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FIXED POINTS OF GERAGHTY φ-RATIONAL TYPE CONTRACTIONS IN ORBITALLY COMPLETE PARTIALLY ORDERED METRIC SPACES

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Abstract. In this paper, we introduce Geraghty φ-rational type contraction mappings and prove the existence of fixed points in orbitally complete partially ordered metric spaces. We provide examples in support of the validity of our results. Our results extend the results of Amini-Harindi and Emami[1] and Harjani, Lopez and Sadarangani[4] to a more general Geraghty φ-rational type contraction mappings.

Keywords: Geraghty contraction; Jaggi contraction; orbitally complete; orbitally continuous; Geraghty φ-rational type contraction.

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1. Introduction and preliminaries

One among the generalizations of contraction condition is Geraghty contraction, in which the contraction constant is replaced by a function having some specified properties. We use the following notation introduced by Geraghty, namely
$S = \{\beta : [0, \infty) \to [0,1) / \beta(t_n) \to 1 \Rightarrow t_n \to 0\}$.

Definition 1.1.[3] A selfmap $f : X \to X$ is said to be a *Geraghty contraction* if there exists

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$\beta \in S$ such that

$$d(f(x), f(y)) \leq \beta(d(x,y))d(x,y) \tag{1.1}$$

for all $x, y \in X$.

Theorem 1.2.[3] Let X be a complete metric space. Assume that $f : X \to X$ is a Geraghty contraction. Then f has a unique fixed point in X.

In 1977, Jaggi [5] introduced a new concept namely 'rational type contraction mappings' and proved the existence of fixed points of such mappings, which is a generalization of Banach contraction principle.

Theorem 1.3.[5] Let f be a continuous selfmap defined on a complete metric space (X,d). Suppose that f satisfies the following condition: there exist $\alpha,\ \gamma \in [0,1)$ with $\alpha + \gamma < 1$ such that

$$d(fx, fy) \leq \alpha \frac{d(x,fx)d(y,fy)}{d(x,y)} + \gamma d(x,y) \tag{1.2}$$

for all $x, y \in X$, $x \neq y$. Then f has a fixed point in X.

Here we note that a mapping $f : X \to X, X$ a metric space that satisfies (1.2) is called a Jaggi contraction map on X.

In recent years, fixed point theory is developing rapidly in partially ordered metric space setting. Ran and Reurings [6] extended and generalized the Banach contraction theorem to ordered metric spaces.

Let $(X, \preceq)$ be a partially ordered set. If there is a metric d on X such that (X,d) is a metric space then we call X is a partially ordered metric space, and we denote it by $(X, \preceq, d)$. If d is complete on X then we call $(X, \preceq, d)$ is a partially ordered complete metric space.

Definition 1.4. Let $(X, \preceq)$ be a partially ordered set. A map $f : X \to X$ is said to be non-decreasing if for any $x, y \in X$ with $x \preceq y$ then $fx \preceq fy$.

In this direction, in 2010, Amimi-Harindi and Emami[1] extended Theorem 1.2 to partially ordered metric spaces.

Theorem 1.5. [1] Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Let $f : X \to X$ be an increasing mapping such that there exists $x_0 \in X$ with $x_0 \preceq f(x_0)$. Suppose that there exists $\beta \in S$ such that

$$d(f(x),f(y)) \leq \beta(d(x,y))d(x,y) \tag{1.3}$$

for all $x,\ y \in X$ with $x \succeq y$. Assume that either

(i) f is continuous; (or)

(ii) X is such that if an increasing sequence $\{x_n\} \to x$ in X then $x_n \preceq x$ for all n.

Further, if for each $x,y \in X$, there exists $z \in X$ such that z is comparable to x and y, then f has a unique fixed point in X.

Harjani, Lopez and Sadarangani [4] extended Theorem 1.3 to the context of partially ordered complete metric spaces.

Theorem 1.6.[4] Let $(X,\preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X,d) is a complete metric space. Let $f: X \to X$ be a non-decreasing mapping such that

$$d(fx,fy) \leq \alpha \frac{d(x,fx)d(y,fy)}{d(x,y)} + \gamma d(x,y) \tag{1.4}$$

for all $x,y \in X$ with $x \succeq y$, $x \neq y$ where $0 \leq \alpha,\gamma < 1$ with $\alpha+\gamma < 1$.
Also, assume either

(i) f is continuous; (or)

(ii) if a non-decreasing sequence $\{x_n\}$ in X is such that $x_n \to x$ as $n \to \infty$ then $x = sup\{x_n\}$.

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

A map f that satisfies the inequality (1.4) is called *Jaggi contraction map* in partially ordered metric spaces.

In 2013, Samet, Vetro and Vetro [7] introduced a new type of contraction condition and proved fixed point theorems in complete metric spaces that generalize Banach contraction principle.

Theorem 1.7.[7] Let $(X,\preceq)$ be a complete metric space, $\varphi: X \to [0,\infty)$ be a lower semi continuous function and $f: X \to X$ be a given mapping. Suppose that there exists a constant $\gamma \in (0,1)$ such that for all $x,y \in X$,

$$d(fx,fy)+\varphi(fx)+\varphi(fy) \leq \gamma[d(x,y)+\varphi(x)+\varphi(y)].$$

Then T has a unique fixed point $x^* \in X$. Moreover, we have $\varphi(x^*) = 0$.

Let X be a nonempty set and f be a selfmap of X. Let $x \in X$, we define the *orbit* of x w. r. t. f by $O_f(x) = \{f^n x / n = 0, 1, 2, ...\}$. Here $f^0 = I$, I is the identity map of X. From here onwards we denote $O_f(x)$ by $O(x)$.

Definition 1.8.[8] Let (X,d) be a metric space. Let $f : X \to X$ be a selfmap of X. A metric space X is said to be *f-orbitally complete* if every Cauchy sequence which is contained in $O(x)$ for all x in X converges to a point of X.

Here we note that every complete metric space is f-orbitally complete for any f; but every f-orbitally complete metric space need not be a complete metric space [8].

Definition 1.9.[8] A selfmap f of X is said to be *orbitally continuous* at a point z in X with respect to x in X, if for any sequence $\{x_n\} \subset O(x)$ with $x_n \to z$ as $n \to \infty$ implies $fx_n \to fz$ as $n \to \infty$.

Clearly, any continuous mapping of a metric space is orbitally continuous, but its converse is not true. For more details and discussion on these concepts we refer Turkoglu, Ozer and Fisher[8].

We use the following lemma in our main result.

Lemma 1.9. [2] Suppose that (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ *as* $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$, $d(x_{m(k)-1}, x_{n(k)}) < \varepsilon$ and

(i) $\lim\limits_{k\to\infty} d(x_{n(k)-1}, x_{m(k)+1}) = \varepsilon$, (ii) $\lim\limits_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \varepsilon$,
(iii) $\lim\limits_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \varepsilon$, (iv) $\lim\limits_{k\to\infty} d(x_{n(k)}, x_{m(k)+1}) = \varepsilon$,
(v) $\lim\limits_{k\to\infty} d(x_{m(k)}, x_{n(k)+1}) = \varepsilon$, and (vi) $\lim\limits_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \varepsilon$.

In the following, we define Geraghty $\varphi-$rational type contraction maps in partially ordered metric spaces.

Definition 1.11. Let $(X, \preceq)$ be a partially ordered metric space and suppose that $f : X \to X$ be a mapping. If there exist two functions $\varphi : X \to [0,\infty)$ lower semi continuous, $\beta \in S$ and a point $x_0 \in X$ such that

$$d(fx, fy) + \varphi(fx) + \varphi(fy) \leq \beta(M(x,y))M(x,y), \tag{1.5}$$

where

$M(x,y) = \max\{d(x,y)+\varphi(x)+\varphi(y), \frac{(d(x,fx)+\varphi(x)+\varphi(fx))(d(y,fy)+\varphi(y)+\varphi(fy))}{d(x,y)+\varphi(x)+\varphi(y)}\}$

for all $x,y \in \overline{O(x_0)}$ with $x \preceq y$ and $x \neq y$, then we say that f is a *Geraghty φ-rational type contraction with respect to* $x_0 \in X$.

Remark 1.12. If $\varphi = 0$ in the inequality (1.5), then we say that f is a Geraghty rational type contraction.

Note: In the context of partially ordered metric spaces, if f satisfies $(1.6.1)$ with $\alpha+\gamma < 1$ then f is a Geraghty φ-rational type contraction with $\varphi \equiv 0$ and $\beta \equiv \alpha+\gamma$ so that every Jaggi contraction is a Geraghty φ-rational type contraction. But, the following example suggests that its converse is not true.

Example 1.13. Let $X = [0,1)$ with the usual metric. We define partial order $\preceq$ on X as follows: $\preceq := \{(x,y) \in X \times X : x = y\} \cup \{(x,y) \in X \times X / x \preceq y$ implies $x \leq y$, where $\leq$ is the usual order$\}$.

We define $f : X \to X$, by $fx = \begin{cases} 0 & if\ x = 0 \\ \frac{x+1}{2} & if\ x \in (0, \frac{2}{5}) \\ \frac{4}{5} & if\ x \in [\frac{2}{5}, 1). \end{cases}$

We define $\varphi : X \to [0,\infty)$ by $\varphi(x) = \begin{cases} \frac{x}{2} & if\ x \in [0, \frac{4}{5}) \\ x - \frac{4}{5} & if\ x \in [\frac{4}{5}, 1) \end{cases}$

and $\beta : [0,\infty) \to [0,1)$ by $\beta(t) = \begin{cases} 0 & if\ t = 0 \\ \frac{4+t}{4+2t} & if\ t > 0. \end{cases}$

Let $x_0 = \frac{1}{4},\ fx_0 = \frac{5}{8}$ then $x_0 \preceq fx_0$. Here $O(x_0) = \{\frac{1}{4}, \frac{5}{8}, \frac{4}{5}, \frac{4}{5}, ...\}$ and $\overline{O(x_0)} = \{\frac{1}{4}, \frac{5}{8}, \frac{4}{5}\} = O(x_0)$. Let $x,y \in O(x_0)$.

The following three cases arise to verify the inequality (1.5).

<u>Case (i):</u> $x = \frac{1}{4}$ and $y = \frac{5}{8}$.

In this case, $d(fx,fy)+\varphi(fx)+\varphi(fy) = \frac{39}{80}$ and $M(x,y) = \frac{13}{16}$.

$d(fx,fy)+\varphi(fx)+\varphi(fy) = \frac{39}{80} \leq \beta(\frac{13}{16})(\frac{13}{16}) = \beta(M(x,y))M(x,y)$.

<u>Case (ii):</u> $x = \frac{5}{8}$ and $y = \frac{4}{5}$.

In this case, the inequality (1.5) holds trivially.

<u>Case (iii):</u> $x = \frac{1}{4}$ and $y = \frac{4}{5}$.

In this case, $d(fx,fy)+\varphi(fx)+\varphi(fy) = \frac{39}{80}$ and $M(x,y) = \frac{27}{40}$.

$d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{39}{80}\leq\beta(\frac{27}{40})(\frac{27}{40})=\beta(M(x,y))M(x,y).$

Hence f is a Geraghty φ-rational type contraction.

Also we observe that the inequality (1.4) fails to hold.

For, by choosing $x=0$ and $y=\frac{4}{5}$ we have

$d(f0,f(\frac{4}{5}))=\frac{4}{5}\nleq\alpha(0)+\gamma(\frac{4}{5})=\alpha\frac{d(0,f0)d(\frac{4}{5},f\frac{4}{5})}{d(0,\frac{4}{5})}+\gamma d(0,\frac{4}{5})$ for any $\alpha,\gamma\in[0,1)$.

i.e., f is not a *Jaggi contraction map*.

Further, we observe that the inequality (1.3) fails to hold by choosing $x=0$ and $y=\frac{4}{5}$.

For, $d(f0,f(\frac{4}{5}))=\frac{4}{5}\nleq\beta(\frac{4}{5})\frac{4}{5}=\beta(d(0,\frac{4}{5}))d(0,\frac{4}{5})$ for any $\beta\in S$.

Hence f is not a *Geraghty contraction*.

Thus we conclude that the class of all Geraghty φ-rational type contractions is more general than the class of Jaggi contraction maps.

In Section 2, we prove the existence of fixed points of Geraghty φ-rational type contraction maps in orbitally complete partially ordered metric spaces. In Section 3, we draw some corollaries from our main results and provide examples in support of our results.

2. MAIN RESULTS

Theorem 2.1. Let $(X,d,\preceq)$ be a partially ordered metric space. Suppose that $f:X\to X$ is a non-decreasing map and $x_0\in X$ such that $x_0\preceq fx_0$. Suppose that f is a Geraghty φ-rational type contraction with respect to $x_0\in X$ and X is f-orbitally complete. Then, the sequence $\{x_n\}$ defined by $x_{n+1}=fx_n,\ n=0,1,2,...$, is Cauchy in X. Let $\lim_{n\to\infty}x_n=z,\ z\in X$. Assume that f is orbitally continuous at z. Then z is a fixed point of f and $\varphi(z)=0$.

Proof. Let $x_0\in X$ be such that $x_0\preceq fx_0$. We define $\{x_n\}$ in X such that

$$x_{n+1}=fx_n \text{ for } n=0,1,2,\ldots. \tag{2.1}$$

Since f is nondecreasing, we have $x_0\preceq x_1\preceq x_2\preceq\ldots\preceq x_n\preceq x_{n+1}\preceq\ldots$.

If $x_n=x_{n+1}$ for some n, then the conclusion of the theorem trivially holds.

Hence w. l. g. we assume that $x_n\neq x_{n+1}$ for all n.

$$\text{We denote } r_n=d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n) \text{ for all } n>0. \tag{2.2}$$

We consider
$$\begin{aligned} r_{n+1}&=d(x_n,x_{n+1})+\varphi(x_n)+\varphi(x_{n+1})\\ &=d(fx_{n-1},fx_n)+\varphi(fx_{n-1})+\varphi(fx_n)\\ &\leq\beta(M(x_{n-1},x_n))(M(x_{n-1},x_n)), \end{aligned}\tag{2.3}$$

where

$$M(x_{n-1},x_n) = \max\{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n), \frac{(d(x_{n-1},fx_{n-1})+\varphi(x_{n-1})+\varphi(fx_{n-1}))(d(x_n,fx_n)+\varphi(x_n)+\varphi(fx_n))}{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)}\}$$
$$= \max\{r_n, \frac{r_n.r_{n+1}}{r_n}\} = \max\{r_n, r_{n+1}\}.$$

If $\max\{r_n,r_{n+1}\} = r_{n+1}$ then from (2.3) we have

$r_{n+1} \leq \beta(r_{n+1})r_{n+1} < r_{n+1}$,

a contradiction.

Hence $\max\{r_n,r_{n+1}\} = r_n$. Hence from (2.3) we have

$$r_{n+1} \leq \beta(r_n)r_n \tag{2.4}$$

which implies that $r_{n+1} < r_n$.

Thus it follows that $\{r_n\}$ is strictly decreasing sequence of non-negative real numbers and hence $\lim_{n\to\infty} r_n$ exists and it is r(say). *i.e.*, $\lim_{n\to\infty} r_n = r \geq 0$.

We now show that $r = 0$.

From (2.1.4), we have

$r_{n+1} \leq \beta(r_n)r_n$.

On letting $n \to \infty$, we have

$1 = \lim_{n\to\infty} \frac{r_{n+1}}{r_n} \leq \lim_{n\to\infty} \beta(r_n) \leq 1$

so that $\beta(r_n) \to 1$ as $n \to \infty$. Since $\beta \in S$, it is follows that

$\lim_{n\to\infty} r_n = 0$. *i.e.*, $r = 0$.

Hence $\lim_{n\to\infty}(d(x_{n+1},x_n)+\varphi(x_{n+1})+\varphi(x_n)) = 0$. *i.e.*, $\lim_{n\to\infty} d(x_{n+1},x_n) = 0$ and $\lim_{n\to\infty}\varphi(x_n) = 0$.

We now show that $\{x_n\}$ is a Cauchy sequence in X.

Suppose that $\{x_n\}$ is not a Cauchy sequence. Then, there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that

$$d(x_{m(k)},x_{n(k)}) \geq \varepsilon. \tag{2.5}$$

We choose $m(k)$, the least positive integer satisfying (2.5). Then, we have

$m(k) > n(k) > k$ with $\quad d(x_{m(k)},x_{n(k)}) \geq \varepsilon, \quad d(x_{m(k)-1},x_{n(k)}) < \varepsilon.$

Now by Lemma 1.10, it follows that $\lim_{k\to\infty} d(x_{m(k)+1},x_{n(k)+1}) = \varepsilon$.

Now from (1.5), we have

$$\begin{aligned} d(x_{m(k)+1},x_{n(k)+1}) &+ \varphi(x_{m(k)+1}) + \varphi(x_{n(k)+1}) \\ &= d(fx_{m(k)},fx_{n(k)}) + \varphi(fx_{m(k)}) + \varphi(fx_{n(k)}) \\ &\leq \beta(M(x_{m(k)},x_{n(k)}))(M(x_{m(k)},x_{n(k)})), \end{aligned} \tag{2.6}$$

where

$M(x_{m(k)},x_{n(k)}) = \max\{d(x_{m(k)},x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}), \frac{(d(x_{m(k)},x_{m(k)+1})+\varphi(x_{m(k)})+\varphi(x_{m(k)+1}))(d(x_{n(k)},x_{n(k)+1})+\varphi(x_{n(k)})+\varphi(x_{n(k)+1}))}{d(x_{m(k)},x_{n(k)})+\varphi(x_{m(k)})+\varphi(x_{n(k)})}\}.$

On letting $k \to \infty$, from the Lemma 1.10, we get

$\lim_{k\to\infty} M(x_{m(k)},x_{n(k)}) = \{\varepsilon, 0\} = \varepsilon.$

On letting $k \to \infty$ from (2.6), we have

$$\begin{aligned}\varepsilon &= \lim_{k\to\infty} d(x_{m(k)+1},x_{n(k)+1}) + \varphi(x_{m(k)+1}) + \varphi(x_{n(k)+1})\\ &\leq \lim_{k\to\infty} \beta(M(x_{m(k)},x_{n(k)})) \lim_{k\to\infty} M(x_{m(k)},x_{n(k)}) = \lim_{k\to\infty} \beta(M(x_{m(k)},x_{n(k)}))\varepsilon.\end{aligned}$$

Hence

$1 \leq \lim_{k\to\infty} \beta(M(x_{m(k)},x_{n(k)})) \leq 1$ so that $\beta(M(x_{m(k)},x_{n(k)})) \to 1$ as $\mathrm{k} \to \infty$.

Since $\beta \in S$, we have $M(x_{m(k)},x_{n(k)}) \to 0$ as $k \to \infty$. *i.e.*, $\varepsilon = 0$,

a contradiction .

Therefore $\{x_n\} \subset O(x_0)$ is a Cauchy sequence in (X,d). Since X is f-orbitally complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{2.7}$$

Since φ is lower semi continuous, we have

$\varphi(z) \leq \liminf_{n\to\infty} \varphi(x_n) = 0.$

Hence $\varphi(z) = 0$.

Since f is orbitally continuous at z w.r.t. x_0, from (2.1), we have

$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f x_n = fz.$

This complete the proof of the theorem. □

In the following theorem, we relax the orbital continuity of f in Theorem 2.1, and in place of this hypothesis, we assume the sequential convergence property.

Theorem 2.2. Let $(X,d,\preceq)$ be a partially ordered metric space. Suppose that $f : X \to X$ is a non-decreasing map and $x_0 \in X$ such that $x_0 \preceq f x_0$. Suppose that $f$is a $\varphi-$rational type contraction with respect to $x_0 \in X$.

Assume the following:

(i) if $x,y,z \in X$ with $x \prec y \prec z$ then $d(x,y) < d(x,z)$ and $d(y,z) < d(x,z)$;

(ii) if $\{x_n\}$ is a non-decreasing sequence converging to $z \in X$, then $x_n \preceq z$, for all n; and

(iii) if $\{x_n\}$ and $\{y_n\}$ are sequences in X with $x_n \preceq y_n$ for all n and $\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ x,\ y \in X$ then $x \preceq y$.

If X is f-orbitally complete then the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n,\ n = 0,1,2,...,$ is Cauchy in X. Let $\lim_{n\to\infty} x_n = z,\ z \in X$. Then z is a fixed point of f and $\varphi(z) = 0$. Further, f is orbitally continuous at z.

Proof. By the proof of Theorem 2.1, we have $\{x_n\} \subset O(x_0)$ is a Cauchy sequence in (X,d). Since X is f-orbitally complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{2.8}$$

Since φ is lower semi continuous, we have

$$\varphi(z) \le \liminf_{n\to\infty} \varphi(x_n) = 0.$$

Hence $\varphi(z) = 0$.

Since $\{x_n\}$ is a non-decreasing sequence and $x_n \to z$, by (ii) we have $x_n \preceq z$ for all n. Since f is non-decreasing, we have $fx_n \preceq fz$ for all n. *i.e.*, $x_{n+1} \preceq fz$ for all n. Moreover, as $x_n \preceq x_{n+1} \preceq fz$ for all n and by using (iii), we get $z \preceq fz$.

We now define a sequence $\{y_n\}$ as $y_0 = z,\ y_{n+1} = fy_n, n = 0,1,2,\dots$. Then $y_0 \preceq fy_0$. Since f is non-decreasing, we obtain that $\{y_n\}$ is a non-decreasing sequence and $\{y_n\}$ is Cauchy (similar to the argument to show $\{x_n\}$ is Cauchy) $y_n \to y$ (say), $y \in X$. Again, by condition (ii), we have $y_n \preceq y$. Since $x_n \preceq z = y_0 \preceq fz = fy_0 \preceq y_n \preceq y$ for all n, we have $x_n \preceq y_n$ for all n, and hence $z \preceq y$ (by (iii)).

If $x_n = y_n$ for some n, then $x_n \preceq z = y_0 \preceq fz = fy_0 \preceq y_n = x_n$ so that $fz = z$.

Hence we assume that $x_n \neq y_n$ for all n.

Now, from (1.5) we have

$$\begin{aligned} d(x_{n+1}, y_{n+1}) + \varphi(x_{n+1}) + \varphi(y_{n+1}) &= d(fx_n, fy_n) + \varphi(fx_n) + \varphi(fy_n) \\ &\le \beta(M(x_n, y_n))M(x_n, y_n), \end{aligned} \tag{2.9}$$

where

$$M(x_n, y_n) = \max\{d(x_n, y_n) + \varphi(x_n) + \varphi(y_n), \tfrac{(d(x_n, fx_n) + \varphi(x_n) + \varphi(fx_n))(d(y_n, fy_n) + \varphi(y_n) + \varphi(fy_n))}{d(x_n, y_n) + \varphi(x_n) + \varphi(y_n)}\}.$$

On letting $n \to \infty$, we have

$$\lim_{n\to\infty} M(x_n, y_n) = \max\{d(z,y), 0\} = d(z,y).$$

On letting $n \to \infty$ in (2.9), we have

$1 = \frac{d(z,y)}{d(z,y)} \le \lim_{n\to\infty} \beta(M(x_n, y_n)) \le 1$ which implies that $\lim_{n\to\infty} \beta(M(x_n, y_n)) = 1$ and hence

$\lim_{n\to\infty} M(x_n,y_n)=0. \quad i.e., d(z,y)=0.$

Hence $z=y$ and we have $z\preceq fz=y_0\preceq y_n\preceq y=z$.

Therefore z is a fixed point of f. □

Now we prove the uniqueness of fixed point of f by using 'condition (H)' and it is the following: Condition (H): For all $x,y\in X$ there exists $z\in X$ such that $x\preceq z$ and $y\preceq z$.

Theorem 2.3 In addition to the hypotheses of Theorem 2.1 $(Theorem\ 2.2)$ if condition (H) holds, then f has a unique fixed point.

Proof. By Theorem 2.1, we have f has a fixed point. Suppose that $x,y\in X$ are two fixed point of f. By condition (H), there exists $z\in X$ such that

$x\preceq z$ and $y\preceq z$.

Put $z=z_0,\ z_1=fz_0$. and define a sequence $\{z_n\}$ in X by $z_{n+1}=fz_n$ for all $n\geq 0$. Then $x\preceq z_0$ and $y\preceq z_0$. By using the non-decreasing property of f, we have

$fx\preceq fz_0$ and $fy\preceq fz_0$. Hence $x\preceq z_1$ and $y\preceq z_1$.

On continuing this process, we have

$$x\preceq z_n \text{ and } y\preceq z_n \text{ for } n\geq 0. \qquad (2.10)$$

In (2.10), if $x=z_n$ for some n, then $fx=fz_n$ so that $x=z_{n+1}$. In fact, we have $x=z_m$ for $m\geq n$ so that $\lim_{n\to\infty} z_n=x$.

If $x\neq z_n$ for all $n=0,1,2,\ldots$ then by using (1.1), we have

$$\begin{aligned} d(x,z_{n+1})+\varphi(x)+\varphi(z_{n+1}) &= d(fx,fz_n)+\varphi(fx)+\varphi(fz_n)\\ &\leq \beta(M(x,z_n))M(x,z_n), \end{aligned} \qquad (2.11)$$

where

$$\begin{aligned} M(x,z_n) &= \max\{d(x,z_n)+\varphi(x)+\varphi(z_n), \tfrac{(d(x,fx)+\varphi(x)+\varphi(fx))(d(z_n,fz_n)+\varphi(z_n)+\varphi(fz_n))}{d(x,z_n)+\varphi(x)+\varphi(z_n)}\}) \\ &= \max\{d(x,z_n)+\varphi(z_n),0\}=d(x,z_n)+\varphi(z_n), \text{ and hence} \end{aligned}$$

$d(x,z_{n+1})+\varphi(z_{n+1})\leq \beta(d(x,z_n)+\varphi(z_n))(d(x,z_n)+\varphi(z_n))$.

On letting $n\to\infty$ in (2.11), we get

$d(x,z)\leq \lim_{n\to\infty}\beta(d(x,z_n)+\varphi(z_n))d(x,z)$, which implies that either

$d(x,z)=0$ or $\beta(d(x,z_n))=1$.

$$\text{Therefore } \lim_{n\to\infty} z_n=x. \qquad (2.12)$$

By applying the similar argument to the sequence $\{z_n\}$ and y, it follows that

$$\lim_{n\to\infty} z_n=y. \qquad (2.13)$$

From (2.12) and (2.13) we have $x = y$ and the conclusion of the theorem follows.

□

3. Corollaries and Examples

We deduce the following corollaries from the main results of Section 2.

Corollary 3.1. Let $(X, \preceq)$ be a partially ordered set and d a metric on X. Suppose that $f : X \to X$ is a non-decreasing map and $x_0 \in X$ such that $x_0 \preceq fx_0$. Suppose that there exists a constant $\beta \in S$ such that

$$d(fx, fy) \leq \beta(M(x,y))M(x,y), \tag{3.1}$$

where

$M(x,y) = \max\{d(x,y), \frac{d(x,fx)d(y,fy)}{d(x,y)}\}$ for all $x, y \in \overline{O(x_0)}$ with $x \preceq y$ and $x \neq y$.

Assume that X is f-orbitally complete. Then, the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n,\ n = 0, 1, 2, ...,$ is Cauchy in X. Let $\lim_{n\to\infty} x_n = z,\ z \in X$. Suppose that f is orbitally continuous at z. Then z is a fixed point of f.

Proof. The inequality (3.1) implies the inequality (1.5) with $\varphi \equiv 0$ on X, and hence the conclusion of the corollary follows from Theorem 2.1. □

Corollary 3.2. Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose that $f : X \to X$ is a non-decreasing map and $x_0 \in X$ such that $x_0 \preceq fx_0$ and $\beta \in S$ such that

$$d(fx, fy) \leq \beta(M(x,y))M(x,y), \tag{3.2}$$

where

$M(x,y) = \max\{d(x,y), \frac{d(x,fx)d(y,fy)}{d(x,y)}\}$ for all $x, y \in \cup_{x_0 \preceq fx_0,\ x_0 \in X} \overline{O(x_0)}$ with $x \preceq y$ and $x \neq y$. Assume the following:

(i) if $x, y, z \in X$ with $x \prec y \prec z$ then $d(x,y) < d(x,z)$ and $d(y,z) < d(x,z)$;
(ii) if $\{x_n\}$ is a non-decreasing sequence converging to $z \in X$, then $x_n \preceq z$, for all n; and
(iii) if $\{x_n\}$ and $\{y_n\}$ are sequences in X with $x_n \preceq y_n$, for all n and $\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ x,\ y \in X$ then $x \preceq y$.

Assume that X is f-orbitally complete. Then, the sequence $\{x_n\}$ defined by

$x_{n+1} = fx_n,\ n = 0,1,2,...$, is Cauchy in X. Let $\lim\limits_{n\to\infty} x_n = z,\ z \in X$.

Then z is a fixed point of f.

Proof. The inequality (3.2) implies the inequality (1.5) with $\varphi \equiv 0$ on X, and hence the conclusion of the corollary follows from Theorem 2.2. □

In the following, we provide examples in support of the results proved in Section 2.

Example 3.3. Let $X = [0,2)$ with the usual metric. We define partial order $\preceq$ on X by $\preceq := \{(x,y) \in X \times X : x = y\} \cup \{(x,y)/x,y \in X,\ x \preceq y \Leftrightarrow x \geq y$, where $\geq$ is the usual order$\}$.

We define $f : X \to X$ by $fx = \begin{cases} 0 & if\ x = 0 \\ \frac{1}{16} & if\ x \in [\frac{1}{2},1) \\ \frac{x}{4} & if\ x \in [\frac{1}{4},\frac{1}{2}) \\ \frac{1}{2^{n+2}} & if\ x \in [\frac{1}{2^{n+1}},\frac{1}{2^n}), n \geq 2 \\ 2-x & if\ x \in [1,2). \end{cases}$

We define $\varphi : X \to [0,\infty)$ by $\varphi(x) = \begin{cases} \frac{x}{3} & if\ x \in [0,\frac{5}{16}) \\ x - \frac{5}{16} & if\ x \in [\frac{5}{16},1). \end{cases}$

and $\beta : [0,\infty) \to [0,1)$ by $\beta(t) = \begin{cases} 0 & if\ t = 0 \\ \frac{4+t}{4+2t} & if\ t > 0. \end{cases}$

Let $x_0 = \frac{3}{8}$ then $x_0 \preceq fx_0$. Here $O(x_0) = \{\frac{3}{8}, \frac{3}{32}, \frac{1}{2^5}, \frac{1}{2^6}..., \frac{1}{2^{n+5}}, ...\} = \{\frac{3}{8}, \frac{3}{32}\} \cup \{\frac{1}{2^n}/n \geq 5\}$ and $\overline{O(x_0)} = O(x_0) \cup \{0\}$.

The following four cases arise to verify the inequality (1.5).

<u>Case (i)</u>: $x = \frac{3}{8}$ and $y = \frac{3}{32}$.

In this case, $d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{10}{96}$ and $M(x,y) = \frac{14}{32}$.

$d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{10}{96} \leq \beta(\frac{14}{32})(\frac{14}{32}) = \beta(M(x,y))M(x,y)$.

<u>Case (ii)</u>: $x = \frac{3}{32}$ and $y = \frac{1}{2^i},\ i \geq 5$.

In this case, $d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{2^i - 2^3}{3.2^{i+3}}$ and $M(x,y) = \frac{3.2^i - 2^4}{3.2^{i+3}}$.

$d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{2^i - 2^3}{3.2^{i+3}} \leq \beta(\frac{3.2^i - 2^4}{3.2^{i+3}})(\frac{3.2^i - 2^4}{3.2^{i+3}}) = \beta(M(x,y))M(x,y)$.

<u>Case (iii)</u>: $x = \frac{3}{8}$ and $y = \frac{1}{2^i},\ i \geq 5$. In this case,

$d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{3.2^i - 2^3}{3.2^{i+3}}$ and $M(x,y) = \frac{12.2^i - 2^4}{3.2^{i+3}}$.

$d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{3.2^i - 2^3}{3.2^{i+3}} \leq \beta(\frac{12.2^i - 2^4}{3.2^{i+3}})(\frac{12.2^i - 2^4}{3.2^{i+3}}) = \beta(M(x,y))M(x,y)$.

Case (iv): $x=\frac{1}{2^i}$ and $y=\frac{1}{2^j}$, $i\geq 5$ and $j\geq i$.

In this case, $d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2.2^j-2^i}{3.2^{i+j}}$ and $M(x,y)=\frac{4.2^j-2.2^i}{3.2^{i+j}}$.

$d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2.2^j-2^i}{3.2^{i+j}}\leq\beta(\frac{4.2^j-2.2^i}{3.2^{i+j}})(\frac{4.2^j-2.2^i}{3.2^{i+j}})=\beta(M(x,y))M(x,y)$.

Hence, all the hypotheses of Theorem 2.1 hold and 0, 1 are two fixed point of f in $\overline{O(x_0)}$. Also $\varphi(0)=0$.

Here, we observe that at $x=0$ and $y=1$

$d(f0,f1)=1\nleq\alpha(0)+\gamma(1)=\alpha\frac{d(0,f0)d(1,f1)}{d(0,1)}+\gamma d(0,1)$.

Hence, the inequality (1.4) does not hold for any α and γ. *i.e.*, f is not a *Jaggi contraction map*. Therefore Theorem 1.6 is not applicable.

Thus, it suggests that Theorem 2.1 is a generalization of Theorem 1.6.

Also, we observe that at $x=0$ and $y=1$ we have

$d(f0,f1)=1\nleq\beta(1).1=\beta d(0,1)d(1,0)$.

Hence, the inequality (1.3) does not hold for any β. *i.e.*, f is not a *Geraghty contraction*. Therefore Theorem 1.5 is not applicable.

Remark 3.4. For $x=0$ and $y=1$, and for any $z\in X$ either $0\npreceq z$ or $1\npreceq z$. Hence condition (H) of Theorem 2.3 fails to hold and f has more than one fixed point namely 0 and 1.

Example 3.5. Let $X=[0,2)$ with the usual metric. We define partial order $\preceq$ on X by $\preceq:=\{(x,y)\in X\times X : x=y\}\cup\{(x,y)/x,y\in X,\ x\preceq y\Leftrightarrow x\geq y$, where $\geq$ is the usual order$\}$. We define $f:X\to X$ by $fx=\begin{cases}\frac{x}{5} & if\ x\in[0,\frac{1}{2}]\\ x-\frac{2}{5} & if\ x\in(\frac{1}{2},2).\end{cases}$

We define $\varphi:X\to[0,\infty)$ by $\varphi(x)=\frac{x}{3}$ for all $x\in X$ and

$\beta:[0,\infty)\to[0,1)$ by $\beta(t)=\begin{cases}0 & if\ t=0\\ \frac{4+t}{4+2t} & if\ t>0.\end{cases}$

Let $x_0=1$ then $x_0\preceq fx_0$, $O(x_0)=\{1,\frac{3}{5},\frac{1}{5},\frac{1}{5^2}\dots,\frac{1}{5^n},\dots.\}=\{1,\frac{3}{5}\}\cup\{\frac{1}{5^n}/n\geq 1\}$ and $\overline{O(x_0)}=O(x_0)\cup\{0\}$.

The following three cases arise to verify the inequality (1.5).

Case (i): $x=1$ and $y=\frac{3}{5}$.

In this case, $d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2}{3}$ and $M(x,y)=\frac{14}{15}$.

$d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2}{3}\leq\beta(\frac{14}{15})(\frac{14}{15})=\beta(M(x,y))M(x,y)$.

Case (ii): $x=\frac{3}{5}$ and $y=\frac{1}{5}$.

In this case, $d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{6}{25}$ and $M(x,y)=\frac{2}{3}$.

$d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{6}{25}\leq\beta(\frac{2}{3})(\frac{2}{3})=\beta(M(x,y))M(x,y)$.

Case (iii): $x=\frac{1}{5^i}$ and $y=\frac{1}{5^j}$, $i\geq 1$ and $j\geq i$.

In this case, $d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{4.5^j-2.5^i}{15.5^{i+j}}$ and $M(x,y)=\frac{4.5^j-2.5^i}{3.5^{i+j}}$.

$d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{4.5^j-2.5^i}{15.5^{i+j}}\leq\beta(\frac{4.5^j-2.5^i}{3.5^{i+j}})(\frac{4.5^j-2.5^i}{3.5^{i+j}})=\beta(M(x,y))M(x,y)$.

Hence, all the hypotheses of Theorem 2.2 hold and condition (H) holds trivially. Thus, f and X satisfy the hypotheses of Theorem 2.3 and 0 is the unique fixed point of f in X. Also $\varphi(0)=0$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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