Available online at http://jfpt.scik.org
J. Fixed Point Theory, 2020, 2020:3
ISSN: 2052-5338

SOLUTION OF RANDOM SEMILINEAR SYSTEM OF DIFFERENTIAL EQUATIONS WITH IMPULSES

ANIMESH GUPTA[1,*] AND GIRIRAJ KISHORE SAHU[2]

[1]H.No. 93/654, Ward No. 2, Gandhi Chowk Pachmarhi - 461881,

Dist. Hoshangabad, Madhya Pradesh, India

[2]Department of Mathematics and Computer Science,

Govt. Model Science College, Jabalpur, Madhya Pradesh, India

Abstract. In this paper, we study the existence of solutions for systems of random semilinear impulsive differential equations by using tripled fixed point theorem. The existence results are established generalized version of Perovs, a nonlinear alternative of Leray-Schauders fixed point principles combined with a technique based on vector-valued metrics and convergent to zero matrices. Also, we give a random abstract formulation to Sadovskiis fixed point theorem in a vector-valued Banach space. Examples illustrating the results are included.

Keywords: random variable; mild solution; vector-valued norm; condensing; measurable selection.

2010 AMS Subject Classification: 47H10, 47H30, 54H25.

1. INTRODUCTION

The theory of impulsive differential equations has also attracted much attention in recent years. In most cases the available data for the description and evaluation of parameters of a

*Corresponding author

E-mail address: dranimeshgupta10@gmail.com

Received December 16, 2019

dynamic system are inaccurate, imprecise, or confusing. In other words, evaluation of parameters of a dynamical system is not without uncertainties. Differential equations with random coefficients are used as models in many different applications. This is due to a combination of uncertainties, complexities, and ignorance on our part which inevitably cloud our mathematical modeling process (e.g., Kampé de Feriet [9], Becus [3] and their references). This interest is due to the fact that there are many applications of this theory to various applied fields such as control theory, statistics, biological sciences, and others.

In this paper we consider the following system of impulsive differential equations with the random effects (random parameters):

$$
\begin{cases}
x'(t,\omega) = A_1(\omega)x(t,\omega) + f_1(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega), & t \in J = [0,b], \\
y'(t,\omega) = A_2(\omega)y(t,\omega) + f_2(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega), & t \in J = [0,b], \\
z'(t,\omega) = A_3(\omega)z(t,\omega) + f_3(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega), & t \in J = [0,b], \\
x(t_k^+,\omega) - x(t_k^-,\omega) = I_k(x(t_k^-,\omega),y(t_k^-,\omega),z(t_k^-,\omega)), & k = 1,2,\dots,m, \\
y(t_k^+,\omega) - y(t_k^-,\omega) = \overline{I}_k(x(t_k^-,\omega),y(t_k^-,\omega),z(t_k^-,\omega)), & k = 1,2,\dots,m, \\
z(t_k^+,\omega) - z(t_k^-,\omega) = \overline{\overline{I}}_k(x(t_k^-,\omega),y(t_k^-,\omega),z(t_k^-,\omega)), & k = 1,2,\dots,m, \\
x(\omega,0) = \varphi_1(\omega), & \omega \in \Omega, \\
y(\omega,0) = \varphi_2(\omega), & \omega \in \Omega, \\
z(\omega,0) = \varphi_3(\omega), & \omega \in \Omega,
\end{cases}
\tag{1.1}
$$

where $f_i : J \times X \times X \times X \times \Omega \to X$ are given functions, $I_k, \overline{I_k}, \overline{\overline{I_k}} \in C(X \times X \times X, X), k = 1,2, \dots, m, 0 = t_0 < t_1 < \cdots < t_n < t_{m+1} = b, \varphi_1, \varphi_2, \varphi_3$ are three random maps, X is a separable Banach space and $A_i : \Omega \times X \to X, i = 1,2,3$ are random operators.

2. Preliminary

In this section, we recall from the literature some notations, definitions, and auxiliary results which will be used throughout this paper.

2.1. Vector metric space.

If $x,y \in \mathbb{R}^n$, with $x = (x_1,\dots,x_n)$ and $y = (y_1,\dots,y_n)$, then by $x \le y$ we mean $x_i \le y_i$ for all $i = 1,\dots,n$. Also we set $|x| = (|x_1|,\dots,|x_n|)$, $\max(x,y) = (\max(x_1,y_1),\dots,\max(x_n,y_n))$ and $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_i > 0\}$. If $c \in \mathbb{R}$, then $x \le c means x_i \le c$ for each $i = 1,\dots,n$.

Definition 1. *Let X be a nonempty set. By a generalized metric space on X, we mean a map $d : X \times X \to \mathbb{R}^n$ with the following properties:*

(i) $d(u,v) \ge 0$ for all $u,v \in X$; if $d(u,v) = 0$, then u=v;

(ii) $d(u,v) = d(v,u)$ for all $u,v \in X$;

(iii) $d(u,v) \le d(u,w) + d(w,v)$ for all $u,v,w \in X$.

We call the pair (X,d) a generalized metric space with

$$d(x,y) := \begin{pmatrix} d_1(x,y) \\ \vdots \\ d_n(x,y) \end{pmatrix}.$$

Notice that d is a generalized metric space(or a vector-valued metric space) on X if and only if $d_i, i = 1,\dots,n$, are metrics on X.

For $r = (r_1,\dots,r_n) \in \mathbb{R}^n_+$, we will denote by

$$B(x_0,r) = \{x \in X : d(x_0,x) < r\} = \{x \in X : d_i(x_0,x) < r_i, i = 1,\dots,n\}$$

the open ball centered in x_0 with radius r and

$$\overline{B(x_0,r)} = \{x \in X : d(x_0,x) \le r\} = \{x \in X : d_i(x_0,x) \le r_i, i = 1,\dots,n\}$$

the closed ball centered in x_0 with radius r. We mention that for a generalized metric space, the notions of open subset, closed set, convergence, Cauchy sequence, and completeness are similar to those in the usual metric spaces.

Definition 2. *A square matrix of real numbers is said to be convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, this means that all the eigenvalues of M are in the open unit disc, i.e., $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(M - \lambda I) = 0$, where I denotes the unit matrix of $\mathscr{M}_{n\times n}(\mathbb{R})$.*

Lemma 3. *Let $M \in \mathscr{M}_{n\times n}(\mathbb{R}_+)$. Then the following assertions are equivalent:*

(i) *M is convergent towards zero;*

(ii) $M^k \to 0$ *as* $k \to \infty$;

(iii) *The matrix $(I-M)$ is nonsingular and* $(I-M)^{-1} = I + M + M^2 + \cdots + M^k + \cdots$;

(iv) *The matrix $(I-M)$ is nonsingular and $(I-M)^{-1}$ has nonnegative elements.*

Remark 4. *Some examples of matrix convergent to zero are as follows:*

1. *Any matrix* $M = \begin{pmatrix} a & a & a \\ b & b & b \\ c & c & c \end{pmatrix}$, *where* $a,b,c \in \mathbb{R}_+$ *and* $a+b+c < 1$.
2. *Any matrix* $M = \begin{pmatrix} a & b & c \\ a & b & c \\ a & b & c \end{pmatrix}$, *where* $a,b,c \in \mathbb{R}_+$ *and* $a+b+c < 1$.
3. *Any matrix* $M = \begin{pmatrix} a & b & c \\ 0 & b & c \\ 0 & 0 & c \end{pmatrix}$, *where* $a,b,c \in \mathbb{R}_+$ *and* $\max a,b,c < 1$.

2.2. Random variable and some selection theorems.

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis and random variable which are used throughout this paper. Let (X,d) be a metric space or a generalized metric space and Y be a subset of X. We denote:

- $\mathscr{P}(X) = \{Y \subset X : Y \neq \emptyset\}$ and
- $\mathscr{P}_p(X) = \{Y \in \mathscr{P}(X) : Y$ has the property 'p'$\}$, where p could be: cl = closed, b = bounded, cp = compact, etc. Thus
- $\mathscr{P}_{cl}(X) = \{Y \in \mathscr{P}(X) : Y$ closed$\}$,
- $\mathscr{P}_b(X) = \{Y \in \mathscr{P}(X) : Y$ bounded$\}$,
- $\mathscr{P}_{cv}(X) = \{Y \in \mathscr{P}(X) : Y$ convex$\}$, where X is a Banach space,
- $\mathscr{P}_{cp}(X) = \{Y \in \mathscr{P}(X) : Y$ compact$\}$.

Let (Ω, Σ) be a measurable space and $F : \Omega \to \mathscr{P}(X)$ be a multivalued mapping, F is called measurable if $F^+(Q) = \{\omega \in \Omega : F(\omega) \subset Q\}$ for every $Q \in \mathscr{P}_{cl}(X)$; equivalently, for every U open set of X, the set $F^-(Q) = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\}$ is measurable. If X is a metric space, we shall use $\mathscr{B}(X)$ to denote the Borel $\sigma-$algebra on X. The $\Sigma \otimes \mathscr{B}(X)$ denotes the smallest $\sigma-$algebra on $\Omega \times X$ which contains all the sets $A \times S$, where $Q \in \Sigma$ and $S \in \mathscr{B}(X)$. Let $F : X \times X \times X \to \mathscr{P}(Y)$ be a multivalued map. A single-valued map $f : X \times X \times X \to Y$ is said to be a selection of G, and we write $(f \subset F)$ whenever $f(x,y,z) \in F(x,y,z)$ for every $(x,y,z) \in X \times X \times X$.

Definition 5. *Recall that a mapping $F : \Omega \times X \times X \times X \to X$ is said to be a random operator if, for any $(x,y,z) \in X \times X \times X, f(\cdot,x,y,z)$ is measurable.*

Definition 6. *A random tripled fixed point of f is a measurable function $x,y,z : \Omega \to X$ such that*

$$x(\omega) = f\big(\omega, x(\omega), y(\omega), z(\omega)\big)$$
$$y(\omega) = f\big(\omega, y(\omega), x(\omega), y(\omega)\big)$$
$$z(\omega) = f\big(\omega, z(\omega), y(\omega), x(\omega)\big)$$

for all $\omega \in \Omega$.

Equivalently, a measurable selection for the multivalued map $\mathrm{Fix}\, F_\omega : \Omega \to \mathscr{P}(X)$ *is defined by*

$$\mathrm{Fix}\, F_\omega(x,y,z) = \big\{(x,y,z) \in X \times X \times X : x = f(\omega,x,y,z),\;\; y = f(\omega,y,x,y),\;\; z = f(\omega,z,y,x)\big\}.$$

Theorem 7. *Let (Ω,Σ), Y be a separable metric space and $F : \Omega \times \Omega \times \Omega \to \mathscr{P}_{cl}(Y)$ be measurable multivalued. Then F has a measurable selection.*

Lemma 8. *Let (X,d) be a generalized metric space. Then there exists a homeomorphism map $h : X \to X \times X \times X$.*

Proof. Consider $h : X \to X \times X \times X$ defined by

$$h(x) = (x,x,x)$$

for all $x \in X$. Obviously, h is bijective. To prove that h is a continuous map, let $x,y \in X$. Thus

$$d_*(h(x),h(y)) \le \Sigma_{i=1}^{3} d_i(x,y).$$

For $\varepsilon > 0$ we take $\delta = (\varepsilon_1, \varepsilon_2, \varepsilon_3)$, let $x_0 \in X$ be fixed and $B(x_0,\delta) = \{x \in X : d(x_0,x) < \delta\}$. Then for every $x \in B(x_0,\delta)$ we have

$$d_*(h(x_0),h(x)) \le \varepsilon.$$

Now we show that $h^{-1} : X \times X \times X \to X$ defined by

$$h^{-1}(x,x,x) = x(x,x,x) \in X \times X \times X$$

is a continuous map. Let $(x,x,x),(y,y,y) \in X \times X \times X$. Then

$$d(h^{-1}(x,x,x),h^{-1}(y,y,y)) = d(x,y).$$

Let $\varepsilon = (\varepsilon_1,\varepsilon_2,\varepsilon_3) > 0$; we take $\delta = \frac{\min_{1\leq i\leq 3}\varepsilon_i}{n}$ and we $fix(x_0,x_0,x_0) \in X \times X \times X$. Set

$$B((x_0,x_0,x_0),\delta) = \{(x,x,x) \in X \times X \times X : d_*((x_0,x_0,x_0),(x,x,x)) < \delta\}.$$

For $(x,x,x) \in B((x_0,x_0,x_0),\delta)$ we have

$$d_*((x_0,x_0,x_0),(x,x,x)) < \delta \Longrightarrow \Sigma_{i=1}^{3} d_i(x_0,x) < \frac{\min_{1\leq i\leq 3}\varepsilon_i}{n}.$$

Then

$$d_i(x_0,x) < \frac{\min_{1\leq i\leq 3}\varepsilon_i}{n}, i = 1,\ldots,n \Longrightarrow d(x_0,x) < \varepsilon.$$

Hence h^{-1} is continuous. □

As a consequence of above theorems 7, we can conclude the following results.

Theorem 9. *Let (Ω,Σ), Y be a separable generalized metric space and $F : \Omega \times \Omega \times \Omega \to \mathscr{P}_{cl}(Y)$ be measurable multivalued. Then F has a measurable selection.*

Proof. Consider $F_* : \Omega \to P_{cl}(\tilde{Y})$ defined by $F_*(\omega^1,\omega^2,\omega^3) = (h \circ F)(\omega^1,\omega^2,\omega^3)$ for all $(\omega^1,\omega^2,\omega^3) \in \Omega \times \Omega \times \Omega$, where h is defined in Lemma 8. Let $C \subset X \times \tilde{X} \times \tilde{X}$ be an open set. Then

$$\begin{aligned} F_*^{-1}(C) &= \{(\omega^1,\omega^2,\omega^3) \in \Omega \times \Omega \times \Omega : (h \circ F)(\omega) \cap C \neq 0\} \\ &= \{(\omega^1,\omega^2,\omega^3) \in \Omega \times \Omega \times \Omega : F(\omega^1,\omega^2,\omega^3) \cap h^{-1}(C) \neq 0\}. \end{aligned}$$

Since F is a measurable multi-valued function, we have $F_*^{-1}(C) \in \Sigma$. By Theorem 7 there exists a measurable single-valued function $x,y,z : \Omega \to \tilde{X}$ such that

$$x(\omega) \in (h \circ F)(\omega^1,\omega^2,\omega^3) \quad \text{for all} \quad \omega \in \Omega \Longrightarrow (h^{-1} \circ x)(\omega^1,\omega^2,\omega^3) \in F((\omega^1,\omega^2,\omega^3))$$

$$y(\omega) \in (h \circ F)(\omega^2,\omega^1,\omega^2) \quad \text{for all} \quad \omega \in \Omega \Longrightarrow (h^{-1} \circ y)(\omega^2,\omega^1,\omega^2) \in F((\omega^2,\omega^1,\omega^2))$$

$$z(\omega) \in (h \circ F)(\omega^3, \omega^2, \omega^1) \quad \text{for all} \quad \omega \in \Omega \Longrightarrow (h^{-1} \circ z)(\omega^3, \omega^2, \omega^1) \in F((\omega^3, \omega^2, \omega^1))$$

for all $\omega \in \Omega$. Using the fact that h^{-1} is a continuous function, $h^{-1} \circ x : \Omega \to X$, $h^{-1} \circ y : \Omega \to X$, $h^{-1} \circ z : \Omega \to X$ are a measurable selection of F. □

Theorem 10. *Let X be a separable metric space, Y be a metric space, $f : \Omega \times X \times X \times X \to X$ be a Carathéodory function, and U be an open subset of Y. Then the multivalued map F : $\Omega \times \Omega \times \Omega \to \mathscr{P}(X)$ defined by*

$$F(\omega)(x,y,z) = \{\omega \in \Omega : f(\omega,x,y,z) \in U\}$$

is measurable. In particular, if f is real-valued, then

$$F_*(\omega^1, \omega^2, \omega^3) = \{\omega \in \Omega : f(\omega,x,y,z) > \lambda\}, \qquad \widetilde{F}(\omega^1, \omega^2, \omega^3) = \{\omega \in \Omega : f(\omega,x,y,z) < \lambda\}$$

$$F_*(\omega^2, \omega^1, \omega^2) = \{\omega \in \Omega : f(\omega,y,x,y) > \lambda\}, \qquad \widetilde{F}(\omega^2, \omega^1, \omega^2) = \{\omega \in \Omega : f(\omega,y,x,y) < \lambda\}$$

$$F_*(\omega^3, \omega^2, \omega^1) = \{\omega \in \Omega : f(\omega,z,y,x) > \lambda\}, \qquad \widetilde{F}(\omega^3, \omega^2, \omega^1) = \{\omega \in \Omega : f(\omega,z,y,x) < \lambda\}$$

are measurable.

Next, we present some random fixed point theorem in a separable vector Banach space.

Theorem 11. *Let $(\Omega, \mathscr{F}, \mu)$ be a probability space, X be a real separable generalized Banach space and $F : \Omega \times X \times X \times X \to X$ be a continuous random operator, and let $M(\omega) \in \mathscr{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matrix such that $M(\omega)$ converges to 0 a.s. and*

$$d\big(F(\omega,x,y,z), F(\omega,u,v,w)\big) \le M(\omega)[d(x,u) + d(y,v) + d(z,w)]$$

for each $(x,y,z), (u,v,w) \in X \times X \times X, \omega \in \Omega$. Then there exists any random variable x,y,z : $\Omega \to X$ which is the unique tripled random fixed point of F.

Proof. Let E be a set defined as follows

$$E = \{\omega \in \Omega : d\big(F(\omega,x,y,z), F(\omega,u,v,w)\big) \le M(\omega)[d(x,u) + d(y,v) + d(z,w)]\}$$

for each $(x,y,z), (u,v,w) \in X \times X \times X$. Then $\mu(E) = 1$. We have for every fixed $\omega \in E$ that there exists a unique $x(\omega) \in X$ such that

$$F(\omega, x(\omega), y(\omega), z(\omega)) = x(\omega),$$

$$F(\omega, y(\omega), x(\omega), y(\omega)) = y(\omega),$$

$$F(\omega, z(\omega), y(\omega), x(\omega)) = z(\omega).$$

Let $u, v, w : \Omega \to X$ be any arbitrary measurable function, an we define $(x_n(\omega)), (y_n(\omega)), (z_n(\omega))$ $n \in \mathbb{N}$, with $x_0(\omega) = u(\omega)$, $y_0(\omega) = v(\omega)$, $z_0(\omega) = w(\omega)$, by

$$x_n(\omega) = F(\omega, F_{n-1}(\omega, u(\omega), v(\omega), w(\omega)),$$

$$y_n(\omega) = F(\omega, F_{n-1}(\omega, v(\omega), u(\omega), v(\omega)),$$

$$z_n(\omega) = F(\omega, F_{n-1}(\omega, z(\omega), v(\omega), u(\omega)),$$

$n \in \mathbb{N}$. It is clear that x_n, y_n, z_n are a random variables and for each $n, m \in \mathbb{N}$ we have

$$d(x_n(\omega), x_{n+k}(\omega)) \leq (M^k(\omega) + \ldots + M^{n+k}(\omega))d(x_0(\omega), x_1(\omega))$$

$$d(y_n(\omega), y_{n+k}(\omega)) \leq (M^k(\omega) + \ldots + M^{n+k}(\omega))d(y_0(\omega), y_1(\omega))$$

$$d(z_n(\omega), z_{n+k}(\omega)) \leq (M^k(\omega) + \ldots + M^{n+k}(\omega))d(z_0(\omega), z_1(\omega)).$$

By Lemma 8, we get

$$d(x_n(\omega), x_{n+k}(\omega)) \leq M^k(\omega)(I - M(\omega))^{-1} d(x_0(\omega), x_1(\omega))$$

$$d(y_n(\omega), y_{n+k}(\omega)) \leq M^k(\omega)(I - M(\omega))^{-1} d(y_0(\omega), y_1(\omega))$$

$$d(z_n(\omega), z_{n+k}(\omega)) \leq M^k(\omega)(I - M(\omega))^{-1} d(z_0(\omega), z_1(\omega)).$$

Hence $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$, $(z_n)_{n\in\mathbb{N}}$ are a Cauchy sequence. Then there exists a random variable $u_*, v_*, w_* : \Omega \to X$ such that

$$d(x_n(\omega), u_*(\omega)) \to 0$$

$$d(y_n(\omega), v_*(\omega)) \to 0$$

$$d(z_n(\omega), w_*(\omega)) \to 0$$

as $n \to \infty$. We obtain

$$d(u_*(\omega), F(\omega, u_*(\omega), v_*(\omega), w_*(\omega)) \leq d(u_*(\omega), x_n(\omega)) + M(\omega)d(x_n(\omega), u_*(\omega)) \to 0$$

$$d(v_*(\omega), F(\omega, v_*(\omega), u_*(\omega), v_*(\omega)) \leq d(v_*(\omega), y_n(\omega)) + M(\omega)d(y_n(\omega), v_*(\omega)) \to 0$$

$$d(w_*(\omega), F(\omega, w_*(\omega), v_*(\omega), u_*(\omega)) \leq d(w_*(\omega), z_n(\omega)) + M(\omega)d(z_n(\omega), w_*(\omega)) \to 0$$

as $n \to \infty$. Thus

$$u_*(\omega) = F(\omega, u_*(\omega), v_*(\omega), w_*(\omega))$$

$$v_*(\omega) = F(\omega, v_*(\omega), u_*(\omega), v_*(\omega))$$

$$w_*(\omega) = F(\omega, w_*(\omega), v_*(\omega), u_*(\omega))$$

for each $\omega \in E$, so $u_*(\omega) = x(\omega)$, $v_*(\omega) = y(\omega)$, $w_*(\omega) = z(\omega)$, $\omega \in E$. □

By a simple modification we conclude the following result.

Theorem 12. *Let $(\Omega, \mathscr{F})$ be a measurable space, X be a real separable generalized Banach space and $F : \Omega \times X \times X \times X \to X$ be a continuous random operator, and let $M(\omega) \in \mathscr{M}_{n\times n}(\mathbb{R}_+)$ be a random variable matrix such that, for every $\omega \in \Omega$, the matrix $M(\omega)$ converges to 0 and*

$$d\big(F(\omega,x,y,z), F(\omega,u,v,w)\big) \leq M(\omega)[d(x,u) + d(y,v) + d(z,w)]$$

for each $(x,y,z),(u,v,w) \in X \times X \times X, \omega \in \Omega$. Then there exists any random variable $x : \Omega \to X$ which is the unique tripled random fixed point of F.

Theorem 13. *Let X be a separable generalized Banach space, and let $F : \Omega \times X \times X \times X \to X$ be a completely continuous random operator. Then either of the following holds:*

(i) The random equation $F(\omega,x,y,z) = x$, $F(\omega,y,x,y) = y$, $F(\omega,z,y,x) = z$ has a random solution, i.e., there is a measurable function $x,y,z : \Omega \to X$ such that

$$F(\omega, x(\omega), y(\omega), z(\omega)) = x(\omega),$$

$$F(\omega, y(\omega), x(\omega), y(\omega)) = y(\omega),$$

$$F(\omega, z(\omega), y(\omega), x(\omega)) = z(\omega),$$

for all $\omega \in \Omega$, or

(ii) The set $\mathscr{M} = \{x,y,z : \Omega \to X$ is measurable$\}$ satisfying

$$\lambda(\omega)F(\omega,x,y,x) = x,$$

$$\lambda(\omega)F(\omega,y,x,y) = y,$$

$$\lambda(\omega)F(\omega,z,y,y) = z$$

are unbounded for some measurable $\lambda : \Omega \to X$ with $0 < \lambda(\omega) < 1$ on Ω.

Definition 14. *A function $f:[0,b]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\Omega\to\mathbb{R}$ is called random Carathéodory if the following conditions are satisfied:*

(i) The map $(t,\omega)\longmapsto f(t,x,y,z,\omega)$ is jointly measurable for all $x,y,z\in\mathbb{R}$,

(ii) The map

$$x\longmapsto f(t,x,y,z,\omega)$$

$$y\longmapsto f(t,y,x,y,\omega)$$

$$z\longmapsto f(t,z,y,x,\omega)$$

are continuous for all $t\in[0,b]$ and $\omega\in\Omega$.

Lemma 15. *Let X be a separable generalized metric space and $G:\Omega\times X\times X\times X\to X$ be a mapping such that $G(\cdot,x,y,z)$ is measurable for all $(x,y,z)\in X\times X\times X$ and $G(\omega,\cdot,\cdot,\cdot)$ is continuous for all $\omega\in\Omega$. Then the map $(\omega,x)\to G(\omega,x,y,z)$, $(\omega,y)\to G(\omega,y,x,y)$, $(\omega,z)\to G(\omega,z,y,x)$ are jointly measurable.*

Proposition 16. . *Let X be a separable Banach space, and D be a dense linear subspace of X. Let L: $\Omega\times D\times D\times D\to X$ be a closed linear random operator such that, for each $\omega\in\Omega$, $L(\omega)$ is one to one and onto. Then the operator $S:\Omega\times X\times X\times X\to X$ defined by $S(\omega,x,y,z)=L^{-1}(\omega)x$ $S(\omega,y,x,y)=L^{-1}(\omega)y$ $S(\omega,z,y,x)=L^{-1}(\omega)z$ is random.*

2.3. C_0-semigroups.

In all this section, $B(X)$ refers to the Banach space of linear bounded operators from X to X with the norm

$$\|A\|_{B(X)}=\sup\{|A(y)|,|y|=1\}.$$

Definition 17. *A one-parameter family $\{S(t),t\geq 0\}\subset B(X)$ is said to be of class C_0 if it satisfies the conditions:*

(i) $S(0)=I$ *(I is the identity operator on X).*

(ii) $S(t+s)=S(t)\circ S(s)$ *for $t,s\geq 0$ (the semigroup property).*

(iii) *The map $x\longmapsto S(t)x$ is strongly continuous for each $x\in E$, i.e.,*

$$\lim_{t\to 0}S(t)x=x\quad\textit{for all } x\in X.$$

A semigroup of bounded linear operators $T(t)$ is uniformly continuous if

$$\lim_{t\to 0}\left\|S(t)-I\right\|_{B(X)}=0.$$

Theorem 18. *([5]) Let $\{S(t),t\geq 0\}$ be a C_0-semigroup of bounded linear operators, then there exist constants $\alpha\in\mathbb{R}$ and $K>0$ such that*

$$\left\|S(t)\right\|_{B(X)}\leq Ke^{\alpha t}\quad \text{for } t\geq 0.$$

Definition 19. *Let $S(t)$ be a semigroup of class C_0 defined on X. The infinitesimal generator A of $S(t)$ is the linear operator defined by*

$$A(x)=\lim_{h\to 0}\frac{S(h)x-x}{h}\quad \text{for } x\in D(A),$$

where

$$D(A)=\{x\in X|\lim_{h\to 0}\frac{T(h)x-x}{h}\ \text{exists in } X\}.$$

Theorem 20. *([5]) Let $S(t)$ be a C_0-semigroup, and let A be its infinitesimal generator. Then, for $x\in D(A)$, $S(t)x\in D(A)$ and $\frac{d}{dt}S(t)x=AS(t)x=S(t)Ax$.*

More details on evolution systems and their properties could be found in the books of Engel and Nagel [5], Pazy [12], and Vrabie [6].

3. EXISTENCE OF SOLUTION

Let $J_k=(t_k,t_{k+1}],k=0,\ldots,m$, and let y_k be the restriction of a function y to J_k. In order to define mild solutions for problem 1.1, consider the space

$$PC=\big\{y\colon[0,b]\to X, y_k\in C(J_k,X), k=0,\ldots,m, \text{ such that } y(t_k^-) \text{ and } y(t_k^+) \text{ exist and satisfy } y(t_k^-)=y(t_k) \text{ for } k=1,\ldots,m\big\}.$$

Endowed with the norm

$$\|y\|_{PC}=\max\big\{\|y_k\|_\infty, k=0,\ldots,m\big\},$$

PC is a Banach space.

Definition 21. *A function $x,y:\Omega\to PC([0,b],X)$ is called a random mild solution of 1.1 if*

$$(3.1)\quad \begin{cases} x(t,\omega)=S_1(\omega,t)\varphi_1(\omega)+\int_0^t S_1(\omega,t-s)f_1(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega)\,ds\\ \qquad +\sum_{0<t_k<t}S(\omega,t-t_k)I_k(x(t_k,\omega),y(t_k,\omega)),z(t_k,\omega),\quad t\in[0,b],\\ y(t,\omega)=S_2(\omega,t)\varphi_2(\omega)+\int_0^t S_2(\omega,t-s)f_2(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega)\,ds\\ \qquad +\sum_{0<t_k<t}S(\omega,t-t_k)\bar{I}_k(x(t_k,\omega),y(t_k,\omega),z(t_k,\omega)),\quad t\in[0,b]\\ z(t,\omega)=S_3(\omega,t)\varphi_3(\omega)+\int_0^t S_3(\omega,t-s)f_3(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega)\,ds\\ \qquad +\sum_{0<t_k<t}S(\omega,t-t_k)\bar{\bar{I}}_k(x(t_k,\omega),y(t_k,\omega),z(t_k,\omega)),\quad t\in[0,b], \end{cases}$$

where $\{S_1(\omega,t)\}_{t\geq 0}$, $\{S_2(\omega,t)\}_{t\geq 0}$, $\{S_3(\omega,t)\}_{t\geq 0}$ are random C_0-semigroups of bounded linear operators on X with infinitesimal generators A_1, A_2,A_3 respectively.

Theorem 22. *Let $f_1,f_2,f_3:J\times X\times X\times X\times\Omega\to X$ be three Carathéodory functions. Assume that the following conditions hold:*

(H1) There exist $p_1,p_2,p_3,p_4,p_5,p_6,p_7,p_8,p_9:\Omega\to L^1([0,b],\mathbb{R}_+)$ random variables such that

$$\big|f_1(t,x,y,z,\omega)-f_1(t,\widetilde{x},\widetilde{y},\widetilde{z},\omega)\big|$$
$$\leq p_1(\omega,t)|x-\widetilde{x}|+p_2(\omega,t)|y-\widetilde{y}|+p_3(\omega,t)|z-\widetilde{z}|,\quad x,y,z,\widetilde{x},\widetilde{y},\widetilde{z}\in X,t\in J,\omega\in\Omega$$

$$\big|f_2(t,x,y,z,\omega)-f_2(t,\widetilde{x},\widetilde{y},\widetilde{z},\omega)\big|$$
$$\leq p_4(\omega,t)|x-\widetilde{x}|+p_5(\omega,t)|y-\widetilde{y}|+p_6(\omega,t)|z-\widetilde{z}|,\quad x,y,z,\widetilde{x},\widetilde{y},\widetilde{z}\in X,t\in J,\omega\in\Omega$$

and

$$\big|f_3(t,x,y,z,\omega)-f_3(t,\widetilde{x},\widetilde{y},\widetilde{z},\omega)\big|$$
$$\leq p_7(\omega,t)|x-\widetilde{x}|+p_8(\omega,t)|y-\widetilde{y}|+p_9(\omega,t)|z-\widetilde{z}|,\quad x,y,z,\widetilde{x},\widetilde{y},\widetilde{z}\in X,t\in J,\omega\in\Omega$$

(H2) There exist random variables $K_1,K_2,K_3:\Omega\longrightarrow(0,+\infty)$ such that

$$\big\|S_1(\omega,t)\big\|\leq K_1(\omega),$$
$$\big\|S_2(\omega,t)\big\|\leq K_2(\omega),$$
$$\big\|S_3(\omega,t)\big\|\leq K_3(\omega)$$

for each $\omega\in\Omega$. If $M(\omega)$ converges to 0, then problem 1.1 has a unique random solution.

Proof. We are going to study problem 1.1 in the intervals $[0,t_1],(t_1,t_2],\dots,(t_m,b]$, respectively. The proof will be given in three steps and then continued by induction.

Step 1. We consider the following problem:

$$\begin{cases} x'(t,\omega)=A_1(\omega)x(t,\omega)+f_1(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega), & t\in[0,t_1],\\ y'(t,\omega)=A_2(\omega)y(t,\omega)+f_2(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega), & t\in[0,t_1],\\ z'(t,\omega)=A_3(\omega)z(t,\omega)+f_3(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega), & t\in[0,t_1],\\ x(0,\omega)=\varphi_1(\omega), \quad \omega\in\Omega, \\ y(0,\omega)=\varphi_2(\omega), \quad \omega\in\Omega \\ z(0,\omega)=\varphi_3(\omega), \quad \omega\in\Omega. \end{cases} \tag{3.2}$$

Consider the operator

$$N_1: C([0,t_1],X)\times C([0,t_1],X)\times C([0,t_1],X)\times\Omega\to C([0,t_1],X)\times C([0,t_1],X)\times C([0,t_1],X),$$

$$(x,y,z)\mapsto\big(N_1^1(\omega,x,y,z),N_2^1(\omega,x,y,z),N_3^1(\omega,x,y,z)\big),$$

where

$$\begin{aligned} &N_1^1\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big)\\ =&S_1(\omega,t)\varphi_1(\omega)+\int_0^t S_1(\omega,t-s)f_1\big(s,x(s,\omega),y(s,\omega),z(t,\omega),\omega\big)\,ds, \quad t\in[0,t_1] \end{aligned}$$

$$\begin{aligned} &N_2^1\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big)\\ =&S_2(\omega,t)\varphi_2(\omega)+\int_0^t S_2(\omega,t-s)f_2\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\,ds, \quad t\in[0,t_1]. \end{aligned}$$

and

$$\begin{aligned} &N_3^1\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big)\\ =&S_3(\omega,t)\varphi_3(\omega)+\int_0^t S_3(\omega,t-s)f_3\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\,ds, \quad t\in[0,t_1]. \end{aligned}$$

First we show that N_1 is a random operator on $C([0,t_1],X)\times C([0,t_1],X)\times C([0,t_1],X)$. Since f_1,f_2 and f_3 are Carath*é*odory functions, then $\omega\longmapsto f_1(t,x,y,z,\omega)$, $\omega\longmapsto f_2(t,x,y,z,\omega)$ and $\omega\longmapsto f_3(t,x,y,z,\omega)$ are measurable maps in view of Lemma 15. By the Crandall-Liggett formula, we have

$$S_i(\omega,t)=\lim_{n\to\infty}\left(I-\frac{t}{n}A_i(\omega)\right)^{-n}x,\quad i=1,2,3.$$

From Proposition 16, we know that $\omega \to (I-\frac{t}{n}A_i(\omega))^{-n}x$ are measurable operators, thus $\omega \to S_i(\omega,t)$ are measurable. Using the continuity properties of the semigroups $S_1(\omega,\cdot),S_2(\omega,\cdot),S_3(\omega,\cdot)$, we get

$$\omega \to S_i(\omega,t)\phi_i(\omega)\quad \text{and}\quad (s,\omega)\to S_i(\omega,t-s)f_i\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)$$

are measurable. Further, the integral is a limit of a finite sum of measurable functions; therefore, the maps

$$\omega \longmapsto N_1^1\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big),$$

$$\omega \longmapsto N_2^1\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big),$$

$$\omega \longmapsto N_3^1\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big),$$

are measurable. As a result, N_1 is a random operator on $C([0,t_1],X)\times C([0,t_1],X)\times C([0,t_1],X)\times\Omega$ into $C([0,t_1],X)\times C([0,t_1],X)\times C([0,t_1],X)$. We show that N_1 satisfies all the conditions of Theorem 11 on $C([0,t_1],X)\times C([0,t_1],X)\times C([0,t_1],X)$.

Let $(x,y,z),(\widetilde{x},\widetilde{y},\widetilde{z})\in C([0,t_1],X)\times C([0,t_1],X)\times C([0,t_1],X)$, then

$$\begin{aligned}
&\big|N_1^1\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big)-N_1^1\big(\widetilde{x}(t,\omega),\widetilde{y}(t,\omega),\widetilde{z}(t,\omega),\omega\big)\big|\\
&=\bigg|S_1(\omega,t)\varphi_1(\omega)+\int_0^t S_1(\omega,t-s)f_1\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\,ds\\
&\quad -S_1(\omega,t)\varphi_1(\omega)-\int_0^t S_1(\omega,t-s)f_1\big(s,\widetilde{x}(s,\omega),\widetilde{y}(s,\omega),\widetilde{z}(s,\omega),\omega\big)\,ds\bigg|\\
&\le \int_0^t \big\|S_1(\omega,t-s)\big\|\big|f_1\big(s,x(s,x),y(s,\omega),z(s,\omega),\omega\big)-f_1\big(s,\widetilde{x}(s,\omega),\widetilde{y}(s,\omega),\widetilde{z}(s,\omega),\omega\big)\big|\,ds\\
&\le K_1(\omega)\bigg(\int_0^t p_1(\omega,t)\big|x(s,\omega)-\widetilde{x}(s,\omega)\big|\,ds+\int_0^t p_2(\omega,t)\big|y(s,\omega)-\widetilde{y}(s,\omega)\big|\,ds\\
&+\int_0^t p_3(\omega,t)\big|z(s,\omega)-\widetilde{z}(s,\omega)\big|\,ds\bigg).
\end{aligned}$$

Thus

$$\left\|N_1^1(x,y,z,\omega)-N_1^1(\widetilde{x},\widetilde{y},\widetilde{z},\omega)\right\|_* \le \frac{K_1(\omega)}{\tau}\|x-\widetilde{x}\|_* + \frac{K_1(\omega)}{\tau}\|y-\widetilde{y}\|_* + \frac{K_1(\omega)}{\tau}\|z-\widetilde{z}\|_*,$$

where

$$\|x\|_* = \sup_{t\in[0,t_1]} e^{-\tau\int_0^t p(s,\omega)\,ds}\left|x(t)\right|, \qquad p(t,\omega)=\sum_{i=1}^{9} p_i(t,\omega), \quad \tau > K_1(\omega)+K_2(\omega)+K_3(\omega).$$

Similarly, we obtain

$$\left\|N_2^1(x,y,z,\omega)-N_2^1(\widetilde{x},\widetilde{y},\widetilde{z},\omega)\right\|_* \le \frac{K_2(\omega)}{\tau}\|x-\widetilde{x}\|_* + \frac{K_2(\omega)}{\tau}\|y-\widetilde{y}\|_* + \frac{K_2(\omega)}{\tau}\|z-\widetilde{z}\|_*,$$

and

$$\left\|N_3^1(x,y,z,\omega)-N_3^1(\widetilde{x},\widetilde{y},\widetilde{z},\omega)\right\|_* \le \frac{K_3(\omega)}{\tau}\|x-\widetilde{x}\|_* + \frac{K_3(\omega)}{\tau}\|y-\widetilde{y}\|_* + \frac{K_3(\omega)}{\tau}\|z-\widetilde{z}\|_*,$$

Hence

$$d_0\big(N_1(x,y,z,\omega),N_1(\widetilde{x},\widetilde{y},\widetilde{z},\omega)\big) \le M_0(\omega)d_0\big((x,y,z),(\widetilde{x},\widetilde{y},\widetilde{z})\big),$$

where

$$d_0(x,y)=\begin{pmatrix}\|x-y\|_*\\ \|x-y\|_*\\ \|x-y\|_*\end{pmatrix}$$

and

$$M_0(\omega)=\begin{pmatrix}\frac{K_1(\omega)}{\tau} & \frac{K_1(\omega)}{\tau} & \frac{K_1(\omega)}{\tau}\\ \frac{K_2(\omega)}{\tau} & \frac{K_2(\omega)}{\tau} & \frac{K_2(\omega)}{\tau}\\ \frac{K_3(\omega)}{\tau} & \frac{K_3(\omega)}{\tau} & \frac{K_3(\omega)}{\tau}\end{pmatrix}.$$

It is clear that the radius spectral $\rho(M(\omega))=\frac{K_1(\omega)+K_2(\omega)+K_3(\omega)}{\tau}<1$. By Lemma 3, $M_0(\omega)$ converges to zero. From Theorem 12 there exists a unique random solution of problem 3.2. We denote by $(x_1(t,\omega),y_1(t,\omega),z_1(t,\omega))$ the mild solution of 3.2.

Step 2. We consider the problem

$$
\begin{cases}
x'(t,\omega) = A_1^1(\omega)x(t,\omega) + f_1(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega) & t \in (t_1,t_2],\\
y'(t,\omega) = A_2^1(\omega)y(t,\omega) + f_2(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega) & t \in (t_1,t_2],\\
z'(t,\omega) = A_3^1(\omega)z(t,\omega) + f_3(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega) & t \in (t_1,t_2],\\
x(t_1^+,\omega) = x_1(t_1,\omega) + I_1(x_1(t_1,\omega),y_1(t_1,\omega),z_1(t_1,\omega)),\\
y(t_1^+,\omega) = y_1(t_1,\omega) + \bar{I}_1(x_1(t_1,\omega),y_1(t_1,\omega),z_1(t_1,\omega))\\
z(t_1^+,\omega) = z_1(t_1,\omega) + \bar{\bar{I}}_1(x_1(t_1,\omega),y_1(t_1,\omega),z_1(t_1,\omega)).
\end{cases} \tag{3.3}
$$

Let

$$C_*\big([t_1,t_2],X\big) = \big\{y \in C\big((t_1,t_2],X\big) : y\big(t_1^+\big) \text{ exists}\big\}.$$

Consider the operator

$$N_2 : C_*([t_1,t_2],X) \times C_*([t_1,t_2],X) \times C_*([t_1,t_2],X) \times \Omega \longrightarrow C_*([t_1,t_2],X) \times C_*([t_1,t_2],X) \times C_*([t_1,t_2],X),$$

$$(x,y,z) \longmapsto \big(N_2^1(\omega,x,y,z), N_2^2(\omega,x,y,z), N_2^3(\omega,x,y,z)\big),$$

where

$$
\begin{aligned}
N_2^1(x,y,z,\omega) =& S_1(\omega,t)x_1(\omega,t_1) + I_1\big(x_1(t_1,\omega),y(t_1,\omega),z_1(t_1,\omega)\big)\\
&+ \int_{t_1}^t S_1(\omega,t-s)f_1\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\,ds \quad t \in (t_1,t_2]
\end{aligned}
$$

and

$$
\begin{aligned}
N_2^2(x,y,z,\omega) =& S_2(\omega,t)y_1(t_1,\omega) + I_1\big(x_1(t_1,\omega),y_1(t_1,\omega),z_1(t_1,\omega)\big)\\
&+ \int_{t_1}^t S_2(\omega,t-s)f_2\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\,ds \quad t \in (t_1,t_2].
\end{aligned}
$$

$$
\begin{aligned}
N_2^3(x,y,z,\omega) =& S_3(\omega,t)z_1(t_1,\omega) + I_1\big(x_1(t_1,\omega),y_1(t_1,\omega),z_1(t_1,\omega)\big)\\
&+ \int_{t_1}^t S_3(\omega,t-s)f_3\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\,ds \quad t \in (t_1,t_2]
\end{aligned}
$$

N_2 is a random operator on $C_*([t_1,t_2],X)\times C_*([t_1,t_2],X)\times C_*([t_1,t_2],X)$. Now we show that N_2 satisfies all the conditions of Theorem 11 on $C_*([t_1,t_2],X)\times C_*([t_1,t_2],X)\times C_*([t_1,t_2],X)$.

Let $(x,y,z),(\widetilde{x},\widetilde{y},\widetilde{z})\in C_*([t_1,t_2],X)\times C_*([t_1,t_2],X)\times C_*([t_1,t_2],X)$, then

$$\begin{aligned}
&\big|N_2^1\big(x(\cdot,\omega),y(\cdot,\omega),z(\cdot,\omega),\omega\big)-N_2^1\big(\widetilde{x}(\cdot,\omega),\widetilde{y}(\cdot,\omega),\widetilde{z}(\cdot,\omega),\omega\big)\big|\\
&\le K_1(\omega)\int_{t_1}^{t}p_1(\omega,t)\big|x(s,\omega)-\widetilde{x}(s,\omega)\big|\,ds\\
&\quad+K_1(\omega)\int_{t_1}^{t_2}p_2(\omega,t)\big|y(s,\omega)-\widetilde{y}(s,\omega)\big|\,ds\\
&\quad+K_1(\omega)\int_{t_1}^{t_2}p_3(\omega,t)\big|z(s,\omega)-\widetilde{z}(s,\omega)\big|\,ds..
\end{aligned}$$

Then

$$\begin{aligned}
&\big\|N_2^1\big(x(\cdot,\omega),y(\cdot,\omega),z(\cdot,\omega),\omega\big)-N_2^1\big(\omega,\widetilde{x}(\cdot,\omega),\widetilde{y}(\cdot,\omega),\widetilde{z}(\cdot,\omega),\omega\big)\big\|_{**}\\
&\le\frac{K_1(\omega)}{\tau}\|x-\widetilde{x}\|_{**}+\frac{K_1(\omega)}{\tau}\|y-\widetilde{y}\|_{**}+\frac{K_1(\omega)}{\tau}\|z-\widetilde{z}\|_{**},
\end{aligned}$$

where

$$\|x\|_*=\sup_{t\in[0,t_1]}e^{-\tau\int_{t_1}^{t}p(s,\omega)\,ds}\big|x(t)\big|,\qquad p(t,\omega)=\sum_{i=1}^{9}p_i(t,\omega),\quad \tau>K_1(\omega)+K_2(\omega)+K_3(\omega).$$

Similarly, we have

$$\begin{aligned}
&\big\|N_2^2\big(x(\cdot,\omega),y(\cdot,\omega),z(\cdot,\omega),\omega\big)-N_2^2\big(\widetilde{x}(\cdot,\omega),\widetilde{y}(\cdot,\omega),\widetilde{z}(\cdot,\omega),\omega\big)\big\|_{**}\\
&\le\frac{K_2(\omega)}{\tau}\|x-\widetilde{x}\|_{**}+\frac{K_2(\omega)}{\tau}\|y-\widetilde{y}\|_{**}\frac{K_2(\omega)}{\tau}\|z-\widetilde{z}\|_{**}.
\end{aligned}$$

and

$$\begin{aligned}
&\big\|N_2^3\big(x(\cdot,\omega),y(\cdot,\omega),z(\cdot,\omega),\omega\big)-N_2^3\big(\widetilde{x}(\cdot,\omega),\widetilde{y}(\cdot,\omega),\widetilde{z}(\cdot,\omega),\omega\big)\big\|_{**}\\
&\le\frac{K_3(\omega)}{\tau}\|x-\widetilde{x}\|_{**}+\frac{K_3(\omega)}{\tau}\|y-\widetilde{y}\|_{**}\frac{K_3(\omega)}{\tau}\|z-\widetilde{z}\|_{**}.
\end{aligned}$$

Therefore

$$d_1\big(N_2(x,y,z,\omega),N_2(\widetilde{x},\widetilde{y},\widetilde{z},\omega)\big)\le M(\omega)d_1\big((x,y,z),(\widetilde{x},\widetilde{y},\widetilde{z})\big),$$

where

$$d_1(x,y) = \begin{pmatrix} \|x-y\|_{**} \\ \|x-y\|_{**} \\ \|x-y\|_{**} \end{pmatrix}$$

and

$$M(\omega) = \begin{pmatrix} \frac{K_1(\omega)}{\tau} & \frac{K_1(\omega)}{\tau} & \frac{K_1(\omega)}{\tau} \\ \frac{K_2(\omega)}{\tau} & \frac{K_2(\omega)}{\tau} & \frac{K_2(\omega)}{\tau} \\ \frac{K_3(\omega)}{\tau} & \frac{K_3(\omega)}{\tau} & \frac{K_3(\omega)}{\tau} \end{pmatrix}.$$

From Theorem 12 there exists a unique random solution of problem 3.3, we denote it by $(x_2(t,\omega), y_2(t,\omega), z_2(t,\omega))$.

Step 3. We consider the problem

$$(3.4)\quad \begin{cases} x'(t,\omega) = A_1^1(\omega)x(t,\omega) + f_1(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega) \quad t \in (t_2,t_3], \\ y'(t,\omega) = A_2^1(\omega)y(t,\omega) + f_2(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega) \quad t \in (t_2,t_3], \\ z'(t,\omega) = A_3^1(\omega)z(t,\omega) + f_3(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega) \quad t \in (t_2,t_3], \\ x(t_1^+,\omega) = x_1(t_2,\omega) + I_1(x_1(t_2,\omega),y_1(t_2,\omega),z_1(t_2,\omega)), \\ y(t_1^+,\omega) = y_1(t_2,\omega) + \bar{I}_1(x_1(t_2,\omega),y_1(t_2,\omega),z_1(t_2,\omega)) \\ z(t_1^+,\omega) = z_1(t_2,\omega) + \bar{\bar{I}}_1(x_1(t_2,\omega),y_1(t_2,\omega),z_1(t_2,\omega)). \end{cases}$$

Let

$$C_{**}\big([t_2,t_3],X\big) = \big\{y \in C\big((t_2,t_3],X\big) : y\big(t_2^+\big) \text{ exists}\big\}.$$

Consider the operator

$$N_3 : C_{**}([t_2,t_3],X) \times C_{**}\big([t_2,t_3],X\big) \times C_{**}\big([t_2,t_3],X\big) \times \Omega \longrightarrow C_{**}\big([t_2,t_3],X\big) \times C_{**}\big([t_2,t_3],X\big) \times C_{**}\big([t_2,t_3],X\big),$$

$$(x,y,z) \longmapsto \big(N_3^1(\omega,x,y,z), N_3^2(\omega,x,y,z), N_3^3(\omega,x,y,z)\big),$$

where

$$N_3^1(x,y,z,\omega) = S_1(\omega,t)x_1(\omega,t_2) + I_1\big(x_1(t_2,\omega), y(t_2,\omega), z_1(t_2,\omega)\big) + \int_{t_2}^{t_3} S_1(\omega,t-s) f_1\big(s, x(s,\omega), y(s,\omega), z(s,\omega), \omega\big)\, ds \quad t \in (t_2,t_3]$$

$$N_3^2(x,y,z,\omega) = S_2(\omega,t)y_1(t_2,\omega) + I_1\big(x_1(t_2,\omega), y_1(t_2,\omega), z_1(t_2,\omega)\big) + \int_{t_2}^{t_3} S_2(\omega,t-s) f_2\big(s, x(s,\omega), y(s,\omega), z(s,\omega), \omega\big)\, ds \quad t \in (t_2,t_3].$$

and

$$N_3^3(x,y,z,\omega) = S_3(\omega,t)z_1(t_2,\omega) + I_1\big(x_1(t_2,\omega), y_1(t_2,\omega), z_1(t_2,\omega)\big) + \int_{t_2}^{t_3} S_3(\omega,t-s) f_3\big(s, x(s,\omega), y(s,\omega), z(s,\omega), \omega\big)\, ds \quad t \in (t_2,t_3]$$

N_3 is a random operator on $C_{**}\big([t_2,t_3],X\big) \times C_{**}\big([t_2,t_3],X\big) \times C_{**}\big([t_2,t_3],X\big)$. Now we show that N_3 satisfies all the conditions of Theorem 11 on $C_{**}\big([t_2,t_3],X\big) \times C_{**}\big([t_2,t_3],X\big) \times C_{**}\big([t_2,t_3],X\big)$.

Let $(x,y,z), (\widetilde{x},\widetilde{y},\widetilde{z}) \in C_{**}\big([t_2,t_3],X\big) \times C_{**}\big([t_2,t_3],X\big) \times C_{**}\big([t_2,t_3],X\big))$, then

$$\begin{aligned}&\big|N_3^1\big(x(\cdot,\omega), y(\cdot,\omega), z(\cdot,\omega), \omega\big) - N_3^1\big(\widetilde{x}(\cdot,\omega), \widetilde{y}(\cdot,\omega), \widetilde{z}(\cdot,\omega), \omega\big)\big| \\ &\le K_1(\omega)\int_{t_2}^{t_3} p_1(\omega,t)\big|x(s,\omega) - \widetilde{x}(s,\omega)\big|\, ds \\ &+ K_1(\omega)\int_{t_2}^{t_3} p_2(\omega,t)\big|y(s,\omega) - \widetilde{y}(s,\omega)\big|\, ds \\ &+ K_1(\omega)\int_{t_2}^{t_3} p_3(\omega,t)\big|z(s,\omega) - \widetilde{z}(s,\omega)\big|\, ds..\end{aligned}$$

Then

$$\begin{aligned}&\big\|N_3^1\big(x(\cdot,\omega), y(\cdot,\omega), z(\cdot,\omega), \omega\big) - N_3^1\big(\omega, \widetilde{x}(\cdot,\omega), \widetilde{y}(\cdot,\omega), \widetilde{z}(\cdot,\omega), \omega\big)\big\|_{***} \\ &\le \frac{K_1(\omega)}{\tau}\|x-\widetilde{x}\|_{***} + \frac{K_1(\omega)}{\tau}\|y-\widetilde{y}\|_{***} + \frac{K_1(\omega)}{\tau}\|z-\widetilde{z}\|_{***},\end{aligned}$$

where

$$\|x\|_* = \sup_{t\in[0,t_1]} e^{-\tau\int_{t_2}^{t} p(s,\omega)\,ds}\big|x(t)\big|, \qquad p(t,\omega) = \sum_{i=1}^{9} p_i(t,\omega), \quad \tau > K_1(\omega) + K_2(\omega) + K_3(\omega).$$

Similarly, we have

$$\begin{aligned}&\left\|N_3^2\big(x(\cdot,\omega),y(\cdot,\omega),z(\cdot,\omega),\omega\big)-N_3^2\big(\widetilde{x}(\cdot,\omega),\widetilde{y}(\cdot,\omega),\widetilde{z}(\cdot,\omega),\omega\big)\right\|_{***}\\&\le\frac{K_2(\omega)}{\tau}\|x-\widetilde{x}\|_{***}+\frac{K_2(\omega)}{\tau}\|y-\widetilde{y}\|_{***}\frac{K_2(\omega)}{\tau}\|z-\widetilde{z}\|_{***}.\end{aligned}$$

and

$$\begin{aligned}&\left\|N_3^3\big(x(\cdot,\omega),y(\cdot,\omega),z(\cdot,\omega),\omega\big)-N_3^3\big(\widetilde{x}(\cdot,\omega),\widetilde{y}(\cdot,\omega),\widetilde{z}(\cdot,\omega),\omega\big)\right\|_{***}\\&\le\frac{K_3(\omega)}{\tau}\|x-\widetilde{x}\|_{***}+\frac{K_3(\omega)}{\tau}\|y-\widetilde{y}\|_{***}\frac{K_3(\omega)}{\tau}\|z-\widetilde{z}\|_{***}.\end{aligned}$$

Therefore

$$d_1\big(N_3(x,y,z,\omega),N_3(\widetilde{x},\widetilde{y},\widetilde{z},\omega)\big)\le M(\omega)d_1\big((x,y,z),(\widetilde{x},\widetilde{y},\widetilde{z})\big),$$

where

$$d_1(x,y)=\begin{pmatrix}\|x-y\|_{***}\\ \|x-y\|_{***}\\ \|x-y\|_{***}\end{pmatrix}$$

and

$$M(\omega)=\begin{pmatrix}\frac{K_1(\omega)}{\tau}&\frac{K_1(\omega)}{\tau}&\frac{K_1(\omega)}{\tau}\\ \frac{K_2(\omega)}{\tau}&\frac{K_2(\omega)}{\tau}&\frac{K_2(\omega)}{\tau}\\ \frac{K_3(\omega)}{\tau}&\frac{K_3(\omega)}{\tau}&\frac{K_3(\omega)}{\tau}\end{pmatrix}.$$

From Theorem 12 there exists a unique random solution of problem 3.4, we denote it by $(x_3(t,\omega),y_3(t,\omega),z_3(t,\omega))$.

Step 4. We continue this process until we arrive at the random variable

$$\omega\to(x_{m+1}(\cdot,\omega),y_{m+1}(\cdot,\omega),z_{m+1}(\cdot,\omega))$$

as a solution of the problem

$$
(3.5)\quad \begin{cases}
x'(t,\omega) = A_1^1(\omega)x(t,\omega) + f_1(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega) & t\in(t_m,b],\\
y'(t,\omega) = A_2^1(\omega)y(t,\omega) + f_2(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega) & t\in(t_m,b],\\
z'(t,\omega) = A_3^1(\omega)z(t,\omega) + f_3(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega) & t\in(t_m,b],\\
x(t_m^+,\omega) = x_1(t_m,\omega) + I_m(x_m(t_m,\omega),y_m(t_m,\omega),z_m(t_m,\omega)),\\
y(t_m^+,\omega) = y_m(t_m,\omega) + \bar{I}_m(x_m(t_m,\omega),y_m(t_m,\omega),z_m(t_m,\omega)\\
z(t_m^+,\omega) = z_m(t_m,\omega) + \bar{\bar{I}}_m(x_m(t_m,\omega),y_m(t_m,\omega),z_m(t_m,\omega).
\end{cases}
$$

Then a random solution of problem 1.1 is defined by

$$
\big(x(t,\omega),y(t,\omega),z(t,\omega)\big) = \begin{cases}
(x_1(t,\omega),y_1(t,\omega),z_1(t,\omega)), & \text{if } t\in[0,t_1],\\
(x_2(t,\omega),y_2(t,\omega),z_2(t,\omega)), & \text{if } t\in(t_1,t_2],\\
\cdots & \cdots\\
(x_{m+1}(t,\omega),y_{m+1}(t,\omega),z_{m+1}(t,\omega)), & \text{if } t\in(t_m,b].
\end{cases}
$$

□

3.1. Existence and compactness results. In this subsection, we prove the existence and compactness of a solution set of problem 1.1. For this we assume that the $C_0 - \textit{semigroup}$ $S_1(\cdot,t), S_2(\cdot,t), S_3(\cdot,t)$, $t>0$ is compact. Now, we consider the following set of hypotheses in what follows:

(H3) The functions f_1, f_2 and f_3 are random Carath*é*odory on $[0,b]\times X\times X\times X\times\Omega$.

(H4) There exist bounded measurable functions $\gamma_1,\gamma_2,\gamma_3 : \Omega\to L^1([0,b],\mathbb{R}_+)$ and nondecreasing continuous functions $\psi_1,\psi_2,\psi_3 : \mathbb{R}_+\to(0,\infty)$ such that

$$\big|f_1(t,x,y,z,\omega)\big| \le \gamma_1(t,\omega)\psi_1\big(|x|+|y|+|z|\big) \quad \text{a.e. } t\in[0,b]$$

$$\big|f_2(t,x,y,z,\omega)\big| \le \gamma_2(t,\omega)\psi_2\big(|x|+|y|+|z|\big) \quad \text{a.e. } t\in[0,b]$$

and

$$\big|f_3(t,x,y,z,\omega)\big| \le \gamma_3(t,\omega)\psi_3\big(|x|+|y|+|z|\big) \quad \text{a.e. } t\in[0,b]$$

for all $\omega\in\Omega$ and $x,y,z\in X$.

(H5) There exist constants $\alpha_i \geq 0$ and $\lambda_i \geq 1$ for $i \in \{1,2,3\}$ such that

$$\left\|S_i(t,\omega)\right\| \leq \lambda_i e^{\alpha_i t} \quad \text{for all } \omega \in \Omega.$$

(H6) There exist constants $c_k,\ \bar{c}_k, \bar{\bar{c}}_k > 0$ with $k = 1,\dots,n$ and continuous functions $\phi_k, \overline{\phi}_k, \overline{\overline{\phi}}_k \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

$$\begin{aligned}
\left|I_k(x,y,z)\right| &\leq c_k \phi_k\big(|x|+|y|+|z|\big) \quad \text{for all } x,y,z \in X,\\
\left|\bar{I}_k(x,y,z)\right| &\leq \bar{c}_k \overline{\phi}_k\big(|x|+|y|+|z|\big) \quad \text{for all } x,y,z \in X\\
\left|\bar{\bar{I}}_k(x,y,z)\right| &\leq \bar{\bar{c}}_k \overline{\overline{\phi}}_k\big(|x|+|y|+|z|\big) \quad \text{for all } x,y,z \in X.
\end{aligned}$$

The following result is known as the Gronwall-Bihari theorem.

Lemma 23. *([7]) Let $u, \bar{g}\colon [a,b] \longrightarrow \mathbb{R}$ be positive real continuous functions. Assume that there exist $c > 0$ and a continuous nondecreasing function $\phi : \mathbb{R}_+ \longrightarrow (0,+\infty)$ such that*

$$u(t) \leq c + \int_a^t \bar{g}(s)\phi\big(u(s)\big)\,ds, \quad \forall t \in J.$$

Then

$$u(t) \leq H^{-1}\left(\int_a^t \bar{g}(s)\,ds\right), \quad \forall t \in J$$

provided that

$$\int_c^{+\infty} \frac{dy}{\phi(y)} > \int_a^b \bar{g}(s)\,ds.$$

Here, H^{-1} refers to the inverse of the function $H(u) = \int_c^u \frac{dy}{\phi(y)} for u \geq c$.

Now, we give our existence and compactness results for problem 1.1.

Theorem 24. *Assume that (H3)-(H6) are satisfied and*

$$\int_0^b \Gamma(s,\omega)\,ds < \int_c^\infty \frac{du}{\psi(u)} \quad \textit{for all } \omega,$$

$$\psi = \psi_1 + \psi_2 + \psi_3 \quad \textit{and} \quad \Gamma = \gamma_1 + \gamma_2 + \gamma_3.$$

Then problem 1.1 has a random solution defined on $[0,b]$.

Proof. Consider the operator

$$T : C([0,b],X)\times C([0,b],X)\times C([0,b],X)\times\Omega \longrightarrow C([0,b],X)\times C([0,b],X)\times C([0,b],X),$$

such that

$$(x,y,z)\longmapsto \big(T_1(\omega,x,y,z),T_2(\omega,x,y,z),T_3(\omega,x,y,z)\big),$$

$(x,y,z)\in PC\times PC\times PC$, where

$$\begin{aligned}T_1\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big) =&S_1(\omega,t)\varphi_1(\omega)+\int_0^t S_1(\omega,t-s)f_1\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\,ds\\ &+\sum_{0<t_k<t} S_1(\omega,t-t_k)I_k\big(x(t_k,\omega),y(t_k,\omega),z(t_k,\omega)\big),\quad t\in[0,b]\end{aligned}$$

$$\begin{aligned}T_2\big(\omega,x(t,\omega),y(t,\omega),z(t,\omega),\big) =&S_2(\omega,t)\varphi_2(\omega)+\int_0^t S_2(\omega,t-s)f_2\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\,ds\\ &+\sum_{0<t_k<t} S_2(\omega,t-t_k)\bar{I}_k\big(x(t_k,\omega),y(t_k,\omega),z(t_k,\omega)\big),\quad t\in[0,b].\end{aligned}$$

and

$$\begin{aligned}T_3\big(\omega,x(t,\omega),y(t,\omega),z(t,\omega),\big) =&S_3(\omega,t)\varphi_3(\omega)+\int_0^t S_3(\omega,t-s)f_2\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\,ds\\ &+\sum_{0<t_k<t} S_3(\omega,t-t_k)\bar{\bar{I}}_k\big(x(t_k,\omega),y(t_k,\omega),z(t_k,\omega)\big),\quad t\in[0,b].\end{aligned}$$

Clearly fixed points of the operator T are random mild solutions of problem 1.1. For $\omega\in\Omega$ fixed, consider $T_\omega : PC\times PC\times PC\to PC\times PC\times PC$ by

$$\begin{aligned}T_\omega\big(x(t,\omega),y(t,\omega),z(t,\omega)\big) =&\big(T_1\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big),T_2\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big),\\ &T_3\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big)\big),\end{aligned}$$

$(x,y,z)\in PC\times PC$.

We shall show that T satisfies assumptions of Theorem 13. We split the proof into several steps. First we show that T_ω is completely continuous.

Step 1. T maps bounded sets into bounded sets in $PC\times PC\times PC$. Let

$$B_p \times B_q \times B_r = \left\{ (x,y,z) \in PC \times PC \times PC : \left\| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\| \leq \begin{pmatrix} p \\ q \\ r \end{pmatrix} \right\},$$

where

$$\left\| \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\| = \begin{pmatrix} \|x\|_{PC} \\ \|y\|_{PC} \\ \|z\|_{PC} \end{pmatrix}.$$

Let $(x,y,z) \in B_p \times B_q \times B_r$, then for each $t \in [0,b]$,

$$\begin{aligned}
&\big|T_1\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big)\big| \\
=&\bigg|S_1(\omega,t)\varphi_1(\omega) + \int_0^t S_1(\omega,t-s) f_1\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\,ds \\
&+ \sum_{0<t_k<t} S_1(\omega,t-t_k) I_k\big(x(t_k,\omega),y(t_k,\omega),z(t_k,\omega),\big)\bigg| \\
\leq& \lambda_1 e^{\alpha_1 b}\big|\varphi_1(\omega)\big| + \lambda_1 e^{\alpha_1 b}\int_0^t \big|f_1\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\big|\,ds \\
&+ \lambda_1 e^{\alpha_1 b}\sum_{k=1}^m c_k \phi_k\big(\big|x(t,\omega)\big| + \big|y(t,\omega)\big| + \big|z(t,\omega)\big|\big) \\
\leq& \lambda_1 e^{\alpha_1 b}|\varphi_1| + \lambda_1 e^{\alpha_1 b}\int_0^t \gamma_1(s,\omega)\psi_1\big(\big|x(s,\omega)\big| + \big|y(s,\omega)\big| + \big|z(s,\omega)\big|\big)\,ds \\
&+ e^{\alpha_1 b}\sum_{k=1}^m c_k \phi_k\big(\|x\|_{PC} + \|y\|_{PC} + \|z\|_{PC}\big).
\end{aligned}$$

Then

$$\big|T_1\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big)\big| \leq \lambda_1 e^{\alpha_1 b}\left(\big|\varphi_1(\omega)\big| + \psi_1(\rho)\|\gamma_1\|_{L_1} + \sum_{k=1}^m c_k \phi_k(\rho)\right) := l_1 < \infty.$$

Similarly, for T_2 and T_3, we have

$$\big\|T_2(x,y,z,\omega)\big\| \leq \lambda_2 e^{\alpha_2 b}\left(\big|\varphi_2(\omega)\big| + \psi_2(\rho)\|\gamma_2\|_{L_1} + \sum_{k=1}^m \bar{c}_k \overline{\phi}_k(\rho)\right) := l_2 < \infty.$$

$$\big\|T_3(x,y,z,\omega)\big\| \leq \lambda_3 e^{\alpha_3 b}\left(\big|\varphi_3(\omega)\big| + \psi_3(\rho)\|\gamma_3\|_{L_1} + \sum_{k=1}^{m} \bar{\bar{c}}_k \overline{\overline{\phi}}_k(\rho)\right) := l_3 < \infty.$$

Step 2. T_ω maps bounded sets into equicontinuous sets of $PC \times PC \times PC$. Let $B_p \times B_q \times B_r$ be a bounded set in $PC \times PC \times PC$ as in Step 1 is an equicontinuous set of $PC \times PC \times PC$. Let $\tau_1, \tau_2 \in [0,b]$ such that $0 < \tau_1 < \tau_2 \leq b$, and $(x,y,z) \in (B_p, B_q, B_r)$. Then

$$\begin{aligned}
&\big|h_1(\tau_1) - h_1(\tau_2)\big| \leq \big|S_1(\omega,\tau_1)\varphi_1(\omega) - S_1(\omega,\tau_2)\varphi_1(\omega)\big| \\
&+ \int_0^{\tau_1} \big\|S_1(\omega,\tau_1 - s) - S_1(\omega,\tau_2 - s)\big\|\big|f_1\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\,ds \\
&+ \sum_{0<t_k<\tau_1} \big\|S_1(\omega,\tau_1 - t_k) - S_1(\omega,\tau_2 - t_k)\big\|\big|I_k\big(\omega,x(\omega,t_k),y(\omega,t_k),z(\omega,t_k)\big)\big| \\
&- \int_{\tau_1}^{\tau_2} \big\|S_1(\omega,\tau_2 - s)\big\|\big|f_1\big(s,x(s,\omega),y(s,\omega),z(s,\omega,\omega)\big|\,ds \\
&- \sum_{\tau_1<t_k<\tau_2} \big\|S_1(\omega,\tau_2 - t_k)\big\|\big|I_k\big(x(t_k,\omega),y(t_k,\omega),z(t_k,\omega)\big)\big|.
\end{aligned}$$

Then

$$\begin{aligned}
&\big|h_1(\tau_1) - h_1(\tau_2)\big| \leq \big\|S_1(\omega,\tau_2 - \tau_1) - I\big\|\big|S_1(\omega,\tau_1)\varphi_1(\omega)\big| \\
&+ \int_0^{\tau_1} \big\|S_1(\omega,\tau_2 - \tau_1) - I)\big\|\big\|S_1(\omega,\tau_1 - s)\big\|\big|f_1\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\big|\,ds \\
&+ \int_{\tau_1}^{\tau_2} \big\|S_1(\omega,\tau_2 - s)\big\|\big|f_1\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\big|\,ds \\
&+ \sum_{0<t_k<\tau_1} \big\|S_1(\omega,\tau_2 - \tau_1) - I)\big\|\big\|S_1(\omega,\tau_1 - t_k)\big\|\big|I_k\big(x(t_k,\omega),y(t_k,\omega),z(t_k,\omega)\big)\big| \\
&+ \sum_{\tau_1<t_k<\tau_2} \big\|S_1(\omega,\tau_2 - t_k)\big\|\big|I_k\big(x(t_k,\omega),y(t_k,\omega),z(t_k,\omega),\big)\big|,
\end{aligned}$$

where

$$h_1(\tau_i) = T_1\big(\omega,x(\omega,\tau_i),y(\omega,\tau_i),z(\omega,\tau_i)\big), \quad i = 1,2,3.$$

By (H5), we get

$$\begin{aligned}\big|h_1(\tau_1)-h_1(\tau_2)\big| \leq &\lambda_1 e^{\alpha_1 b}\big\|S_1(\omega,\tau_2-\tau_1)-I\big\| \\ &+\lambda_1 e^{\alpha_1 b}\psi_1(p+q)\big\|S_1(\omega,\tau_2-\tau_1)-I\big\|\int_0^{\tau_1}\gamma_1(s,\omega)\,ds \\ &+\lambda_1 e^{\alpha_1 b}\psi_1(p+q)\int_{\tau_1}^{\tau_2}\gamma_1(s,\omega)\,ds \\ &+\lambda_1 e^{\alpha_1 b}\sum_{0<t_k<\tau_1} c_k\phi_k(p+q)\big\|S_1(\omega,\tau_2-\tau_1)-I\big\| \\ &+\sum_{\tau_1<t_k<\tau_2} c_k\phi_k(p+q)\big\|S_1(\omega,\tau_1+h-t_k)\big\|.\end{aligned}$$

Since $\{S_1(\omega,t)\}_{t\geq 0}$ is uniformly continuous, then $\|S_1(h)-I\|\to 0$ as $h\to 0^+$. Thus the right-hand side tends to zero as $\tau_2\to\tau_1$. This proves equicontinuity for the case where $t\neq t_i, i=1,\dots,m$.

Now we prove equicontinuity at $t=t_i^-$. Let $\xi_1>0$ such that $\{t_k : k\neq i\}\cap[t_i-\xi_1,t_i+\xi_1]\neq\emptyset$. For $0<\varepsilon<\xi_1$, we get

$$\begin{aligned}\big|h_1(t_i)-h_1(t_i-\varepsilon)\big| \leq &\big|S_1(\omega,t_i)\varphi_1(\omega)-S_1(\omega,t_i-\varepsilon)\varphi_1(\omega)\big| \\ &+\psi_1(p+q+r)\int_0^{t_i}\big\|S_1(\omega,t_i-s)-S_1(\omega,t_i-\varepsilon-s)\big\|\gamma_1(s,\omega)\,ds \\ &+\sum_{k=1}^{i-1}\big\|S_1(\omega,t_i-t_k)-S_1(\omega,t_i-\varepsilon-t_k)\big\|\phi_k(p+q) \\ &+\psi_1(p+q+r)\int_{t_i-\varepsilon}^{t_i}\big\|S_1(\omega,t_i-\varepsilon-s)\big\|\gamma_1(s,\omega)\,ds.\end{aligned}$$

The right-hand side tends to zero as $\varepsilon\longrightarrow 0$. Next we prove equicontinuity at $t=t_i^+$. Fix $\xi_2>0$ such that $\{t_k : k\neq i\}\cap[t_i-\xi_2,t_i+\xi_2]\neq\emptyset$. For $0<\varepsilon<\xi_2$, we have

$$\begin{aligned}\big|h_1(t_i+\varepsilon)-h_1(t_i)\big| \leq &\big|S_1(\omega,t_i+\varepsilon)\varphi_1(\omega)-S_1(\omega,t_i)\varphi_1(\omega)\big| \\ &+\psi_1(p+q+r)\int_0^{t_i}\big\|S_1(\omega,t_i+\varepsilon-s)-S_1(\omega,t_i-s)\big\|\gamma_1(s,\omega)\,ds \\ &+\sum_{0<t_k<t_k}\big\|S_1(\omega,t_i+\varepsilon-t_k)-S_1(\omega,t_i-t_k)\big\|c_k\phi_k(p+q) \\ &+\psi_1(p+q+r)\int_{t_i}^{t_i+\varepsilon}\big\|S_1(\omega,t_i-s)\big\|\gamma_1(s,\omega)\,ds \\ &+\sum_{t_i<t_k<t_i+\varepsilon} c_k\big\|S_1(\omega,t_i+\varepsilon-t_k)\big\|\phi_k(p+q+r).\end{aligned}$$

The right-hand side tends to zero as $h \longrightarrow 0$. By a similar way we can prove the equicontinuity for $T_2(B_p, B_q, B_r)$ and $T_3(B_p, B_q, B_r)$.

Step 3. Now we will prove that $T_\omega(B_p \times B_q \times B_r)(t)$ for $t \in [0,b]$ is relatively compact in PC. For $0 < \varepsilon < t$ and $t \in [0,b]$, let

$$T_\varepsilon(x,y,z,\omega) = \big(T_1^\varepsilon(x,y,z,\omega), T_2^\varepsilon(x,y,z,\omega), T_3^\varepsilon(x,y,z,\omega)\big),$$

where

$$\begin{aligned}
&T_1^\varepsilon\big(\omega, x(t,\omega), y(t,\omega), z(t,\omega)\big)\\
&\quad = S_1(\omega,t)\varphi_1(\omega) + \int_0^{t-\varepsilon} S_1(\omega,t-s) f_1\big(s, x(s,\omega), y(s,\omega), z(s,\omega), \omega\big)\, ds\\
&\qquad + \sum_{0<t_k<t-\varepsilon} S_1(\omega,t-t_k) I_k\big(\omega, x(t_k,\omega), y(t_k,\omega), z(t_k,\omega)\big)\\
&\quad = S_1(\omega,t)\varphi_1(\omega) + S_1(\omega,\varepsilon)\int_0^{t-\varepsilon} S_1(\omega,t-s-\varepsilon) f_1\big(s, x(s,\omega), y(s,\omega), z(s,\omega), \omega\big)\, ds\\
&\qquad + S_1(\omega,\varepsilon) \sum_{0<t_k<t-\varepsilon} S_1(\omega,t-t_k) I_k\big(x(t_k,\omega), y(t_k,\omega), z(t_k,\omega),\big)
\end{aligned}$$

$$\begin{aligned}
&T_2^\varepsilon\big(\omega, x(t,\omega), y(t,\omega), z(t,\omega)\big)\\
&\quad = S_2(\omega,t)\varphi_2(\omega) + S_2(\omega,\varepsilon)\int_0^{t-\varepsilon} S_2(\omega,t-s-\varepsilon) f_2\big(s, x(s,\omega), y(s,\omega), z(s,\omega), \omega\big)\, ds\\
&\qquad + S_2(\omega,\varepsilon) \sum_{0<t_k<t-\varepsilon} S_2(\omega,t-t_k) \bar{I}_k\big(x(t_k,\omega), y(t_k,\omega), z(t_k,\omega)\big).
\end{aligned}$$

and

$$\begin{aligned}
&T_3^\varepsilon\big(\omega, x(t,\omega), y(t,\omega), z(t,\omega)\big)\\
&\quad = S_3(\omega,t)\varphi_3(\omega) + S_3(\omega,\varepsilon)\int_0^{t-\varepsilon} S_3(\omega,t-s-\varepsilon) f_3\big(s, x(s,\omega), y(s,\omega), z(s,\omega), \omega\big)\, ds\\
&\qquad + S_3(\omega,\varepsilon) \sum_{0<t_k<t-\varepsilon} S_3(\omega,t-t_k) \bar{\bar{I}}_k\big(x(t_k,\omega), y(t_k,\omega), z(t_k,\omega)\big).
\end{aligned}$$

The compactness of the semigroup $\{S_i(\omega,t)\}_{t>0}$ for $i = 1,2,3$ implies that the set $T_\varepsilon(B_p \times B_q \times B_r)(t)$ is precompact. Moreover,

$$\begin{aligned}
&\big|T_1^{\varepsilon}\big(\omega,x(t,\omega),y(t,\omega),z(t,\omega)\big)-T_1\big(\omega,x(t,\omega),y(t,\omega),z(t,\omega)\big)\big|\\
&\quad=\bigg|S_1(\omega,\varepsilon)\int_{t-\varepsilon}^{t}S_1(\omega,t-s-\varepsilon)f_1\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\,ds\\
&\qquad+S_1(\omega,\varepsilon)\sum_{t-\varepsilon<t_k<t}S_1(\omega,t-t_k)I_k\big(x(t_k,\omega),y(t_k,\omega),z(t_k,\omega)\big)\bigg|\\
&\quad\le\lambda_1e^{\alpha_1 b}\int_{t-\varepsilon}^{t}\gamma_1(\omega,s)\psi_1\big(\big|x(s,\omega)\big|+\big|y(\omega,s)\big|+\big|z(\omega,s)\big|\big)\,ds\\
&\qquad+\lambda_1e^{\alpha_1 b}\sum_{t-\varepsilon<t_k<t}c_k\phi_k\big(\big|x(s,\omega)\big|+\big|y(s,\omega)\big|+\big|z(s,\omega)\big|\big)\\
&\quad\le\lambda_1e^{\alpha_1 b}\psi_1(p+q+r)\int_{t-\varepsilon}^{t}\gamma_1(\omega,s)\,ds+\lambda_1e^{\alpha_1 b}\sum_{t-\varepsilon<t_k<t}c_k\phi_k(p+q+r).
\end{aligned}$$

The right-hand term tends to 0 uniformly in t as $\varepsilon\longrightarrow 0$. This implies that the set $T_1^{\varepsilon}(B_p\times B_q\times B_r)(t)$ is relatively compact for $t\in[0,b]$.

By a similar way as above, we prove that $T_2^{\varepsilon}(B_p\times B_q\times B_r)(t)$ and $T_3^{\varepsilon}(B_p\times B_q\times B_r)(t)$ are also relatively compact. This implies that $T_{\varepsilon}(B_p\times B_q\times B_r)(t)$ is relatively compact.

Step 4. Now we show that the operator T is continuous.

Let (x_n,y_n,z_n) be a sequence such that$(x_n,y_n,z_n)\longrightarrow(x,y,z)$ in $PC\times PC\times PC$ as $n\longrightarrow\infty$. By (H4), (H5) we obtain

$$\begin{aligned}
&\big|T_1\big(x_n(t,\omega),y_n(t,\omega),z_n(t,\omega)\omega\big)-T_1\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big)\big|\\
&\quad\le\lambda_1e^{\alpha_1 b}\int_0^t\big|f_1\big(s,x_n(\omega,s),y_n(s,\omega),z_n(s,\omega),\omega\big)-f_1\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\big|\,ds\\
&\qquad+\lambda_1e^{\alpha_1 b}\sum_{0<t_k<t}\big|I_k\big(\omega,x_n(t_k,\omega),y_n(t_k,\omega),z_n(t_k,\omega)\big)-I_k\big(x(t_k,\omega),y(t_k,\omega),z(t_k,\omega)\big)\big|.
\end{aligned}$$

Hence

$$\begin{aligned}&\big\|T_1\big(x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega),\omega\big)-T_1\big(x(\cdot,\omega),y(\cdot,\omega),z(\cdot,\omega),\omega\big)\big\|_{PC}\\&\quad\le\lambda_1e^{\alpha_1 b}\int_0^b\big|f_1\big(s,x_n(s,\omega),y_n(s,\omega),z_n(s,\omega),\omega\big)-f_1\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\big|\,ds\\&\qquad+\lambda_1e^{\alpha_1 b}\sum_{k=1}^m\big|I_k\big(x_n(t_k,\omega),y_n(t_k,\omega),z_n(t_k,\omega)\big)-I_k\big(x(t_k,\omega),y(t_k,\omega),z(t_k,\omega)\big)\big|.\end{aligned}$$

Since f is a Carathéodory function, by the Lebesgue dominated convergence theorem and the continuity of I_k, we get

$$\big\|T_1(x_n,y_n,z_n,\omega)-T_1(x,y,z,\omega)\big\|_{PC}\longrightarrow 0\quad\text{as } n\longrightarrow\infty.$$

Similarly

$$\big\|T_2(x_n,y_n,z_n\omega)-T_2(x,y,z,\omega)\big\|_{PC}\longrightarrow 0\quad\text{as } n\longrightarrow\infty.$$

and

$$\big\|T_3(x_n,y_n,z_n\omega)-T_3(x,y,z,\omega)\big\|_{PC}\longrightarrow 0\quad\text{as } n\longrightarrow\infty.$$

Thus T is continuous.

Step 5. Now, we show that the set

$$\mathscr{M}=\big\{(x,y,z)\in PC\times PC\times PC:(x,y,z)=\lambda(\omega)T(x,y,z),\lambda(\omega)\in(0,1)\big\}$$

is bounded for some measurable function $\lambda:\Omega\longrightarrow\mathbb{R}$. Let $(x,y,z)\in\mathscr{M}$, then for each $t\in[0,t_1]$,

$$x(t,\omega)=\lambda(\omega)T_1\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big),$$

$$y(t,\omega)=\lambda(\omega)T_2\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big)$$

$$z(t,\omega)=\lambda(\omega)T_3\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big)$$

such that

$$T_1\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big)=S_1(\omega,t)\varphi_1(\omega)+\int_0^t S_1(\omega,t-s)f_1\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\,ds$$

$$T_2\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big) = S_2(\omega,t)\varphi_2(\omega)+\int_0^t S_2(\omega,t-s)f_2\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\,ds.$$

and

$$T_3\big(x(t,\omega),y(t,\omega),z(t,\omega),\omega\big) = S_3(\omega,t)\varphi_3(\omega)+\int_0^t S_3(\omega,t-s)f_3\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\,ds.$$

For some $0<\lambda(\omega)<1$, we have

$$\begin{aligned}\big|x(t,\omega)\big| &= \big|\lambda(\omega)T_1\big(\omega,x(t,\omega),y(t,\omega),z(t,\omega),\big)\big|\\ &\leq \big|\lambda(\omega)\big|\bigg(\big|S_1(\omega,t)\varphi_1(\omega)\big|+\int_0^t \big\|S_1(\omega,t-s)\big\|\big|f_1\big(s,x(s,\omega),y(s,\omega),z(s,\omega),\omega\big)\big|\,ds\bigg)\\ &\leq \lambda_1 e^{\alpha_1 b}\big|\varphi_1(\omega)\big|+\lambda_1 e^{\alpha_1 b}\int_0^t \gamma_1(s,\omega)\psi_1\big(\big|x(s,\omega)\big|+\big|y(s,\omega)\big|+\big|z(s,\omega)\big|\big)\,ds.\end{aligned}$$

Thus

$$\big|x(t,\omega)\big| \leq \lambda_1 e^{\alpha_1 b}\big|\varphi_1(\omega)\big|+\lambda_1 e^{\alpha_1 b}\int_0^t \gamma_1(s,\omega)\psi_1\big(\big|x(s,\omega)\big|+\big|y(s,\omega)\big|+\big|z(s,\omega)\big|\big)\,ds.$$

Similarly

$$\big|y(t,\omega)\big| \leq \lambda_2 e^{\alpha_2 b}\big|\varphi_2(\omega)\big|+\lambda_2 e^{\alpha_2 b}\int_0^t \gamma_2(s,\omega)\psi_2\big(\big|x(s,\omega)\big|+\big|y(s,\omega)\big|+\big|z(s,\omega)\big|\big)\,ds.$$

and

$$\big|z(t,\omega)\big| \leq \lambda_3 e^{\alpha_2 b}\big|\varphi_3(\omega)\big|+\lambda_3 e^{\alpha_3 b}\int_0^t \gamma_3(s,\omega)\psi_3\big(\big|x(s,\omega)\big|+\big|y(s,\omega)\big|+\big|z(s,\omega)\big|\big)\,ds.$$

By the three above inequalities, we get

$$\big|x(t,\omega)\big|+\big|y(t,\omega)\big|+\big|z(t,\omega)\big| \leq c+\lambda_1 e^{\alpha_1 b}\int_0^t \Gamma(s,\omega)\psi\big(\big|x(s,\omega)\big|+\big|y(s,\omega)\big|+\big|z(s,\omega)\big|\big)\,ds.$$

Applying the Bihari lemma, we obtain

(3.6) $$\left|x(t,\omega)\right| + \left|y(t,\omega)\right| + \left|z(t,\omega)\right| \leq H^{-1}\left(\int_c^t \Gamma(s,\omega)\,ds\right) \quad \text{for each } t \in [0,b],$$

where $H(u) = \int_c^u \frac{du}{\psi(u)}$. Finally from 3.6 there exists a constant $\sigma > 0$ such that

$$\|x\|_{PC} \leq \sigma, \qquad \|y\|_{PC} \leq \sigma \quad \text{and} \quad \|z\|_{PC} \leq \sigma.$$

This shows that $\mathscr{M}$ is bounded. Thus by Theorem 13 the operator T has at least one fixed point which is a random mild solution of problem 1.1. □

4. Random Sadovskiis Fixed Point Theorem Type

In this section, we present the random Sadovskiis fixed point theorem in a vector Banach space. First, we give definitions and properties for a measure of noncompactness.

Definition 25. *Let X be a generalized Banach space and $(\mathscr{A},\leq)$ be a partially ordered set. A map $\beta : \mathscr{P}(X) \to \mathscr{A} \times \mathscr{A} \times \ldots \times \mathscr{A}$ is called a generalized measure of noncompactness (m.n.c.) on X if*

$$\beta(\overline{\text{co}}C) = \beta(C) \quad \textit{for every } C \in \mathscr{P}(X),$$

$C \in \mathscr{P}(X)$, where

$$\beta(C) := \begin{pmatrix} \beta_1(C) \\ \vdots \\ \beta_n(C) \end{pmatrix}.$$

Definition 26. *A measure of noncompactness β is called*

(a) Monotone if $C_0, C_1, C_2 \in \mathscr{P}(X), C_0 \subset C_1 \subset C_2$ implies $\beta(C_0) \leq \beta(C_1) \leq \beta(C_2)$.

(b) Nonsingular if $\beta(\{a\} \cup C) = \beta(C)$ for every $a \in X, C \in \mathscr{P}(X)$.

(c) Invariant with respect to the union with compact sets if $\beta(K \cup C) = \beta(C)$ for every relatively compact set $K \subset X$, and $C \in \mathscr{P}(X)$.

(d) Real if $\mathscr{A} = \overline{\mathbb{R}}_+$ and $\beta(C) < \infty$ for every $i = 1,\ldots,n$ and every bounded C.

(e) Semi-additive if $\beta(C_0 \cup C_1) = \max\{\beta(C_0), \beta(C_1), \beta(C_2)\}$ for every $C_0, C_1, C_2 \in \mathscr{P}(X)$.

(f) Lower-additive if β is real and $\beta(C_0 + C_1 + C_2) \leq \beta(C_0) + \beta(C_1) + \beta(C_2)$ for every $C_0, C_1, C_2 \in \mathscr{P}(X)$.

(g) Regular if the condition $\beta(C) = 0$ is equivalent to the relative compactness of C.

A typical example of an *MNC* is the Hausdorff measure of noncompactness χ defined, for all $C \subset X$, by

$$\chi(C) := \inf\big\{\varepsilon \in \mathbb{R}^n_+ : \text{there exists } n \in \mathbb{N} \text{ such that } C \text{ has finite } \varepsilon\text{-net}\big\}.$$

Definition 27. *Let X,Y be two generalized normed spaces and $F : X \times X \times X \to \mathscr{P}(Y)$ be a multivalued map. F is called an M-contraction (with respect to β) if there exists $M \in \mathscr{M}_{n\times n}(\mathbb{R}_+)$ converging to zero such that, for every $D \in \mathscr{P}(X)$, we have*

$$\beta\big(F(D,D,D)\big) \le M\beta(D).$$

The next result is concerned with β-condensing or M-contractivity.

Theorem 28. *Let $V \subset X$ be a bounded closed convex subset and $N : V \times V \times V \to V$ be a generalized β-condensing continuous mapping, where β is a nonsingular measure of noncompactness defined on the subsets of X. Then the set*

$$\text{Fix}(N) = \big\{(x,y,z) \in V \times V \times V : x = N(x,y,z)\big\}$$

is nonempty.

As a consequence of Theorem 28, we present versions of Schaefers fixed point theorem and the nonlinear alternative Leray-Schauder-type theorem for β-condensing operators in a generalized Banach space.

Theorem 29. *Let E be a generalized Banach space and $N : E \times E \times E \to E$ be a continuous and β-condensing operator. Moreover, assume that the set*

$$A = \big\{(x,y,z) \in E \times E \times E : x = \lambda N(x,y,z) \textit{ for some } \lambda \in (0,1)\big\}$$

is bounded. Then N has a tripled fixed point.

Now Theorems 28, 29 establish the results.

Theorem 30. *Let $(\Omega, \mathscr{F})$ be a measurable space, C be a closed, convex, bounded subset of a separable vector Banach space, and $F : \Omega \times C \times C \times C \to C$ be a continuous condensing random operator. Then F has at least one random tripled fixed point.*

Proof. Let $\omega \in \Omega$. Consider $F_\omega : C \times C \times C \to C$ defined by $F_\omega(x,y,z) = F(\omega,x,y,z)$ By Theorem 29, there exists $(x(\omega), y(\omega), z(\omega)) \in C \times C \times C$ such that

$$(x(\omega), y(\omega), z(\omega)) = F\big(\omega, x(\omega), y(\omega), z(\omega)\big).$$

Define $\mathscr{T} : C \to \mathscr{P}_{cl}(C)$ by

$$\mathscr{T}(\omega) = \big\{x \in X : x = F(\omega,x,y,z)\big\}.$$

Since F is a Carath*é*odory function, then the function $\Psi : \Omega \times C \to \mathbb{R}^n_+$ defined by $\Psi(\omega,x) = d\big(x, F(\omega,x,y,z)\big)$ is also a Carath*é*odory operator. From Theorem 10, the set multivalued map G_p is measurable, so

$$\overline{G_p(\omega)} = \overline{\big\{x \in \Omega : x - F(\omega,x,y,z) \in B(0,\varepsilon_p)\big\}}, \qquad \varepsilon_p = \begin{pmatrix} \frac{1}{p} \\ \vdots \\ \frac{1}{p} \end{pmatrix}, \quad p \in \mathbb{N}.$$

Moreover,

$$\mathscr{T}(\omega) = \bigcap_{n=1}^{\infty} \overline{G_p(\omega)}, \quad \omega \in \Omega.$$

From Theorem 7, there exists a measurable selection $x,y,z : \Omega \to C$ of $\mathscr{T}$ which is a random fixed point of F. □

We can also prove the following result.

Theorem 31. *Let X be a separable generalized Banach space, and let $F : \Omega \times X \times X \times X \to X$ be a condensing continuous random operator. Then either of the following holds:*

(i) The random equation $F(\omega,x,y,z) = x$ has a random solution, i.e., there is a measurable function $x : \Omega \to X$ such that $F(\omega, x(\omega), y(\omega), z(\omega)) = x(\omega)$ for all $\omega \in \Omega$, or

(ii) The set

$$\mathscr{M} = \big\{x : \Omega \to X \text{ is measurable} | \lambda(\omega) F(\omega,x,y,z) = x\big\}$$

is unbounded for some measurable function $\lambda : \Omega \to X$ with $0 < \lambda(\omega) < 1$ on Ω.

Lemma 32. *Let E be a Banach space and $N\colon L^1([a,b],E) \to C([a,b],E)$ be an abstract operator satisfying the following conditions:*

$(\mathscr{S}_1)$: *N is ξ-Lipschitz: there exists $\xi > 0$ such that, for every*

$$f,g \in L^1([a,b],E), \left|Nf(t) - Ng(t)\right| \leq \xi \int_a^b \left|f(s) - g(s)\right| ds \quad \text{for all } t \in [a,b].$$

$(\mathscr{S}_2)$: *N is weakly-strongly sequentially continuous on compact subsets: for any compact $K \subset E$ and any sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([a,b],E)$ such that $\{f_n(t)\}_{n=1}^{\infty} \subset K$ for a.e. $t \in [a,b]$, the weak convergence $f_n \rightharpoonup f_0$ implies the strong convergence $N(f_n) \to N(f_0)$ as $n \to +\infty$.*

Then, for every semi-compact sequence $\{f_n\}_{n=1}^{\infty} \subset L^1([0,b],E)$, the image sequence $N(\{f_n\}_{n=1}^{\infty})$ is relatively compact in $C([a,b],E)$.

Corollary 33. *Let $N : L^1([0,b],E) \to C([0,b],E)$ be defined by*

$$N(f)(t) = \int_0^t S(t-s)f(s)\,ds, \quad t \in [0,b],$$

where $(S(t))_{t\geq 0}$ is a C_0-semigroup, then N satisfies $\mathscr{S}_1$ and $\mathscr{S}_2$.

Lemma 34. *([8], Theorem 5.2.2) Let an operator $N\colon L^1([a,b],E) \to C([a,b],E)$ satisfy conditions $(\mathscr{S}_1) - (\mathscr{S}_2)$ together with*

$(\mathscr{S}_3)$: *There exists $\eta \in L^1([a,b])$ such that, for every integrably bounded sequence $\{f_n\}_{n=1}^{\infty}$, we have*

$$\chi\big(\{f_n(t)\}_{n=1}^{\infty}\big) \leq \eta(t) \quad \text{for a.e. } t \in [a,b],$$

where χ is the Hausdorff MNC.

Then

$$\chi\big(\{N(f_n)(t)\}_{n=1}^{\infty}\big) \leq 2\xi \int_a^b \eta(s)\,ds \quad \text{for all } t \in [a,b],$$

where ξ is the constant in $(\mathscr{S}_1)$.

Now we give our main existence result for problem 1.1 without the compactness of a C_0-semigroup, and there exists $M > 0$ such that

$$\left\|S(t)\right\| \leq M \quad \text{for all } t \in [0,b].$$

We will need to introduce the following hypothesis which is assumed thereafter:

(H7) There exists $p_i : \Omega \to L^1([0,b],\mathbb{R}_+)$ random variable such that, for every bounded D, D' in X,

$$\chi\big(f_i(t,D,D',\omega)\big) \le \overline{p}_i(t,\omega)\chi(D) + p_i(t,\omega)\chi\big(D'\big).$$

Theorem 35. *Under the conditions of Theorem 24 and (H7), problem 1.1 has at least one random mild solution.*

Proof. We are going to study problem 1.1 respectively in the intervals $[0,t_1],(t_1,t_2],\dots,(t_m,b]$. The proof will be given in three steps and then continued by induction.

Step 1. It is clear that all the random mild solutions of problem 3.2 are fixed points of the operator N_1 defined in Theorem 22. For applied Theorem 29, first we prove that N_1 is a $\beta_{0,1}$-condensing operator for a suitable MNC. Given a bounded subset $D \subset C([0,t_1],X)$, let $\mathrm{mod}_C(D)$ the modulus of quasi-equicontinuity of the set of functions D denote

$$\mathrm{mod}_C(D) = \lim_{\delta\to 0}\sup_{x\in D}\max_{|\tau_2-\tau_1|\le\delta}\big|x(\tau_1)-x(\tau_2)\big|.$$

It is well known (see, e.g., Example 2.1.2 in [4] that $\mathrm{mod}_C(D)$ defines an MNC in $C([0,t_1],X)$, which satisfies all of the properties in Definition 26 except regularity. Given the Hausdorff MNC *Chi*, let $\overline{\gamma}_1$ be the real MNC defined on bounded subsets on $C([0,t_1],X)$ by

$$\overline{\gamma}_1(D) = \sup_{t\in[0,t_1]} e^{\frac{2M}{\tau}\int_0^t p(s,\omega)\,ds}\chi\big(D(t)\big), \qquad p(\cdot,\omega) = p_1(\cdot,\omega)+p_2(\cdot,\omega).$$

Finally, define the following MNC on bounded subsets of $D\times D_*\times D_{**} \subset C([0,t_1],X)\times C([0,t_1]\times C([0,t_1],X)$ by

$$\beta_{0,1}(D\times D_*\times D_{**}) := \begin{pmatrix}\beta_1(D)\\ \beta_1(D_*)\\ \beta_1(D_{**})\end{pmatrix},$$

$$\beta_1(D) = \max_{D\in\Delta(C([0,t_1],X))}\big(\gamma_1(D),\mathrm{mod}_C(D)\big),$$

where $\Delta(C([0,t_1],X)\times C([0,t_1],X)\times C([0,t_1],X))$ is the collection of all denumerable subsets of $D\times D_*\times D_{**}$. Then the MNC β is monotone, regular, and nonsingular (see Example 2.1.4 in [4]). This measure is also used in [1] in the discussion of semi-linear evolution differential inclusions. To show that N_1 is $\beta_{0,1}$-condensing, let $B = D\times D\times D \subset C([0,t_1],X)\times C([0,t_1],X)\times C([0,t_1],X)$ be a bounded set in $C([0,t_1],X)\times C([0,t_1],X)\times C([0,t_1],X)$ such that

$$\beta_{0,1}(B) \leq \beta_{0,1}\big(N(B)\big). \tag{4.1}$$

We will show that B is relatively compact. Let $\{(x_n,y_n,z_n) : n\in\mathbb{N}\}\subset B$, and let $N_i^i = L_1^i + L_2^i, i = 1,2,3$ where $L_1^i : C([0,t_1],X) \to C([0,t_1],X)$ is defined by

$$L_1^1\big(x(t,\omega),y(t,\omega),z(t,\omega)\big) = S(t)x_0(\omega),$$

$$L_2^1\big(x(t,\omega),y(t,\omega),z(t,\omega)\big) = S(t)y_0(\omega),$$

and

$$L_3^1\big(x(t,\omega),y(t,\omega),z(t,\omega)\big) = S(t)z_0(\omega),$$

$L_2^i : C([0,t_1],X) \to C([0,t_1],X)$ is defined by

$$L_2^i\big(x(t,\omega),y(t,\omega,z(t,\omega)\big) = \int_0^t S(t-s)f_i\big(s,x(s,\omega),y(s,\omega),z(s,\omega)\big)\,ds, \quad t\in[0,t_1], i = 1,2,3.$$

From assumption (H7), it holds that for a.e. $t\in[0,t_1]$,

$$\begin{aligned}
&\chi\Bigg(f_1\Bigg(s,\bigcup_{n\in\mathbb{N}}\{x_n(t,\omega)\},\bigcup_{n\in\mathbb{N}}\{y_n(t,\omega)\},\bigcup_{n\in\mathbb{N}}\{z_n(t,\omega)\},\omega\Bigg)\\
&\quad\leq \chi\big(f_1\big(s,\{x_n(s,\omega)\}_{n\in\mathbb{N}},\{y_n(s,\omega)\}_{n\in\mathbb{N}},\{z_n(s,\omega)\}_{n\in\mathbb{N}},\omega\big)\big)\\
&\quad\leq p_1(s,\omega)\chi\big(\{x_n(s,\omega)\}_{n\in\mathbb{N}}\big)\\
&\qquad+\overline{p}_1(s,\omega)\chi\big(\{y_n(s,\omega)\}_{n\in\mathbb{N}}\big)+\overline{\overline{p}}_1(s,\omega)\chi\big(\{z_n(s,\omega)\}_{n\in\mathbb{N}}\big)\\
&\quad\leq p_1(s,\omega)e^{3M\tau\int_0^s p(r,\omega)\,dr}e^{-3M\tau\int_0^s p(r,\omega)\,dr}\chi\big(\{x_n(s,\omega)\}_{n\in\mathbb{N}}\big)\\
&\qquad+\overline{p}_1(s,\omega)e^{3M\tau\int_0^s\overline{p}_1(r,\omega)\,dr}e^{-3M\tau\int_0^s p_1(r,\omega)\,dr}\chi\big(\{y_n(s,\omega)\}_{n\in\mathbb{N}}\big)\\
&\qquad+\overline{\overline{p}}_1(s,\omega)e^{3M\tau\int_0^s\overline{\overline{p}}_1(r,\omega)\,dr}e^{-3M\tau\int_0^s p_1(r,\omega)\,dr}\chi\big(\{z_n(s,\omega)\}_{n\in\mathbb{N}}\big)\\
&\quad\leq p_1(t,\omega)e^{3M\tau\int_0^s p(r,\omega)\,dr}\overline{\gamma}_1\big(\{x_n(s,\omega)\}_{n\in\mathbb{N}}\big)\\
&\qquad+e^{3M\tau\int_0^s p(r,\omega)\,dr}\overline{p}_1(s,\omega)\overline{\gamma}_1\big(\{y_n(s,\omega)\}_{n\in\mathbb{N}}\big)\\
&\qquad+e^{3M\tau\int_0^s p(r,\omega)\,dr}\overline{\overline{p}}_1(s,\omega)\overline{\gamma}_1\big(\{z_n(s,\omega)\}_{n\in\mathbb{N}}\big).
\end{aligned}$$

Hence

$$
\begin{aligned}
&\chi\Big(f_1\Big(s,\bigcup_{n\in\mathbb{N}}\big\{\big(x_n(t,\omega),y_n(t,\omega),z_n(t,\omega)\big)\big\},\omega\Big)\\
&\leq p_1(t,\omega)e^{3M\tau\int_0^s p(r,\omega)\,dr}\overline{\gamma}_1\big(\big\{x_n(s,\omega)\big\}_{n\in\mathbb{N}}\big)\\
&\quad+\overline{p}_1(s,\omega)e^{3M\tau\int_0^s p(r,\omega)\,dr}\overline{\gamma}_1\big(\big\{y_n(s,\omega)\big\}_{n\in\mathbb{N}}\\
&\quad+\overline{\overline{p}}_1(s,\omega)e^{3M\tau\int_0^s p(r,\omega)\,dr}\overline{\gamma}_1\big(\big\{z_n(s,\omega)\big\}_{n\in\mathbb{N}}\big).
\end{aligned}
\tag{4.2}
$$

Lemmas 32 and 34 imply that

$$
\begin{aligned}
&\chi\big(\big\{N_1^1\big(x_n(t,\omega),y_n(t,\omega),z_n(t,\omega)\big)\big\}_{n=1}^{\infty}\big)\\
&\leq\overline{\gamma}_1\big(\{x_n\}_{n=1}^{\infty}\big)3M\int_0^t p_1(s,\omega)\,ds\\
&\quad+\overline{\gamma}_1\big(\{y_n\}_{n=1}^{\infty}\big)3M\int_0^t \overline{p}_1(s)\,ds\\
&\quad+\overline{\overline{\gamma}}_1\big(\{z_n\}_{n=1}^{\infty}\big)3M\int_0^t \overline{\overline{p}}_1(s)\,ds\\
&\leq\overline{\gamma}_1\big(\{x_n\}_{n=1}^{\infty}\big)3M\int_0^t p(s,\omega)e^{3M\tau\int_0^s p(r,\omega)\,dr}\,ds\\
&\quad+\overline{\gamma}_1\big(\{y_n\}_{n=1}^{\infty}\big)3M\int_0^t e^{3M\tau\int_0^s p(r,\omega)\,dr}p(s,\omega)\,ds\\
&\quad+\overline{\overline{\gamma}}_1\big(\{z_n\}_{n=1}^{\infty}\big)3M\int_0^t e^{3M\tau\int_0^s p(r,\omega)\,dr}p(s,\omega)\,ds.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&e^{-3M\int_0^t p(s)\,ds}\chi\big(\big\{N_1\big(x_n(t,\omega),y_n(t,\omega),z_n(t,\omega)\big)\big\}_{n=1}^{\infty}\big)\\
&\leq\frac{3M}{\tau}\overline{\gamma}_1\big(\{x_n\}_{n=1}^{\infty}\big)+\frac{3M}{\tau}\overline{\gamma}_1\big(\{y_n\}_{n=1}^{\infty}\big)+\frac{3M}{\tau}\overline{\gamma}_1\big(\{z_n\}_{n=1}^{\infty}\big)
\end{aligned}
$$

and

$$
\chi\big(L_1^1\big(\big\{x_n(t)\big\}_{n=1}^{\infty},\big\{y_n(t)\big\}_{n=1}^{\infty}\big),\big\{z_n(t)\big\}_{n=1}^{\infty}\big)=0.
$$

Therefore

$$
\overline{\gamma}_1\big(\big\{N_1^1\big(x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega)\big)\big\}_{n=1}^{\infty}\big)\leq\frac{3M}{\tau}\overline{\gamma}_1\big(\{x_n\}_{n=1}^{\infty}\big)+\frac{3M}{\tau}\overline{\gamma}_1\big(\{y_n\}_{n=1}^{\infty}\big)+\frac{3M}{\tau}\overline{\gamma}_1\big(\{z_n\}_{n=1}^{\infty}\big).
$$

Similarly, we have

$$
\overline{\gamma}_1\big(\big\{N_2\big(x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega)\big)\big\}_{n=1}^{\infty}\big)\leq\frac{3M}{\tau}\overline{\gamma}_1\big(\{x_n\}_{n=1}^{\infty}\big)+\frac{3M}{\tau}\overline{\gamma}_1\big(\{y_n\}_{n=1}^{\infty}\big)+\frac{3M}{\tau}\overline{\gamma}_1\big(\{z_n\}_{n=1}^{\infty}\big).
$$

and

$$\overline{\gamma}_1\left(\left\{N_3\left(x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega)\right)\right\}_{n=1}^{\infty}\right) \leq \frac{3M}{\tau}\overline{\gamma}_1\left(\{x_n\}_{n=1}^{\infty}\right)+\frac{3M}{\tau}\overline{\gamma}_1\left(\{y_n\}_{n=1}^{\infty}\right)+\frac{3M}{\tau}\overline{\gamma}_1\left(\{z_n\}_{n=1}^{\infty}\right).$$

So

$$\begin{pmatrix}\overline{\gamma}_1(N_1(\{x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega)\}_{n=1}^{\infty}))\\ \overline{\gamma}_1(N_2(\{x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega)\}_{n=1}^{\infty}))\\ \overline{\gamma}_1(N_3(\{x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega)\}_{n=1}^{\infty}))\end{pmatrix} \leq \begin{pmatrix}\frac{3M}{\tau} & \frac{3M}{\tau} & \frac{3M}{\tau}\\ \frac{3M}{\tau} & \frac{3M}{\tau} & \frac{3M}{\tau}\\ \frac{3M}{\tau} & \frac{3M}{\tau} & \frac{3M}{\tau}\end{pmatrix}\begin{pmatrix}\overline{\gamma}_1(\{x_n(\cdot,\omega)\}_{n=1}^{\infty})\\ \overline{\gamma}_1(\{y_n(\cdot,\omega)\}_{n=1}^{\infty})\\ \overline{\gamma}_1(\{z_n(\cdot,\omega)\}_{n=1}^{\infty})\end{pmatrix}.$$

From 4.1, we have

$$\begin{pmatrix}\overline{\gamma}_1(N_1^1(\{x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega)\}_{n=1}^{\infty}))\\ \overline{\gamma}_1(N_2^1(\{x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega)\}_{n=1}^{\infty}))\\ \overline{\gamma}_1(N_3^1(\{x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega)\}_{n=1}^{\infty}))\end{pmatrix} \leq A\begin{pmatrix}\overline{\gamma}_1(N_1^1(\{x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega)\}_{n=1}^{\infty}))\\ \overline{\gamma}_1(N_2^1(\{x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega)\}_{n=1}^{\infty}))\\ \overline{\gamma}_1(N_3^1(\{x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega)\}_{n=1}^{\infty}))\end{pmatrix},$$

where

$$A=\begin{pmatrix}\frac{3M}{\tau} & \frac{3M}{\tau} & \frac{3M}{\tau}\\ \frac{3M}{\tau} & \frac{3M}{\tau} & \frac{3M}{\tau}\\ \frac{3M}{\tau} & \frac{3M}{\tau} & \frac{3M}{\tau}\end{pmatrix}.$$

Since the spectral radius $\rho(A)=\frac{9M}{\tau}<1$, then

$$\overline{\gamma}_1\left(N_1^1\left(\left\{x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega)\right\}_{n=1}^{\infty}\right)\right)=0,$$

$$\overline{\gamma}_1\left(N_2^1\left(\left\{x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega)\right\}_{n=1}^{\infty}\right)\right)=0,$$

$$\overline{\gamma}_1\left(N_3^1\left(\left\{x_n(\cdot,\omega),y_n(\cdot,\omega),z_n(\cdot,\omega)\right\}_{n=1}^{\infty}\right)\right)=0,$$

This implies that

$$\begin{aligned}&\overline{\gamma}_1\left(N_1^1\left(\left\{x_n(t,\omega),y_n(t,\omega),z_n(t,\omega)\right\}_{n=1}^{\infty}\right)\right)=0,\\ &\overline{\gamma}_1\left(N_2^1\left(\left\{x_n(t,\omega),y_n(t,\omega),z_n(t,\omega)\right\}_{n=1}^{\infty}\right)\right)=0\\ &\overline{\gamma}_1\left(N_3^1\left(\left\{x_n(t,\omega),y_n(t,\omega),z_n(t,\omega)\right\}_{n=1}^{\infty}\right)\right)=0 \quad \text{for } t\in[0,t_1].\end{aligned} \tag{4.3}$$

Now, we show that $\mathrm{mod}_C(B) = 0$, i.e., the set

$$B_n = \{ \big(N_1\big(x_n(t,\omega), y_n(t,\omega), z_n(t,\omega)\big),$$

$$N_2\big(\{x_n(t,\omega), y_n(t,\omega), z_n(t,\omega))\big), N_3\big(\{x_n(t,\omega), y_n(t,\omega), z_n(t,\omega))\big)\big)\}_{n=1}^{\infty})]\}$$

for $t \in [0,t_1]$ is equicontinuous, we proceed as in the proof of Theorem 24. It follows that $\mathrm{mod}_C(B_n) = 0$, which implies, by 4.3, that $\beta_{0,1}(B_n) = 0$. We have proved that B is relatively compact. Hence $N_1 : C([0,t_1],X) \times C([0,t_1],X) \times C([0,t_1],X) \to C([0,t_1],X) \times C([0,t_1],X) \times C([0,t_1],X)$ is $\beta_{0,1}$-condensing. As in Theorem 24, N_1 is continuous and, for some random variable $\lambda : \Omega \to (0,1)$, we have

$$\mathscr{M}_1 = \big\{(x,y,z) : C\big([0,t_1],X\big) \times C\big([0,t_1],X\big) \times C\big([0,t_1],X\big) : \lambda(\omega) N_1(\omega,x,y,z) = (x,y,z)\big\}$$

is bounded. As a consequence of Theorem 31, we deduce that N_1 has a fixed point (x,y,z) in $C([0,t_1],X) \times C([0,t_1],X) \times C([0,t_1],X)$, which is a solution to problem 1.1 on $[0,t_1]$. Denote this by (x_0,y_0,z_0).

Step 2. We consider problem 1.1 on $(t_1,t_2]$. It is clear that the fixed points of the operator defined in Theorem 22 are the solutions of 3.3. Thus we only prove that N_1 is a $\beta_{1,2}$-condensing operator. For a bounded subset $B \times B \times B \subset C_*([t_1,t_2],X) \times C_*([t_1,t_2],X) \times C_*([t_1,t_2],X)$, let $\mathrm{mod}_C(B)$ be the modulus of quasi-equicontinuity of the set of functions B, $\overline{\gamma}_2$ be the real MNC defined on a bounded subset on $C_*([t_1,t_2],X)$ by

$$\overline{\gamma}_2(B) = \sup_{t \in [t_1,t_2]} e^{-3M\tau \int_{t_1}^{t} p(r,\omega)\,dr} \chi\big(B(t)\big),$$

and β_2 the MNC defined on $C_*([t_1,t_2],X)$ by

$$\beta_2(B) = \max_{B \in \Delta(C_*([t_1,t_2],X))} \big(\overline{\gamma}_2(B), \mathrm{mod}_C(B)\big),$$

where $\Delta(C_*([t_1,T_2],X)$ is the collection of all denumerable subsets of $D \times D_* \times D_{**}$. So, we define MNC on bounded sets $C_*([t_1,t_2],X) \times C_*([t_1,t_2],X) \times C_*([t_1,t_2],X)$ by

$$\beta_{1,2}(D \times D_* \times D_{**}) := \begin{pmatrix} \beta_2(D) \\ \beta_2(D_*) \\ \beta_2(D_{**}) \end{pmatrix}.$$

As in Step 1, we can prove that N_2 is continuous and $\beta_{1,2}$-condensing. From Theorem 31, we deduce that N_1 has a fixed point (x,y,z) in $C_*([t_1,t_2],X)\times C_*([t_1,t_2],X)\times C_*([t_1,t_2],X)$ denoted by (x_1,y_1,z_1).

Step 3. We consider problem 1.1 on $(t_2,t_3]$. It is clear that the fixed points of the operator defined in Theorem 22 are the solutions of 3.3. Thus we only prove that N_1 is a $\beta_{2,3}$-condensing operator. For a bounded subset $B\times B\times B\subset C_{**}([t_2,t_3],X)\times C_{**}([t_2,t_3],X)\times C_{**}([t_2,t_3],X)$, let $\mathrm{mod}_C(B)$ be the modulus of quasi-equicontinuity of the set of functions B, $\overline{\overline{\gamma}}_3$ be the real MNC defined on a bounded subset on $C_{**}([t_2,t_3],X)$ by

$$\overline{\overline{\gamma}}_3(B)=\sup_{t\in[t_2,t_3]} e^{-3M\tau\int_{t_2}^{t}p(r,\omega)\,dr}\chi\big(B(t)\big),$$

and β_3 the MNC defined on $C_{**}([t_2,t_3],X)$ by

$$\beta_3(B)=\max_{B\in\Delta(C_{**}([t_2,t_3],X))}\big(\overline{\overline{\gamma}}_3(B),\mathrm{mod}_C(B)\big),$$

where $\Delta(C_{**}([t_2,t_3],X)))$ is the collection of all denumerable subsets of $D\times D_*\times D_{**}$. So, we define MNC on bounded sets $C_{**}([t_2,t_3],X)\times C_{**}([t_2,t_3],X)\times C_{**}([t_2,t_3],X)$ by

$$\beta_{2,3}(D\times D_*\times D_{**}):=\begin{pmatrix}\beta_3(D)\\ \beta_3(D_*)\\ \beta_3(D_{**})\end{pmatrix}.$$

As in Step 1, we can prove that N_3 is continuous and $\beta_{2,3}$-condensing. From Theorem 31, we deduce that N_3 has a fixed point (x,y,z) in $C_{**}([t_2,t_3],X)\times C_{**}([t_2,t_3],X)\times C_{**}([t_2,t_3],X)$ denoted by (x_1,y_1,z_1).

Step 4. We continue this process taking into account that $(x_m,y_m,z_m):=(x|_{[t_m,b]},y|_{[t_m,b]},z|_{[t_m,b]})$ is a solution of problem 3.5. A random mild solution (x,y,z) of problem 1.1 is ultimately defined by

$$\big(x(t,\omega),y(t,\omega),z(t,\omega)\big)=\begin{cases}(x_0(t,\omega),y_0(t,\omega),z_0(t,\omega)), & \text{if } t\in[0,t_1],\\ (x(t,\omega),y_1(t,\omega),z_1(t,\omega)), & \text{if } t\in(t_1,t_2],\\ \dots & \dots\\ (x_m(t,\omega),y_m(t,\omega),zkmm_m(t,\omega)), & \text{if } t\in(t_m,b].\end{cases}$$

□

5. EXAMPLES

In this section we use the abstract results proved in the above section to study the existence of a mild solution for random impulsive Stokes and hyperbolic differential equations.

Example 36. *Let (Ω,Σ) be a measurable space and $G \subset \mathbb{R}^3$ be a bounded open domain with the smooth boundary ∂G. Consider the following system of impulsive stochastic Stokes-type partial differential inclusions:*

$$\begin{cases} u_t(t,\xi,\omega) - P(\Delta u(t,\xi,\omega)) = f(t,u(t,\xi,\omega),v(t,x,\omega),w(t,x,\omega),\omega), & a.e.\ t \in [0,b], \xi \in G, \\ v_t(t,\xi,\omega) - P(\Delta v(t,\xi,\omega)) = g(t,u(t,\xi,\omega),v(t,\xi,\omega),w(t,x,\omega)), & a.e.\ t \in [0,b], \xi \in G, \\ w_t(t,\xi,\omega) - P(\Delta w(t,\xi,\omega)) = h(t,u(t,\xi,\omega),v(t,\xi,\omega),w(t,\xi,\omega)), & a.e.\ t \in [0,b], \xi \in G, \\ u(t_k^+,\xi,\omega) - u(t_k^-,\xi,\omega) = I_k(u(t_k,\xi,\omega)), & \\ v(t_k^+,\xi,\omega) - v(t_k,\xi,\omega) = \overline{I}_k(v(t_k,\xi,\omega)), & k = 1,\ldots,m, \\ w(t_k^+,\xi,\omega) - w(t_k,\xi,\omega) = \overline{\overline{I}}_k(w(t_k,\xi,\omega)), & k = 1,\ldots,m, \\ \nabla u = \nabla v = \nabla w = 0, & (t,\xi) \in [0,b] \times \partial G, \\ u(t,\xi,\cdot) = v(t,\xi,\cdot) = w(t,\xi,\cdot) = 0, & (t,\xi) \in [0,b] \times \partial G, \\ u(0,\xi,\cdot) = u(b,\xi,\cdot), \qquad v(0,\xi,\cdot) = v(b,x,\cdot), \qquad w(0,\xi,\cdot) = w(b,x,\cdot), & \xi \in G, \end{cases} \tag{5.1}$$

where $n(x)$ is the outward normal to D at the point $\xi \in \partial G$. Let

$$E = \left\{u \in \left(C_c^\infty(G)\right)^3 : \nabla u = 0 \text{ in } \Omega \text{ and } n \cdot u = 0 \text{ on } \partial G\right\},$$

and let $X = \overline{E}^{(L^2(D))^3}$ be the closure of Y in $(L^2(G))^3$. It is clear that, endowed with the standard inner product of the space $(L^2(G))^3$, defined by

$$\langle u,v\rangle = \sum_{i=1}^{3} \langle u_i,v_i\rangle_{L^2(G)},$$

X is a Hilbert space. Let $P : (L^2(G))^3 \to E$ denote the orthogonal projection of $(L^2(G))^3$ onto E, where $P(\Delta)$ is the Stokes operator. Let $A : D(A) \subset X \to X$ be defined by

$$\begin{cases} D(A) = (H^2(G) \cap H_0^1(G))^3 \cap X, \\ Au = -P(\Delta u), \quad u \in D(A). \end{cases}$$

Lemma 37. *The operator A, defined as above, is the generator of a compact and analytic C_0-semigroup of contractions in X. Let us assume that*

$(\mathscr{K}_1)$:

$$f_i, g_i, h_i : [0,b] \times G \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}, i = 1,2,3,$$

are Carathéodory functions.

$(\mathscr{K}_2)$: $\phi, \psi, \varphi : \Omega \to L^1([0,b], \mathbb{R}_+)$ *are random functions such that*

$$\big|f(t,x,u,v,w,\omega)\big| \le \phi_i(t,\omega),$$

$$\big|g(t,x,u,v,w,\omega)\big| \le \psi_i(t,\omega)$$

and

$$\big|h(t,x,u,v,w,\omega)\big| \le \varphi_i(t,\omega),$$

for $i = 1,2,3$, and for each $(t,x,u,v,w,\omega) \in [0,b] \times G \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \Omega$. Let

$$x(t,\omega)(\xi) = u(t,\xi,\omega), \qquad y(t,\omega)(\xi) = v(t,\xi,\omega), \quad z(t,\omega)(\xi) = w(t,\xi,\omega) \quad t \in [0,b], \xi \in G,$$

$$I_k\big(x(t_k,\omega)\big) = K_k \frac{x(t_k^-,\omega)}{1+|x(t_k^-,\omega)|_X}, \quad \xi \in G, k = 1,\dots,m,$$

$$I_k\big(y(t_k,\cdot,\omega)\big) = \bar{K}_k \frac{y(t_k^-,\omega)}{1+|y(t_k^-,\omega)|_X}, \quad \xi \in \Omega, k = 1,\dots,m,$$

$$I_k\big(z(t_k,\cdot,\omega)\big) = \bar{\bar{K}}_k \frac{z(t_k^-,\omega)}{1+|z(t_k^-,\omega)|_X}, \quad \xi \in \Omega, k = 1,\dots,m,$$

$$x(0,\omega)(\xi) = u(0,\xi,\omega) = u(b,\xi,\omega) = x(b,\omega)(\xi),$$

$$y(0,\omega)(\xi) = v(0,\xi,\omega) = v(b,\xi,\omega) = y(b,\omega)(\xi),$$

$$z(0,\omega)(\xi) = w(0,\xi,\omega) = w(b,\xi,\omega) = z(b,\omega)(\xi), \xi \in G,$$

where $K_k, \bar{K}_k, \bar{\bar{K}}_K \in \mathbb{R}, k = 1, \ldots, m$. Assume that $(\mathscr{K}_1) - (\mathscr{K}_2)$ are satisfied. Thus problem 5.1 can be written in the abstract form

$$(5.2) \quad \begin{cases} x'(t,\omega) - A_1 x(t,\omega) = f_1(t, x(t,\omega), y(t,\omega), z(t,\omega), \omega), & t \in [0,b], \\ y'(t,\omega) - A_2 y(t,\omega) = f_2(t, x(t,\omega), y(t,\omega), z(t,\omega), \omega), & t \in [0,b], \\ z'(t,\omega) - A_3 y(t,\omega) = f_3(t, x(t,\omega), y(t,\omega), z(t,\omega), \omega), & t \in [0,b], \\ x(t_k^+,\omega) - x(t_k^-,\omega) = I_k(x(t_k,\omega)), & \\ y(t_k^+,\omega) - y(t_k^-,\omega) = \bar{I}_k(y(t_k,\omega)), & \\ z(t_k^+,\omega) - z(t_k^-,\omega) = \bar{\bar{I}}_k(z(t_k,\omega)), & k = 1, \ldots, m, \\ x(0,\omega) = x_0(\omega), \qquad y(0,\omega) = y_0(\omega), \qquad z(0,\omega) = z_0(\omega), & \end{cases}$$

where $A_1 = A_2 = A_3 = A$. Since, for each $k = 1, \ldots, m$, we have

$$\left|I_k(x)\right| = \left|K_k \frac{x}{1 + |x|_X}\right|_X \leq |K_k|,$$

$$\left|\bar{I}_k(x)\right| = \left|\bar{K}_k \frac{x}{1 + |x|_X}\right|_X \leq |\bar{K}_k|$$

$$\left|\bar{\bar{I}}_k(x)\right| = \left|\bar{\bar{K}}_k \frac{x}{1 + |x|_X}\right|_X \leq |\bar{\bar{K}}_k|$$

for all $x \in X$. Then Theorem 24 ensures that problem 5.1 possesses at least one solution.

Example 38. *Consider the following hyperbolic system of impulsive partial differential equations:*

$$(5.3)\quad \begin{cases} u_{tt}(t,\xi,\omega)-\Delta u(t,\xi,\omega)=f(t,\xi,u(t,\xi,\omega),v(t,x,\omega),w(t,x,\omega)), & a.e.\ t\in J,\xi\in G,\\ v_{tt}(t,\xi,\omega)-\Delta v(t,\xi,\omega)=g(t,u(t,\xi,\omega),v(t,\xi,\omega),w(t,\xi,\omega)), & a.e.\ t\in J,\xi\in G,\\ w_{tt}(t,\xi,\omega)-\Delta w(t,\xi,\omega)=h(t,u(t,\xi,\omega),v(t,\xi,\omega),w(t,\xi,\omega)), & a.e.\ t\in J,\xi\in G,\\ u(t_k^+,\xi,\omega)-u(t_k^-,\xi,\omega)=I_k(u(t_k,\xi,\omega)), & k=1,\dots,m,\\ v(t_k^+,x,\omega)-v(t_k^-,x,\omega)=\overline{I}_k(v(t_k,\xi,\omega)), & k=1,\dots,m,\\ w(t_k^+,x,\omega)-w(t_k^-,x,\omega)=\overline{\overline{I}}_k(w(t_k,\xi,\omega)), & k=1,\dots,m,\\ u(0,\xi,\omega)=v(0,\xi,\omega)=w(0,\xi,\omega)=0, & \\ \quad (t,\xi)\in[0,b]\times\partial G,\omega\in\Omega, & \\ u(0,\xi,\omega)=u_0(\xi,\omega),\qquad u_t(0,x,\omega)=u_1(\xi,\omega), & \xi\in G,\omega\in\Omega,\\ v(0,\xi,\omega)=v_0(\xi,\omega),\qquad v_t(0,x,\omega)=v_1(\xi,\omega), & \xi\in G,\omega\in\Omega,\\ w(0,\xi,\omega)=w_0(\xi,\omega),\qquad w_t(0,x,\omega)=w_1(\xi,\omega), & \xi\in G,\omega\in\Omega, \end{cases}$$

where G is bounded in $\mathbb{R}^d$ with a sufficiently regular boundary. Let $x,y,z:\Omega\to PC([0,b],L^2(G,\mathbb{R}))$ be defined by

$$x(t,\omega)(\xi)=u(t,\xi,\omega),\qquad y(t,\omega)=v(t,\xi,\omega),\qquad z(t,\omega)=w(t,\xi,\omega)$$

and

$$f_1,f_2,f_3:[0,b]\times L^2(G,\mathbb{R})\times L^2(G,\mathbb{R})\times L^2(G,\mathbb{R})\times\Omega\to L^2(G,\mathbb{R})$$

by

$$f_1\big(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega\big)(\xi)=f\big(t,\xi,u(t,\xi,\omega),v(t,\xi,\omega),w(t,\xi,\omega),\omega\big)$$

$$f_2\big(t,x(t,\omega),y(t,\omega),zy(t,\omega),\omega\big)(\xi)=g\big(t,\xi,u(t,\xi,\omega),v(t,\xi,\omega),w(t,\xi,\omega),\omega\big).$$

and

$$f_3\big(t,x(t,\omega),y(t,\omega),zy(t,\omega),\omega\big)(\xi)=h\big(t,\xi,u(t,\xi,\omega),v(t,\xi,\omega),w(t,\xi,\omega),\omega\big).$$

Hence problem 5.3 can be rewritten in the following form:

$$
(5.4)\qquad
\begin{cases}
x''(t,\omega)-A_1x(t,\omega)=f_1(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega), & t\in[0,b],\\
y''(t,\omega)-A_2y(t,\omega)=f_2(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega), & t\in[0,b],\\
z''(t,\omega)-A_3z(t,\omega)=f_3(t,x(t,\omega),y(t,\omega),z(t,\omega),\omega), & t\in[0,b],\\
x(t_k^+,\omega)-x(t_k^-,\omega)=I_k(x(t_k,\omega)), & k=1,\dots,m,\\
y(t_k^+,\omega)-y(t_k^-,\omega)=\bar{I}_k(y(t_k,\omega)), & k=1,\dots,m,\\
z(t_k^+,\omega)-z(t_k^-,\omega)=\bar{\bar{I}}_k(z(t_k,\omega)), & k=1,\dots,m,\\
x(0,\omega)=x_0(\omega), \qquad x'(0,\omega)=\bar{x}_0(\omega), & \\
y(0,\omega)=y_0(\omega), \qquad y'(0,\omega)=\bar{y}_0(\omega), & \\
z(0,\omega)=z_0(\omega), \qquad z'(0,\omega)=\bar{z}_0(\omega) &
\end{cases}
$$

where

$$A_1=A_2=A_3=A:D(A)=W^{2,2}(G,\mathbb{R})\cap W_0^{1,2}(G,\mathbb{R})\subset L^2(G,\mathbb{R})\to\mathbb{R}$$

defined as

$$Au=\Delta u,\quad u\in W^{2,2}(G,\mathbb{R})\cap W_0^{1,2}(G,\mathbb{R}).$$

The following linear operator

$$\mathscr{A}=\begin{pmatrix}0&0&I\\0&I&0\\A&0&0\end{pmatrix},\qquad D(\mathscr{A})=D(A)\times W_0^{1,2}(G,\mathbb{R}),$$

generates a strongly continuous semigroup (see [5]*). So, we can transform problem 5.4 to a first-order system of random semilinear differential equations in X.*

$$
(5.5)\quad \begin{cases}
X'(t,\omega)-\mathscr{A}X(t,\omega)=F_1(t,X(t,\omega),Y(t,\omega),Z(t,\omega),\omega), & t\in[0,b],\\
Y'(t,\omega)-\mathscr{A}Y(t,\omega)=F_2(t,X(t,\omega),Y(t,\omega),Z(t,\omega),\omega), & t\in[0,b],\\
Z'(t,\omega)-\mathscr{A}Z(t,\omega)=F_3(t,X(t,\omega),Y(t,\omega),Z(t,\omega),\omega), & t\in[0,b],\\
X(t_k^+,\omega)-X(t_k^-,\omega)=R_k(X(t_k,\omega)), & k=1,\dots,m,\\
Y(t_k^+,\omega)-Y(t_k^-,\omega)=\overline{R}_k(Y(t_k,\omega)), & k=1,\dots,m,\\
Z(t_k^+,\omega)-Z(t_k^-,\omega)=\overline{\overline{R}}_k(Z(t_k,\omega)), & k=1,\dots,m,\\
X(0,\omega)=X_0(\omega),\\
Y(0,\omega)=Y_0(\omega),\\
Z(0,\omega)=Z_0(\omega),
\end{cases}
$$

where $F_1,F_2,F_3:[0,b]\times X\times X\times X\times\Omega\to X$ *is defined as*

$$
F_1(t,X,Y,Z,\omega)=\begin{pmatrix}0\\ f_1(t,x_1,y_1,z_1,\omega)\end{pmatrix},
$$

$$
F_2(t,X,Y,Z,\omega)=\begin{pmatrix}0\\ f_2(t,x_1,y_1,z_1,\omega)\end{pmatrix},
$$

$$
F_3(t,X,Y,Z,\omega)=\begin{pmatrix}0\\ f_3(t,x_1,y_1,z_1,\omega)\end{pmatrix},
$$

$$
X=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},\qquad Y=\begin{pmatrix}y_1\\y_2\\y_3\end{pmatrix},\qquad Z=\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix},
$$

$$
R_k\big(X(t_k,\omega)\big)=\begin{pmatrix}0\\ I_k(x_1(t_k,\omega))\end{pmatrix},\qquad \overline{R}_k\big(Y(t_k,\omega)\big)=\begin{pmatrix}0\\ \overline{I}_k(y_1(t_k,\omega))\end{pmatrix},\qquad \overline{\overline{R}}_k\big(Z(t_k,\omega)\big)=\begin{pmatrix}0\\ \overline{\overline{I}}_k(z_1(t_k,\omega))\end{pmatrix},
$$

and

$$
X_0(0,\omega)=\begin{pmatrix}x_0(\omega)\\ \bar{x}_0(\omega)\\ \bar{\bar{x}}_0(\omega)\end{pmatrix},\qquad Y=\begin{pmatrix}y_0(\omega)\\ \bar{y}_0(\omega)\\ \bar{\bar{y}}_0(\omega)\end{pmatrix},\qquad Z=\begin{pmatrix}z_0(\omega)\\ \bar{z}_0(\omega)\\ \bar{\bar{z}}_0(\omega)\end{pmatrix}.
$$

Observe that the semigroup generated by $\mathscr{A}$ is noncompact. Assume that $(\mathscr{K}_1)-(\mathscr{K}_2)$, (H7) are satisfied and $I_k, \bar{I}_k, \bar{\bar{I}}_k$ are continuous functions. Then from Theorem 35 problem 5.3 has at least one random mild solution.

ACKNOWLEDGEMENTS

Author are thankful to referee for their valuable suggestions to improve this paper.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] Graef, JR, Henderson, J, Ouahab, A: Impulsive Differential Inclusions - a Fixed Point Approach. De Gruyter Series in Nonlinear Analysis and Applications, vol. 20. de Gruyter, Berlin (2013).

[2] Bihari, I: A generalisation of a lemma of Bellman and its application to uniqueness problems of differential equations. Acta Math. Acad. Sci. Hung. 7 (1956), 81-94.

[3] Becus, G: Random generalized solutions to the heat equation. J. Math. Anal. Appl. 90 (1977), 93-102.

[4] Djebali, S, Gorniewicz, L, Ouahab, A: Solutions Sets for Differential Equations and Inclusions. De Gruyter Series in Nonlinear Analysis and Applications, vol. 18. de Gruyter, Berlin (2013).

[5] Engel, KJ, Nagel, R: One-Parameter Semigroups for Linear Evolution Equations. Springer, New York (2000).

[6] Vrabie, II: C_0-Semigroups and Applications. North-Holland Mathematics Studies, vol. 191. North-Holland, Amsterdam (2003).

[7] Henderson, J, Ouahab, A, Slimani, M: Existence results for a semilinear system of discrete equations. Int. J. Difference Equ. 12 (2017) , 235-253.

[8] Kamenskii, M, Obukhovskii, V, Zecca, P: Condensing Multi-Valued Maps and Semilinear Differential Inclusions in Banach Spaces. de Gruyter, Berlin (2001).

[9] Kamp*é* de Feriet, J: Random Solutions of Partial Differential Equations. Proc. 3rd Berkeley Sympos. Math. Stat. and Probab., vol. III, pp. 199-208. University of California Press, Berkeley (1956).

[10] Kisielewicz, M: Differential Inclusions and Optimal Control. Kluwer Academic, Dordrecht (1991).

[11] Krawaritis, D, Stavrakakis, N: Perturbations of maximal monotone random operators. Linear Algebra Appl. 84 (1986), 301-310.

[12] Pazy, A: Semigroups of Linear Operators and Applications to Partial Differential Equations. Springer, New York (1983).

[13] Kuratowski, K, Ryll-Nardzewski, C: A general theorem on selectors. Bull. Acad. Pol. Sci., Sr. Sci. Math. Astron. Phys. 13 (1965), 397-403.

[14] Sinacer, ML, Nieto, JJ, Ouahab, A: Random fixed point theorem in generalized Banach space and applications. Random Oper. Stoch. Equ. 24 (2016), 93-112.