14), we get

$$\begin{aligned} kS(z,z,Tz) &\le \lim_{n\to\infty} F(\psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5),\psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5),\varphi(t_1^n,t_2^n,t_3^n,t_4^n,t_5))\\ &= F(\lim_{n\to\infty}\psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5),\lim_{n\to\infty}\psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5),\lim_{n\to\infty}\varphi(t_1^n,t_2^n,t_3^n,t_4^n,t_5))\\ &= F(\psi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0),\\ &\qquad \psi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0),\varphi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0))\\ &\le k\psi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0)\\ &\le k\eta(S(z,z,Tz)) \le kS(z,z,Tz). \end{aligned}$$

That is

$$F(\psi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0),\psi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0),$$
$$\varphi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0))$$
$$= k\psi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0)$$

which implies that either $\psi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0) = 0$

or $\varphi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0) = 0$.

Hence $S(z,z,Tz) = 0$ so that $z = Tz$.

Therefore z is a common fixed point of f,G and T.

Similarly, if the pair (f,T) satisfies property (E. A.) and since $f(X)$ is closed, then by Lemma 3.6, f and T have a coincidence point in X. Now proceeding as above we can prove that f,G and T have a common fixed point.

Let z and z' be two common fixed points of f,G and T. By taking $x = y = z$ in (3.9), we get

$$\begin{aligned}
kS(z,z,z') &= kS(Gz,Gz,Tz') \\
&\le F(\psi(t_1,t_2,t_3,t_4,t_5),\psi(t_1,t_2,t_3,t_4,t_5),\varphi(t_1,t_2,t_3,t_4,t_5)) \\
&= F(\psi(S(fz,fz,fz'),S(fz,fz,Gz),S(fz,fz,Tz'),S(fz',fz',Gz),S(fz,fz,Tz')), \\
&\quad \psi(S(fz,fz,fz'),S(fz,fz,Gz),S(fz,fz,Tz'),S(fz',fz',Gz),S(fz,fz,Tz')), \\
&\quad \varphi(S(fz,fz,fz'),S(fz,fz,Gz),S(fz,fz,Tz'),S(fz',fz',Gz),S(fz,fz,Tz'))) \\
&= F(\psi(S(z,z,z'),0,S(z,z,z'),S(z',z',z),S(z,z,z')),\psi(S(z,z,z'),0,S(z,z,z'),S(z',z',z), \\
&\quad S(z,z,z')),\varphi(S(z,z,z'),0,S(z,z,z'),S(z',z',z),S(z,z,z'))) \\
&\le k\psi(S(z,z,z'),0,S(z,z,z'),S(z',z',z),S(z,z,z')) \\
&\le k\eta(S(z,z,z')) \le kS(z,z,z').
\end{aligned}$$

That is

$$F(\psi(S(z,z,z'),0,S(z,z,z'),S(z,z,z'),S(z,z,z')),\psi(S(z,z,z'),0,S(z,z,z'),S(z,z,z'),S(z,z,z')),$$
$$\varphi(S(z,z,z'),0,S(z,z,z'),S(z',z',z),S(z,z,z')))$$
$$= k\psi(S(z,z,z'),0,S(z,z,z'),S(z,z,z'),S(z,z,z'))$$

i.e., either $\psi(S(z,z,z'),0,S(z,z,z'),S(z,z,z'),S(z,z,z')) = 0$

or $\varphi(S(z,z,z'),0,S(z,z,z'),S(z,z,z'),S(z,z,z')) = 0$

which implies that $S(z,z,z') = 0$ so that $z = z'$.

Hence the common fixed point of f, g and T is unique. $\square$

Theorem 3.8. *Let (X,S) be an S-metric space. Let f,G and T be self mappings on X satisfying* (3.9). *Suppose that*

(a) *the pair (G,T) is asymptotically regular with respect to f at some $x_0 \in X$;*

(b) *X is orbitally complete;*

(c) *the pair (f,G) or (f,T) is weakly compatible.*

Then f,G and T have a unique common fixed point in X.

Proof. Since (G,T) is asymptotically regular with respect to f at x_0, there exists a sequence $\{x_n\}$ in X defined by $Gx_{2n} = fx_{2n+1}$ and $Tx_{2n+1} = fx_{2n+2}$ for $n = 0,1,2,\ldots$ such that

$$\lim_{n\to\infty} S(fx_n, fx_n, fx_{n+1}) = 0. \tag{3.15}$$

We now show that $\{fx_{2n}\}$ is a Cauchy sequence. Suppose $\{fx_{2n}\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ such that for each even integer $2k$, there are even integers $2m_k$ and $2n_k$ such that $S(fx_{2m_k}, fx_{2m_k}, fx_{2n_k}) \geq \varepsilon$.

Now we take $x = x_{2n_k}, y = x_{2n_k}$ and $z = x_{2m_k+1}$ in (3.9) and using Lemma 3.5, we have

(3.16)

$$kS(Gx_{2n_k}, Gx_{2n_k}, Tx_{2m_k+1}) \leq F(\psi(t_1^{n_k}, t_2^{n_k}, t_3^{n_k}, t_4^{n_k}, t_5^{n_k}), \psi(t_1^{n_k}, t_2^{n_k}, t_3^{n_k}, t_4^{n_k}, t_5^{n_k}), \varphi(t_1^{n_k}, t_2^{n_k}, t_3^{n_k}, t_4^{n_k}, t_5^{n_k}))$$

where $t_1^{n_k} = S(fx_{2n_k}, fx_{2n_k}, fx_{2m_k+1})$; $t_2^{n_k} = S(fx_{2n_k}, fx_{2n_k}, Gx_{2n_k}) = S(fx_{2n_k}, fx_{2n_k}, fx_{2n_k+1})$;
$t_3^{n_k} = S(fx_{2n_k}, fx_{2n_k}, Tx_{2m_k+1}) = S(fx_{2n_k}, fx_{2n_k}, fx_{2m_k+2})$;
$t_4^{n_k} = S(fx_{2m_k+1}, fx_{2m_k+1}, Gx_{2n_k}) = S(fx_{2m_k+1}, fx_{2m_k+1}, fx_{2n_k+1})$;
$t_5^{n_k} = S(fx_{2m_k+1}, fx_{2m_k+1}, Tx_{2m_k+1}) = S(fx_{2m_k+1}, fx_{2m_k+1}, fx_{2m_k+2})$.

Here we observe that

$\lim_{k\to\infty} t_1^{n_k} = \varepsilon;\ \lim_{k\to\infty} t_2^{n_k} = 0;\ \lim_{k\to\infty} t_3^{n_k} = \varepsilon;\ \lim_{k\to\infty} t_4^{n_k} = \varepsilon;\ \lim_{k\to\infty} t_5^{n_k} = 0.$

On letting $k \to \infty$ in (3.16), we get

$$\begin{aligned} k\varepsilon = k\lim_{k\to\infty} S(fx_{2n_k+1}, fx_{2n_k+1}, fx_{2m_k+2}) &= k\lim_{k\to\infty} S(Gx_{2n_k}, Gx_{2n_k}, Tx_{2m_k+1}) \\ &\leq F(\psi(\varepsilon,0,\varepsilon,\varepsilon,0), \psi(\varepsilon,0,\varepsilon,\varepsilon,0), \varphi(\varepsilon,0,\varepsilon,\varepsilon,0)) \\ &\leq k\psi(\varepsilon,0,\varepsilon,\varepsilon,0) \\ &\leq k\eta(\varepsilon) \leq k\varepsilon. \end{aligned}$$

i.e., $F(\psi(\varepsilon,0,\varepsilon,\varepsilon,0), \psi(\varepsilon,0,\varepsilon,\varepsilon,0), \varphi(\varepsilon,0,\varepsilon,\varepsilon,0)) = k\psi(\varepsilon,0,\varepsilon,\varepsilon,0)$.

By using the second property of F, we have either $\psi(\varepsilon,0,\varepsilon,\varepsilon,0)=0$ or $\varphi(\varepsilon,0,\varepsilon,\varepsilon,0)=0$, so that $\varepsilon=0$, a contradiction.

Hence $\{fx_{2n}\}$ is a Cauchy sequence. Since X is orbitally complete, the Cauchy sequence $\{fx_{2n}\}$ has a limit z in $f(X)$. From (3.15), we have $\lim_{n\to\infty} S(fx_{2n},fx_{2n},fx_{2n+1})=0$. Hence by Lemma 2.16, we have $\lim_{n\to\infty} fx_{2n+1}=z$. Therefore there exists $u\in X$ such that $fu=z$.

Now we take $x=u,\ y=u$ and $z=x_{2n+1}$ in (3.9). Then we have

$$kS(Gu,Gu,Tx_{2n+1})\leq F(\psi(t_1^n,t_2,t_3^n,t_4^n,t_5^n),\psi(t_1^n,t_2,t_3^n,t_4^n,t_5^n),\varphi(t_1^n,t_2,t_3^n,t_4^n,t_5^n)) \tag{3.17}$$

where $t_1^n=S(fu,fu,fx_{2n+1})$; $t_2=S(fu,fu,Gu)=S(z,z,Gu)$; $t_3^n=S(fu,fu,Tx_{2n+1})=S(z,z,fx_{2n+2})$; $t_4^n=S(fx_{2n+1},fx_{2n+1},Gu)$; $t_5^n=S(fx_{2n+1},fx_{2n+1},Tx_{2n+1})=S(fx_{2n+1},fx_{2n+1},fx_{2n+2})$.

Here we observe that

$\lim_{n\to\infty} t_1^n=S(z,z,z)=0$; $\lim_{n\to\infty} t_3^n=S(z,z,z)=0$; $\lim_{n\to\infty} t_4^n=S(z,z,Gu)$; $\lim_{n\to\infty} t_5^n=0$.

On letting $n\to\infty$ in (3.17), we get

$$\begin{aligned}
kS(Gu,Gu,z)\leq F(&\psi(0,S(Gu,Gu,z),0,S(Gu,Gu,z),0),\psi(0,S(Gu,Gu,z),0,S(Gu,Gu,z),0),\\
&\varphi(0,S(Gu,Gu,z),0,S(Gu,Gu,z),0))\\
&\leq k\psi(0,S(Gu,Gu,z),0,S(Gu,Gu,z),0)\\
&\leq k\eta(S(Gu,Gu,z))\leq kS(Gu,Gu,z).
\end{aligned}$$

That is

$$\begin{aligned}
F(&\psi(0,S(Gu,Gu,z),0,S(Gu,Gu,z),0),\psi(0,S(Gu,Gu,z),0,S(Gu,Gu,z),0),\\
&\varphi(0,S(Gu,Gu,z),0,S(Gu,Gu,z),0))\\
&=k\psi(0,S(Gu,Gu,z),0,S(Gu,Gu,z),0).
\end{aligned}$$

By using the second property of F, we get either $\psi(0,S(Gu,Gu,z),0,S(Gu,Gu,z),0)=0$ or $\varphi(0,S(Gu,Gu,z),0,S(Gu,Gu,z),0)$

which implies that $S(Gu,Gu,z)=0$. Hence $z=Gu$.

Therefore $fu=z=Gu$. By the weak compatibility of the pair (f,G), we have $fGu=Gfu$. That is $fz=Gz$. We take $x=y=z$ in (3.9). Then we have

$$kS(Gz,Gz,Tz)\leq F(\psi(t_1,t_2,t_3,t_4,t_5),\psi(t_1,t_2,t_3,t_4,t_5),\varphi(t_1,t_2,t_3,t_4,t_5)) \tag{3.18}$$

where $t_1=S(fz,fz,fz)=0$; $t_2=S(fz,fz,Gz)=0$; $t_3=S(fz,fz,Tz)=S(Gz,Gz,Tz)$; $t_4=0$; $t_5=S(fz,fz,Tz)=S(Gz,Gz,Tz)$.

From (3.18), we have

$$\begin{aligned} kS(Gz,Gz,Tz) &\le F(\psi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz)),\psi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz)),\\ &\qquad \varphi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz)))\\ &\le k\psi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz))\\ &\le k\eta(S(Gz,Gz,Tz)) \le kS(Gz,Gz,Tz). \end{aligned}$$

i.e., $F(\psi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz)),\psi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz)),$

$$\varphi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz)))$$
$$= k\psi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz))$$

which implies that either $\psi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz)) = 0$

or $\varphi(0,0,S(Gz,Gz,Tz),0,S(Gz,Gz,Tz)) = 0$.

Hence $S(Gz,Gz,Tz) = 0$ so that $Gz = Tz$. Therefore $fz = Gz = Tz$.

We now prove that $Tz = z$. By taking $x = x_{2n},\ y = x_{2n}$ and $z = z$ in (3.9), we have

$$kS(Gx_{2n},Gx_{2n},Gz) \le F(\psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5),\psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5),\varphi(t_1^n,t_2^n,t_3^n,t_4^n,t_5)) \tag{3.19}$$

where $t_1^n = S(fx_{2n},fx_{2n},fz)$; $t_2^n = S(fx_{2n},fx_{2n},Gx_{2n}) = S(fx_{2n},fx_{2n},fx_{2n+1})$; $t_3^n = S(fx_{2n},fx_{2n},Tz)$; $t_4^n = S(fz,fz,Gx_{2n}) = S(fz,fz,fx_{2n+1})$; $t_5 = 0$.

Here we observe that

$\lim_{n\to\infty} t_1^n = S(z,z,Tz)$; $\lim_{n\to\infty} t_2^n = 0$; $\lim_{n\to\infty} t_3^n = S(z,z,Tz)$; $\lim_{n\to\infty} t_4^n = S(Tz,Tz,z)$.

On letting $n \to \infty$ in (3.19), we have

$$\begin{aligned} kS(z,z,Tz) &\le \lim_{n\to\infty} F(\psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5),\psi(t_1^n,t_2^n,t_3^n,t_4^n,t_5),\varphi(t_1^n,t_2^n,t_3^n,t_4^n,t_5))\\ &= F(\psi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0),\psi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0),\\ &\qquad \varphi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0))\\ &\le k\psi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0)\\ &\le k\eta(S(z,z,Tz)) \le kS(z,z,Tz). \end{aligned}$$

i.e., $F(\psi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0),\psi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0),$

$$\varphi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0))$$
$$= k\psi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0)$$

which implies that either $\psi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0) = 0$

or $\varphi(S(z,z,Tz),0,S(z,z,Tz),S(z,z,Tz),0) = 0$.

Hence $S(z,z,Tz) = 0$ so that $Tz = z$.

Therefore z is a common fixed point of f, G and T. Uniqueness of z follows as in Theorem 3.7. □

4. Corollaries and Examples

We get the following corollaries for Theorem 3.7 and Theorem 3.8 if F is a C_k-class function in two variables.

Corollary 4.1. *Let (X,S) be an S-metric space. Let f, G and T be self mappings on X satisfying the following condition: there exists $k \geq 1$ such that*
$kS(Gx,Gy,Tz) \leq F(\psi(t_1,t_2,t_3,t_4,t_5), \varphi(t_1,t_2,t_3,t_4,t_5))$ *for all $x,y \in X$,*
where $\psi, \varphi \in \Psi$ and F is a C_k-class function in two variables with t_1,t_2,t_3,t_4,t_5 are as in (3.9). Suppose that

(a) *either the pair (f,G) or (f,T) satisfies property (E. A.);*
(b) *$f(X)$ is a closed subspace of X;*
(c) *either the pair (f,G) or (f,T) is weakly compatible.*

Then f, G and T have a unique common fixed point in X.

Corollary 4.2. *Let (X,S) be an S-metric space. Let f, G and T be self mappings on X satisfying the following condition: there exists $k \geq 1$ such that*
$kS(Gx,Gy,Tz) \leq F(\psi(t_1,t_2,t_3,t_4,t_5), \varphi(t_1,t_2,t_3,t_4,t_5))$
for all $x,y \in X$, where $\psi, \varphi \in \Psi$ and $F \in C_k$-class function in two variables with t_1,t_2,t_3,t_4,t_5 are as in (3.9). Suppose that

(a) *the pair (G,T) is asymptotically regular with respect to f at some $x_0 \in X$;*
(b) *X is orbitally complete;*
(c) *the pair (f,G) or (f,T) is weakly compatible.*

Then f, G and T have a unique common fixed point in X.

Choosing $k = 1$ in Corollary 4.1 and Corollary 4.2, we get the following corollaries.

Corollary 4.3. *Let (X,S) be an S-metric space. Let f, G and T be self mappings on X satisfying the following condition:*

$S(Gx,Gy,Tz) \leq F(\psi(t_1,t_2,t_3,t_4,t_5),\varphi(t_1,t_2,t_3,t_4,t_5))$

for all $x,y \in X$, where $\psi,\varphi \in \Psi$ and $F \in \mathscr{C}$ with t_1,t_2,t_3,t_4,t_5 are as in (3.9). *Suppose that*

(a) *either the pair (f,G) or (f,T) satisfies property (E. A.);*

(b) *$f(X)$ is a closed subspace of X;*

(c) *either the pair (f,G) or (f,T) is weakly compatible.*

Then f,G and T have a unique common fixed point in X.

Corollary 4.4. *Let (X,S) be an S-metric space. Let f,G and T be self mappings on X satisfying the following condition:*

$S(Gx,Gy,Tz) \leq F(\psi(t_1,t_2,t_3,t_4,t_5),\varphi(t_1,t_2,t_3,t_4,t_5))$

for all $x,y \in X$, where $\psi,\varphi \in \Psi$ and $F \in \mathscr{C}$ with t_1,t_2,t_3,t_4,t_5 are as in (3.9). Suppose that

(a) *the pair (G,T) is asymptotically regular with respect to f at some $x_0 \in X$;*

(b) *X is orbitally complete;*

(c) *the pair (f,G) or (f,T) is weakly compatible.*

Then f,G and T have a unique common fixed point in X.

In Theorem 3.7, if we choose $f = I,\ T = G,\ \psi(t_1,t_2,t_3,t_4,t_5) = \max\{t_1,t_2,t_3,t_4,t_5\}$ and $F(s,r,t) = ms$, for $0 < m < 1$ then we have the following.

Corollary 4.5. *Let (X,S) be an S-metric space. Suppose that a map $G : X \to X$ satisfies the following condition: there exists $0 \leq L < 1$ such that*

$$S(Gx,Gy,Gz) \leq L\max\{t_1,t_2,t_3,t_4,t_5\} \tag{4.1}$$

for all $x,y \in X$ with $t_1 = S(x,y,z)$, $t_2 = \frac{S(x,x,Gx)+S(y,y,Gy)}{2}$, $t_3 = \frac{S(x,x,Gz)+S(y,y,Gz)}{2}$, $t_4 = \frac{S(z,z,Gx)+S(z,z,Gy)}{2}$ and $t_5 = S(z,z,Gz)$.

Then G has a unique fixed point in X.

If we choose $y = x$ and $z = y$ in corollary 4.5, we have the following.

Corollary 4.6. *Let (X,S) be an S-metric space. Let G be self mappings satisfying the following condition: there exists $0 \leq L < 1$ such that*

$$S(Gx,Gx,Gy) \leq L\max\{t_1,t_2,t_3,t_4,t_5\} \tag{4.2}$$

for all $x,y \in X$ *with* $t_1 = S(x,x,y),\ t_2 = S(x,x,Gx),\ t_3 = S(x,x,Gy),\ t_4 = S(y,y,Gx)$ *and* $t_5 = S(y,y,Gy)$.

Then G *has a unique fixed point in* X.

Remark 4.7. Theorem 2.19 follows as a corollary to Corollary 4.6.

Example 4.8. Let $X = [0,7)$ and We define $S : X^3 \to [0,\infty)$ by

$$S(x,y,z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x,y,z\} & \text{otherwise.} \end{cases}$$

Then S is an S-metric on X. Now we define $f, G, T : X \to X$ by

$$fx = \begin{cases} \frac{x}{2} & \text{if } x \in [0,1] \\ 6 & \text{if } x \in (1,7) \end{cases}, \quad Gx = \begin{cases} \frac{x}{16} & \text{if } x \in [0,1] \\ \frac{1}{2} & \text{if } x \in (1,7) \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{x}{5} & \text{if } x \in [0,1] \\ 1 & \text{if } x \in (1,7). \end{cases}$$

We now define functions $\psi, \varphi : [0,\infty)^5 \to [0,\infty)$ by

$\psi(t_1,t_2,t_3,t_4,t_5) = \frac{1}{2}\max\{t_1,t_2,t_3,t_4,t_5\}$ and $\varphi(t_1,t_2,t_3,t_4,t_5) = \frac{1}{3}\max\{t_1,t_2,t_3,t_4,t_5\}$.

We have $f(X)$ is closed. If $\{x_n\} = \frac{1}{n},\ n \geq 0$ then $\lim\limits_{n\to\infty} fx_n = 0$ and $\lim\limits_{n\to\infty} Gx_n = 0$.

Therefore the pair (f,G) satisfies property (E. A.). Also, the pair (f,G) is weakly compatible.

We define $F : [0,\infty)^3 \to \mathbb{R}^+$ such that $F(s,s,t) = ms$ for $0 < m < 1$. Then $F \in \mathscr{C}_k$. We now show that the the inequality (3.9) is true with $m = \frac{3}{4}$ and $k = 2$. That is to verify

$$\begin{aligned} kS(Gx,Gy,Tz) &\leq F(\psi(t_1,t_2,t_3,t_4,t_5), \psi(t_1,t_2,t_3,t_4,t_5), \varphi(t_1,t_2,t_3,t_4,t_5)) \\ &= m\psi(t_1,t_2,t_3,t_4,t_5) \\ &= \frac{m}{2}\max\{t_1,t_2,t_3,t_4,t_5\}. \end{aligned}$$

Therefore it is enough to verify the following inequality:

$S(Gx,Gy,Tz) \leq \frac{3}{16}\max\{t_1,t_2,t_3,t_4,t_5\}$.

<u>*Case*</u> (i): Let $x,y,z \in [0,1]$.

We assume, without loss of generality, that $x > y > z$.

$S(Gx,Gy,Tz) = S(\frac{x}{16}, \frac{y}{16}, \frac{z}{16}) = \frac{x}{16};\ t_1 = S(fx,fy,fz) = S(x,y,z) = x;$

$S(Gx,Gy,Tz) = \frac{x}{16} \leq \frac{3}{16} \times x = \frac{3}{16} \times t_1 \leq \frac{3}{16} \times \max\{t_1,t_2,t_3,t_4,t_5\}$.

<u>*Case*</u> (ii): Let $x,y,z \in (1,7)$.

In this case $S(Gx,Gy,Tz) = S(\frac{1}{2}, \frac{1}{2}, 1) = 1;\ t_5 = S(fz,fz,Tz) = S(6,6,1) = 6;$

$S(Gx,Gy,Tz) = 1 \leq \frac{3}{16} \times 6 = \frac{3}{16} \times t_5 \leq \frac{3}{16} \times \max\{t_1,t_2,t_3,t_4,t_5\}$.

In all the remaining cases, we have $t_1 = S(fx, fy, fz) = 6$ so that

$S(Gx, Gy, Tz) \leq \frac{3}{16} \times t_1 \leq \frac{3}{16} \times \max\{t_1, t_2, t_3, t_4, t_5\}$.

Therefore f, G and T satisfy all the hypotheses of Theorem 3.7 and f, G and T have a unique common fixed point 0.

Example 4.9. Let $X = [0, 7)$ and we define $S : X^3 \to [0, \infty)$ by

$$S(x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x, y, z\} & \text{otherwise.} \end{cases}$$

Then S is an S-metric on X. Now we define $f, G, T : X \to X$ by

$$fx = \begin{cases} \frac{x}{2} & \text{if } x \in [0, 1] \\ 5 & \text{if } x \in (1, 7) \end{cases}, \quad Gx = \begin{cases} \frac{x^3}{5} & \text{if } x \in [0, 1] \\ 2 & \text{if } x \in (1, 7) \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{x^2}{5} & \text{if } x \in [0, 1] \\ \frac{1}{2} & \text{if } x \in (1, 7). \end{cases}$$

We now define functions $\psi, \varphi : [0, \infty)^5 \to [0, \infty)$ by

$\psi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{2} \max\{t_1, t_2, t_3, t_4, t_5\}$ and $\varphi(t_1, t_2, t_3, t_4, t_5) = \frac{1}{3} \max\{t_1, t_2, t_3, t_4, t_5\}$. Clearly, $f(X)$ is closed. If $\{x_n\} = \frac{1}{n+1}$, $n \geq 0$ then the pair (G, T) is asymptotically regular with respect to f at 0. Also, the pair (f, G) is weakly compatible.

We define $F : [0, \infty)^3 \to [0, \infty)$ by $F(s, s, t) = ms$ for $0 < m < 1$. Then $F \in \mathscr{C}_k$. We now verify the inequality (3.9) with $m = \frac{9}{10}$ and $k = 2$. That is to verify

$$\begin{aligned} kS(Gx, Gy, Tz) &\leq F(\psi(t_1, t_2, t_3, t_4, t_5), \psi(t_1, t_2, t_3, t_4, t_5), \varphi(t_1, t_2, t_3, t_4, t_5)) \\ &= m\psi(t_1, t_2, t_3, t_4, t_5) \\ &= \frac{m}{2} \max\{t_1, t_2, t_3, t_4, t_5\}. \end{aligned}$$

Therefore it is enough to verify the following inequality:

$S(Gx, Gy, Tz) \leq \frac{9}{20} \max\{t_1, t_2, t_3, t_4, t_5\}$.

<u>*Case*</u> (i): Let $x,\ y,\ z \in [0, 1]$.

We assume, without loss of generality, that $x > y > z$.

$S(Gx, Gy, Tz) = S(\frac{x^3}{5}, \frac{y^3}{5}, \frac{z^2}{5}) = \max\{\frac{x^3}{5}, \frac{z^2}{5}\}$; $t_1 = S(fx, fy, fz) = \frac{x}{2}$;

If $S(Gx, Gy, Tz) = \frac{x^3}{5} \leq \frac{9}{20} \times \frac{x}{2} = \frac{9}{20} \times t_1 \leq \frac{9}{20} \times \max\{t_1, t_2, t_3, t_4, t_5\}$.

If $S(Gx, Gy, Tz) = \frac{z^2}{5} \leq \frac{9}{20} \times \frac{x}{2} = \frac{9}{20} \times t_1 \leq \frac{9}{20} \times \max\{t_1, t_2, t_3, t_4, t_5\}$.

<u>*Case*</u> (ii): Let $x, y \in [0, 1]$ and $z \in (1, 7)$.

In this case $S(Gx, Gy, Tz) = S(\frac{x^3}{5}, \frac{y^3}{5}, \frac{1}{2}) = \frac{1}{2}$; $t_1 = S(fx, fy, fz) = S(\frac{x}{2}, \frac{y}{2}, 5) = 5$;

$S(Gx,Gy,Tz) = \frac{1}{2} \leq \frac{9}{20} \times 5 = \frac{9}{20} \times t_1 \leq \frac{9}{20} \times \max\{t_1,t_2,t_3,t_4,t_5\}$.

<u>*Case*</u> (iii): Let $x,y,z \in (1,7)$.

Here $S(Gx,Gy,Tz) = S(2,2,\frac{1}{2}) = 2$; $t_5 = S(fz,fz,Tz) = S(5,5,2) = 5$;

$S(Gx,Gy,Tz) = 2 \leq \frac{9}{20} \times 5 = \frac{9}{20} \times t_5 \leq \frac{9}{20} \times \max\{t_1,t_2,t_3,t_4,t_5\}$.

Similarly, in all the remaining cases we get $S(Gx,Gy,Tz) \leq \frac{9}{20} \times \max\{t_1,t_2,t_3,t_4,t_5\}$.

Therefore f, G and T satisfy all the hypotheses of Theorem 3.8 and therefore f, G and T have a unique common fixed point 0.

Example 4.10. Let $X = [0,2]$ and We define $S : X^3 \to [0,\infty)$ by $S(x,y,z) = \max\{|x-z|,|y-z|\}$ for all $x,y,z \in X$. Then S is an S-metric on X. Now we define $G : X \to X$ by

$$Gx = \begin{cases} \frac{3}{2} & \text{if } x \in [0,2) \\ \frac{1}{4} & \text{if } x = 2. \end{cases}$$

We now define functions $\psi, \varphi : [0,\infty)^5 \to [0,\infty)$ by

$\psi(t_1,t_2,t_3,t_4,t_5) = \frac{1}{2}\max\{t_1,t_2,t_3,t_4,t_5\}$ and $\varphi(t_1,t_2,t_3,t_4,t_5) = \frac{1}{3}\max\{t_1,t_2,t_3,t_4,t_5\}$.

We now verify that G satisfies the inequality 4.2 in Corollary 4.6.

<u>*Case*</u> (i): Either $x,y \in [0,2)$ or if $x = y = 2$.

In this case $S(Gx,Gx,Gy) = 0$ so that inequality (4.2) holds trivially.

<u>*Case*</u> (ii): Let $x \in [0,2)$ and $y = 2$.

Here $S(Gx,Gx,Gy) = \frac{5}{4}$ and $t_5 = S(y,y,Gy) = \frac{7}{4}$.

We have $S(Gx,Gx,Gy) = \frac{5}{4} = \frac{5}{7} \times t_5 \leq L\max\{t_1,t_2,t_3,t_4,t_5\}$ with $L = \frac{5}{7}$.

Therefore G satisfies all the hypothesis of Corollary 4.6 and G has a unique fixed point $\frac{3}{4}$.

But we observe that the inequality (2.1) fails to hold. For, we choose $x = 1$ and $y = 2$. Then we have $S(Gx,Gx,Gy) = \frac{5}{4} \nleq L = L\,S(x,x,y)$ for any $0 \leq L < 1$.

Therefore Theorem 2.19 can not be applied. Hence from Remark 4.7 and Example 4.10 we conclude that Corollary 4.6 is a generalization of Theorem 2.19.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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