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FIXED POINT OF JUNGCK-ABBAS ITERATIVE SCHEME ON HILBERT SPACE UNDER CONTRACTIVE-LIKE MAPPINGS

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Abstract. In this paper, we have tried to established the stability and strong convergence results Jungck-Abbas iterative scheme on Hilbert space under contractive-like mappings. Moreover, we will also show the example that the Jungck-Abbas converges faster that to another three-step iterative schemes.

Keywords: Hilbert space; contraction-like mappings; fixed point theorem; Jungck-Abbas iterative scheme.

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1. INTRODUCTION

In 1976, Rhoades [26] introduced the convergence result of Zamfirescu operators using Mann and Ishikawa iterative schemes. Berinde [3] established the class of operators that is more elaborate than the class Zamfirescu operators. It introduced the convergence results of Ishikawa iteration process from this class of operators. After strong convergence of two-step iterative processes, in 2006, Rafiq studied the convergence of quasi-contractive mappings by a three-step iterative scheme. Olatinwo & Imoru [17] introduces convergence results of the class of generalized Zamfirescu operators under the Jungck-Ishikawa and the Jungck-Mann iteration scheme in and Olatinwo [19] studied the convergence for generalized Zamfirescu

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operators by Jungck-Noor iterative scheme. Bosede [5] introduced strong convergence results of contractive-like mappings with the Jungck-Ishikawa and the Jungck-Mann iterative schemes and Chugh, Renu and Kumar, Vivek [9, 10] studied of the strong convergence and stability result for the Jungck-SP and Jungck-Agrawal et al. iteration procedure in 2011.
In the second case of this work we have the concept of fixed point of stability results of iteration methods. The stability of fixed point iteration methods was forst studied by Ostrowaski [22], where they were the stability result of the Picard iteration method by Banach's contraction operator. After many researchers has studied this concept in various ways. Many researchers studying stability the following are Harder and Hicks [11], Rhoades [26], Berinde [4], Olatinwo [18] Osilike and Udomene [21] Osilike [20], and Singh et al. [27] Bosede and Rhoades [6] and Chugh Renu and Kumar Vivek [8, 9, 10]. The final concept of this task will study the intensity of iterative methods.

Let K be a closed convex subset of a Hilbert space H and let $T : K \to K$ be a self-mapping of K. Suppose that $F_T = \{q \in K : Tq = q\}$ is the set of fixed points of T.
In Picard iterative scheme $u_0 \in K$ and $\{u_s\}_{s=0}^{\infty}$ defined by

$$u_{s+1} = Tu_s,\ s = 0, 1, 2, .. \tag{1}$$

i.e.,

$$u_{s+1} = f(T, u_s),\ s = 0, 1, 2, ..$$

and it is used to approximate the fixed points of operators satisfying the inequality

$$\|Tx - Ty\| \leq \alpha \|x - y\|\ , \forall x, y \in K\ \ \&\ \ \ \alpha \in [0, 1) \tag{2}$$

The equation (2) is called the Banach's contraction operator and the equation (2) is called is Banach's Fixed point theorem [2].
In 1953, W. R. Mann [15] introduced the following iterative scheme, for for $u_0 \in K$ and $\{u_s\}_{s=0}^{\infty}$ defined by

$$u_{s+1} = (1 - \alpha_s) u_s + \alpha_s T u_s \tag{3}$$

where $s \in N$, $\{\alpha_s\}$ is sequence of positive numbers in $[0,1]$ and it is called Mann iterative scheme.

In 1974, S. Ishikawa [12] introduced the following iterative scheme, for $u_0 \in K$ and $\{u_s\}_{s=0}^{\infty}$ defined by

$$\begin{aligned} u_{s+1} &= (1-\alpha_s)\,u_s + \alpha_s T v_s \\ v_s &= (1-\beta_s)\,u_s + \beta_s T u_s \end{aligned} \tag{4}$$

where $s \in N$, $\{\alpha_s\}$, $\{\beta_s\}$ are sequence of positive numbers in $[0,1]$ and it is called Ishikawa iterative scheme.

In 2000, Noor [16] introduced the following iterative scheme, for $u_0 \in K$ and $\{u_s\}_{s=0}^{\infty}$ defined by

$$\begin{aligned} u_{s+1} &= (1-\alpha_s)\,u_s + \alpha_s T v_s \\ v_s &= (1-\beta_s)\,u_s + \beta_s T w_s \\ w_s &= (1-\gamma_s)\,u_s + \gamma_s T u_s \end{aligned} \tag{5}$$

where $s \in N$, $\{\alpha_s\}$, $\{\beta_s\}$, $\{\gamma_s\}$ are sequence of positive numbers in $[0,1]$ and it is called Noor iterative scheme.

In 2014, Abbas [1] introduced the following iterative scheme, for $u_0 \in K$ and $\{u_s\}_{s=0}^{\infty}$ defined by

$$\begin{aligned} u_{s+1} &= (1-\alpha_s)\,T v_s + \alpha_s T v w_s \\ v_s &= (1-\beta_s)\,T u_s + \beta_s T w_s \\ w_s &= (1-\gamma_s)\,u_s + \gamma_s T u_s \end{aligned} \tag{6}$$

where $s \in N$, $\{\alpha_s\}$, $\{\beta_s\}$, $\{\gamma_s\}$ are sequence of positive numbers in $[0,1]$ and it is called Abbas iterative scheme.

Theorem 1.1. *Let K be a non-empty closed convex subset of a Hilbert space H and let $T : K \to K$ be a mapping on K. Then the operator T is called Zamfirescu operator if and only if there exists real numbers $r \in [0,1)$, $s,t \in [0,1/2)$ such that for each u and v in K at least one of the following condition hold:*

(a) $\|Tu - Tv\| \leq r\|u - v\|$

(b) $\|Tu - Tv\| \leq s\{\|u - Tu\| + \|v - Tv\|\}$

(c) $\|Tu - Tv\| \leq t\{\|u - Tv\| + \|v - Tu\|\}$

Then T has a unique fixed point q and the Picard iterative scheme $\{u_n\}$ defined by

$$u_{n+1} = Tu_n,\ n = 0, 1, ...$$

converges to q for any arbitrary but fixed $u_0 \in K$.

In 2005, Berinde [3] discussed a new class of operators on metric space, Banach space and Hilbert space and it is given by

$$\|Tu - Tv\| \leq 2\delta\,\|u - Tu\| + L\,\|u - v\| \tag{7}$$

$\forall u, v \in K$ and $\delta, L \in [0, 1)$

$$\delta = \max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\},\ 0 \leq \delta < 1$$

Theorem 1.2. *[11] Let $T : K \to K$ be a mapping satisfying (3.2.8) and $F(T) \neq \phi$. Let $\{u_s\}_{s=0}^{\infty}$ be defined by iteration scheme (3.2.4). If $\{\alpha_s\}$ and $\{\beta_s\}$ are sequences of positive numbers in $[0, 1]$ such that $\sum_{s=0}^{\infty} \alpha_s = \infty$ and $\sum_{s=0}^{\infty} \beta_s = \infty$ respectively.Then $\{u_s\}_{s=0}^{\infty}$ converges strongly to the fixed point of T.*

2. Preliminaries

In 1976, Jungck introduced iterative scheme [13] as Let T and S be an arbitrary mappings on any set C with values in K satisfy $T(C) \subseteq S(C)$, where $S(C)$ is a complete subspace of K. For $u_0 \in C$, let $\{Su_n\}_{n=0}^{\infty}$ follows as

$$Su_{n+1} = Tu_n, \qquad n = 0, 1, 2, ..$$

$$\text{i.e., } Su_{n+1} = f(T, u_n), \qquad n = 0, 1, 2, .. \tag{8}$$

For $S = Id$ (2.1) convert to Picard iterative scheme.

In 2005, Singh et al [27] introduced iteration prosses as follows, for $u_0 \in C$, let $\{Su_n\}_{n=0}^{\infty}$ defined as

$$Su_{s+1} = (1 - \alpha_s)Su_s + \alpha_s Tu_s \tag{9}$$

where $s \in N$, $\{\alpha_s\}$ is sequence of positive numbers in $[0,1]$ and it is called Jungck-Mann iterative scheme.

In 2008, Olatinwo and Imoru [17] introduced iteration prosses as follows, for $u_0 \in C$, let $\{Su_n\}_{n=0}^{\infty}$ defined as

$$
\begin{aligned}
Su_{s+1} &= (1-\alpha_s)Su_s + \alpha_s T v_s \\
Sv_s &= (1-\beta_s)Su_s + \beta_s T u_s
\end{aligned}
\tag{10}
$$

where $s \in N$, $\{\alpha_s\}$, $\{\beta_s\}$ are sequence of positive numbers in $[0,1]$ and it is called Jungck-Ishikawa iterative scheme.

In 2008, Olatinwo [19] introduced iteration prosses as follows,for $u_0 \in C$, let $\{Su_n\}_{n=0}^{\infty}$ defined as

$$
\begin{aligned}
Su_{s+1} &= (1-\alpha_s)Su_s + \alpha_s T v_s \\
Sv_s &= (1-\beta_s)Su_s + \beta_s T w_s \\
Sw_s &= (1-\gamma_s)Su_s + \gamma_s T u_s
\end{aligned}
\tag{11}
$$

where $s \in N$, $\{\alpha_s\}$, $\{\beta_s\}$, $\{\gamma_s\}$ are sequence of positive numbers in $[0,1]$ and it is called Noor iterative scheme.

Using a new idea, Osilike [20] considered the following contractive condition: there exist a real number $L \geq 0$, $a \in [0,1)$ such that for each $u, v \in K$, we have

$$
\|Tu - Tv\| \leq L\|u - Tu\| + a\|u - v\| \tag{12}
$$

Imoru and Olatinwo [17] extended the results of oslike [20] using the following contractive condition: there exists a real number $a \in [0,1)$ and a monotonic increasing function $\phi : R^+ \to R^+$ such that $\phi(0) = 0$ and $\forall u, v \in K$, we have

$$
\|Tu - Tv\| \leq \phi(\|u - Tv\|) + a\|u - v\| \tag{13}
$$

Jungck [13] studied that the mappings T and S satisfying

$$
\|Tu - Tv\| \leq \alpha\|Su - Sv\|,
$$

$\forall u,v\in K$, $0\leq\alpha<1$ has a unique common fixed point. For $C=K$ & $S=Id$ above operator becomes well know contraction mapping.

Olatinwo [18] introduced an operator that is a generalization of the Zamfirescu operator, called contractive-like operators and defined as

(A) There exists a real number $M\geq 0$, $a\in[0,1)$ and a monotonic increasing function $\phi:R^{+}\to R^{+}$ such that $\phi(0)=0$ and $\forall u,v\in K$, we have

$$\|Tu-Tv\|\leq\phi(\|Su-Tu\|)+a\|Su-Sv\|$$

(B) There exists a real number $M\geq 0$, $a\in[0,1)$ and a monotonic increasing function $\phi:R^{+}\to R^{+}$ such that $\phi(0)=0$ and $\forall u,v\in K$, we have

$$\|Tu-Tv\|\leq\frac{\phi(\|Su-Tv\|)+a\|Su-Sv\|}{1+M\|Su-Tv\|}$$

Bosede and Rhoades [6] introduced an operator that is a generalization of the Zamfirescu operator, called contractive-like operators and defined as: there exists a real number $M\geq 0$, $\delta\in[0,1)$ and a monotonic increasing function $\phi:R^{+}\to R^{+}$ such that $\phi(0)=0$ and $\forall u,v\in K$, we have

$$\|Tu-Tv\|\leq e^{M\|Su-Tv\|}\left[\phi\left\{\|Su-Tv\|\right\}+\delta\|Su-Sv\|\right] \tag{14}$$

In this paper, we define following new Jungck-Abbas iteration process and is defined as follows

$$\begin{aligned} Su_{s+1}&=(1-\alpha_s)Tv_s+\alpha_s Tw_s\\ Sv_s&=(1-\beta_s)Tu_s+\beta_s Tw_s\\ Sw_s&=(1-\gamma_s)Su_s+\gamma_s Tu_s \end{aligned} \tag{15}$$

where $s\in N$, $\{\alpha_s\}$, $\{\beta_s\}$, $\{\gamma_s\}$ are sequence of positive numbers in $[0,1]$ and it is called Jungck-Abbas iterative scheme.

We are emplying this contractive-like operators there exists a real number $M\geq 0$, $a\in[0,1)$ and a monotonic increasing function $\phi:R^{+}\to R^{+}$ such that $\phi(0)=0$ and $\forall u,v\in K$, we have

$$\|Tu-Tv\|\leq e^{M\|Su-Tv\|}\left[\phi\left\{\|Su-Tv\|\right\}+a\|Su-Sv\|\right] \tag{16}$$

We shall need the following lemma and definition:

Definition 2.1. Let $T,S:C\to K$ be nonself mappings on C satisfying $T(C)\subseteq S(C)$, where $S(C)$ is a complete subspace of K and S is an injective mapping. Let b be a coincidence point of T and S i.e. $Sb=Tb=q$ (say). Let, for $u_0\in C$, $\{Su_n\}_{n=0}^{\infty}$ be a iterative scheme (1.4) converging to q. Suppose $\{Sv_n\}_{n=0}^{\infty}\subseteq K$ be an arbitrary sequence and $\varepsilon_n=\|Sv_{n+1}-f(T,v_n)\|$. Then the iterative scheme (2.7) will be called (S,T)-stable iff

$$\lim_{n\to\infty}\varepsilon_n=0\Leftrightarrow\lim_{n\to\infty}Sv_n=q.$$

Lemma 2.1. *Let p be a real number such that $0\le p<1$ and $\{\varepsilon_s\}_{s=0}^{\infty}$ be a sequence of non-negative numbers such that $\lim_{s\to\infty}\varepsilon_s=0$, then for any sequence $\{u_s\}_{n=0}^{\infty}$ is satisfying $u_{s+1}\le\delta u_s+\varepsilon_s, s=0,1,2,....,$ we have $\lim_{s\to\infty}u_s=0$.*

Definition 2.2. Let b_s and a_s be convergent sequences of real numbers and b, a be limits of the sequences b_s and a_s respectively. then b_s is said to converge faster than a_s if $\lim_{s\to0}\left|\frac{b_s-b}{a_s-a}\right|=0$.

In this paper, we have studied the results of convergence by new iterative three-step method for contractive-like operators in Hilbert spaces. Also by an example we will show that this new method is compared to other methods change rapidly and we will show that this new iteration method is equivalent to some other iteration method in the sense of convergence and we study the stability result for the fixed point of this iteration method.

3. Main Results

3.1. Strong Convergence Results of Fixed Point Theorem in a Hilbert Space.

Theorem 3.1. *Let K be a non-empty closed convex subset of a Hilbert space H and let $T,S:K\to K$ be non-self mappings on an arbitrary set C such that $T(C)\subseteq S(C)$, where $S(C)$ is a complete subspace of K and S is an injective mapping. Let y be a coincidence point of T and S i.e. $Ty=Sy=q$. Suppose T, S are satisying the condition (16) and for $u_0\in C$, let $\{Su_m\}_{n=0}^{\infty}$ be a Jungck-Abbas iterative scheme defined by (15). Then the sequence $\{Su_m\}_{n=0}^{\infty}$ converges strongly to q.*

Proof. Let Y be the set of coincidence points of T and S. We use condition (16) to establish that T and S have a unique coincidence point y, i.e. $Sy=Ty=q$ (say). Suppose that $\exists y_1,y_2\in Y$ such that $Sy_1 = Ty_1 = q_1$ and $Sy_2 = Ty_2 = q_2$. If $q_1 = q_2$, then $Sy_1 = Sy_2$ and since S is an injective, it follows that $y_1 = y_2$.

If $q_1 \neq q_2$, then using contractive condition (16) we have

$$\begin{aligned}0 \leq \|q_1 - q_2\| &= \|Ty_1 - Ty_2\| \\ &\leq e^{L\|Sy_1 - Ty_2\|}\{\varphi(\|Sy_1 - Ty_2\|) + a\|Sy_1 - Sy_2\|\} \\ &= a\|q_1 - q_2\|\end{aligned}$$

Which leads to $(1-a)\|q_1 - q_2\| \leq 0$, from which it follows that $(1-a) > 0$ since $a \in [0.1)$ but $\|q_1 - q_2\| \leq 0$, since norm is non-negative, which is a contradiction. Therefore, we have that $\|q_1 - q_2\| = 0$ i.e. $q_1 = q_2 = q$. Since $q_1 = q_2$, then we have that $q_1 = Sy_1 = Ty_1 = Sy_2 = Ty_2 = q_2$ leading to $Sy_1 = Sy_2 \Rightarrow y_1 = y_2 = y$ (since S is an injective). Hence y is a unique coincidence point of T and S.

We now prove that $\{Su_m\}_{m=0}^{\infty}$ converges strongly to q. Therefore, from (15) and (16) we have

$$\begin{aligned}\|Su_{m+1} - q\| &= \|(1-\alpha_m)Tv_m + \alpha_m Tw_m - q\| \\ &\leq (1-\alpha_m)\|Tv_m - q\| + \alpha_m\|Tw_m - q\| \\ &= (1-\alpha_m)\|Tv_m - Ty\| + \alpha_m\|Tw_m - Ty\| \\ &= (1-\alpha_m)\|Ty - Tv_m\| + \alpha_m\|Ty - Tw_m\| \\ &\leq (1-\alpha_m)e^{L\|Sy-Ty\|}\{\varphi(\|Sy - Ty\|) + a\|Sy - Sv_m\|\} \\ &\quad + \alpha_m e^{L\|Sy-Ty\|}\{\varphi(\|Sy - Ty\|) + a\|Sy - Sw_m\|\} \\ &= (1-\alpha_m)a\|Sy - Sv_m\| + \alpha_m a\|Sy - Sw_m\| \\ &= (1-\alpha_m)a\|Sv_m - q\| + \alpha_m a\|Sw_m - q\| \qquad (17)\end{aligned}$$

Now, we have the following estimates:

$$\begin{aligned}
\|Sv_m - q\| &= \|(1-\beta_m)Tu_m + \beta_m Tw_m - q\| \\
&\leq (1-\beta_m)\|Tu_m - q\| + \beta_m \|Tw_m - q\| \\
&= (1-\beta_m)\|Tu_m - Ty\| + \beta_m \|Tw_m - Ty\| \\
&= (1-\beta_m)\|Ty - Tu_m\| + \beta_m \|Ty - Tw_m\| \\
&\leq (1-\beta_m)e^{L\|Sy-Ty\|}\{\varphi(\|Sy-Ty\|) + a\|Sy - Su_m\|\} \\
&\quad + \beta_m e^{L\|Sy-Ty\|}\{\varphi(\|Sy-Ty\|) + a\|Sy - Sw_m\|\} \\
&= (1-\beta_m)a\|Sy - Su_m\| + \beta_m a\|Sy - Sw_m\| \\
&= (1-\beta_m)a\|q - Su_m\| + \beta_m a\|q - Sw_m\| \\
&= (1-\beta_m)a\|Su_m - q\| + \beta_m a\|Sw_m - q\| \qquad (18)
\end{aligned}$$

and

$$\begin{aligned}
\|Sw_m - q\| &= \|(1-\gamma_m)Su_m + \gamma_m Tu_m - q\| \\
&\leq (1-\gamma_m)\|Su_m - q\| + \gamma_m \|Tu_m - q\| \\
&= (1-\gamma_m)\|Su_m - q\| + \gamma_m \|Tu_m - Ty\| \\
&= (1-\gamma_m)\|Su_m - q\| + \gamma_m \|Ty - Tu_m\| \\
&\leq (1-\gamma_m)\|Su_m - q\| \\
&\quad + \gamma_m e^{L\|Sy-Ty\|}\{\varphi(\|Sy-Ty\|) + a\|Sy - Su_m\|\} \\
&= (1-\gamma_m)\|Su_m - q\| + \gamma_m a\|Sy - Su_m\| \\
&= (1-\gamma_m)\|Su_m - q\| + \gamma_m a\|Su_m - q\| \\
&= [1-\gamma_m(1-a)]\|Su_m - q\| \qquad (19)
\end{aligned}$$

Putting the value of (19) in equation (18), we get

$$\begin{aligned}
\|Sv_m - q\| &\leq a(1-\beta_m)\|Su_m - q\| + a\beta_m[1-\gamma_m(1-a)]\|Su_m - q\| \\
&= a[1-\beta_m\gamma_m(1-a)]\|Su_m - q\| \qquad (20)
\end{aligned}$$

Putting the value of (19) and (20) in equation (17), we get

$$\begin{aligned}
\|Su_{m+1}-q\| &\leq a^2(1-\alpha_m)[1-\beta_m\gamma_m(1-a)]\|Su_m-q\| \\
&\quad +\alpha_m a[1-\gamma_m(1-a)]\|Su_m-q\| \\
&= a[a(1-\alpha_m)\{1-\beta_m\gamma_m(1-a)\} \\
&\quad +\alpha_m\{1-\gamma_m(1-a)\}]\|Su_m-q\| \\
&\leq [a(1-\alpha_m)\{1-\beta_m\gamma_m(1-a)\} \\
&\quad +\alpha_m\{1-\gamma_m(1-a)\}]\|Su_m-q\| \\
&\leq [(1-\alpha_m)\{1-\gamma_m(1-a)\} \\
&\quad +\alpha_m\{1-\gamma_m(1-a)\}]\|Su_m-q\| \\
&\leq [1-\gamma_m(1-a)]\|Su_m-q\| \\
&\leq \prod_{i=0}^{m}[1-\gamma_i(1-a)]\|Su_0-q\| \\
&\leq e^{-(1-a)\sum_{i=0}^{m}\gamma_i}\|u_0-q\| \qquad (21)
\end{aligned}$$

Since $0 \leq a < 1$, $\gamma_i \in [0,1]$ and $\sum_{m=0}^{\infty}\gamma_m = \infty$, so $e^{-(1-a)\sum_{i=0}^{m}\gamma_i} \to 0$ as $m \to \infty$.
Hence, it follows from (21) that $\lim_{m\to\infty}\|Su_{m+1}-q\| = 0$. Therefore $\{Su_m\}_{m=0}^{\infty}$ converges strongly to q. □

3.2. Stability Results of Fixed Point Theorem in Hilbert Space.

Theorem 3.2. *Let K be a non-empty closed convex subset of a Hilbert space H and let $T,S : K \to K$ be non-self mappings on an arbitrary set C such that $T(C) \subseteq S(C)$, where $S(C)$ is a complete subspace of K and S is an injective mapping. Let y be a coincidence point of T and S i.e. $Ty = Sy = q$. Suppose T, S are satisying the condition (16) and for $u_0 \in C$, let $\{Su_m\}_{n=0}^{\infty}$ be a Jungck-Abbas iterative scheme (15) converging to q. Then the Jungck-Abbas iterative scheme is $(S,T)-$stable.*

Proof. Suppose $\{Sv_m\}_{m=0}^{\infty} \subset K$ be an arbitrary sequence and define

$$\varepsilon_m = \|Sv_{m+1}-(1-\alpha_m)Tb_m-\alpha_m Tc_m\|$$

where $Sb_m = (1-\beta_m)Tv_m + \beta_m Tc_m$, $Sc_m = (1-\gamma_m)Sv_m + \gamma_m Tv_m$ and let $\lim_{m\to\infty} \varepsilon_m = 0$. Now we have to prove that $\lim_{m\to\infty} Sv_m = q$.

Then it follows from (15) and (16) that

$$
\begin{aligned}
\|Sv_{m+1} - q\| &\leq \|Sv_{m+1} - (1-\alpha_m)Tb_m - \alpha_m Tc_m\| \\
&\quad + \|(1-\alpha_m)Tb_m + \alpha_m Tc_m - q\| \\
&\leq \varepsilon_m + (1-\alpha_m)\|Tb_m - q\| + \alpha_m \|Tc_m - q\| \\
&= (1-\alpha_m)\|Tb_m - Ty\| + \alpha_m \|Tc_m - Ty\| + \varepsilon_m \\
&= (1-\alpha_m)\|Ty - Tb_m\| + \alpha_m \|Ty - Tc_m\| + \varepsilon_m \\
&\leq (1-\alpha_m)e^{L\|Sy-Ty\|}\{\varphi(\|Sy-Ty\|) + a\|Sy - Sb_m\|\} + \varepsilon_m \\
&\quad + \alpha_m e^{L\|Sy-Ty\|}\{\varphi(\|Sy-Ty\|) + a\|Sy - Sc_m\|\} + \varepsilon_m \\
&= (1-\alpha_m)a\|Sy - Sb_m\| + \alpha_m a\|Sy - Sc_m\| + \varepsilon_m \\
&= (1-\alpha_m)a\|Sb_m - q\| + \alpha_m a\|Sc_m - q\| + \varepsilon_m \qquad (22)
\end{aligned}
$$

Now, we have the following estimates:

$$
\begin{aligned}
\|Sb_m - q\| &= \|(1-\beta_m)Tv_m + \beta_m Tc_m - q\| \\
&\leq (1-\beta_m)\|Tv_m - q\| + \beta_m \|Tc_m - q\| \\
&= (1-\beta_m)\|Tv_m - Ty\| + \beta_m \|Tc_m - Ty\| \\
&= (1-\beta_m)\|Ty - Tv_m\| + \beta_m \|Ty - Tc_m\| \\
&\leq (1-\beta_m)e^{L\|Sy-Ty\|}\{\varphi(\|Sy-Ty\|) + a\|Sy - Sv_m\|\} \\
&\quad + \beta_m e^{L\|Sy-Ty\|}\{\varphi(\|Sy-Ty\|) + a\|Sy - Sc_m\|\} \\
&= (1-\beta_m)a\|Sy - Sv_m\| + \beta_m a\|Sy - Sc_m\| \\
&= (1-\beta_m)a\|q - Sv_m\| + \beta_m a\|q - Sc_m\| \\
&= (1-\beta_m)a\|Sv_m - q\| + \beta_m a\|Sc_m - q\| \qquad (23)
\end{aligned}
$$

and

$$\begin{aligned}
\|Sc_m - q\| &= \|(1-\gamma_m)Sv_m + \gamma_m Tv_m - q\| \\
&\leq (1-\gamma_m)\|Sv_m - q\| + \gamma_m \|Tv_m - q\| \\
&= (1-\gamma_m)\|Sv_m - q\| + \gamma_m \|Tv_m - Ty\| \\
&= (1-\gamma_m)\|Sv_m - q\| + \gamma_m \|Ty - Tv_m\| \\
&\leq (1-\gamma_m)\|Sv_m - q\| + \gamma_m e^{L\|Sy-Ty\|} \\
&\quad \{\varphi(\|Sy-Ty\|) + a\|Sy - Sv_m\|\} \\
&= (1-\gamma_m)\|Sv_m - q\| + \gamma_m a \|Sy - Sv_m\| \\
&= (1-\gamma_m)\|Sv_m - q\| + \gamma_m a \|Sv_m - q\| \\
&= [1-\gamma_m(1-a)]\|Sv_m - q\| \qquad (24)
\end{aligned}$$

Putting the value of (24) in equation (23), we get

$$\begin{aligned}
\|Sb_m - q\| &\leq a(1-\beta_m)\|Sv_m - q\| + a\beta_m[1-\gamma_m(1-a)]\|Sv_m - q\| \\
&= a[1-\beta_m\gamma_m(1-a)]\|Sv_m - q\| \qquad (25)
\end{aligned}$$

Putting the value of (25) and (24) in equation (22), we get

$$\begin{aligned}
\|Sv_{m+1} - q\| &\leq a^2(1-\alpha_m)[1-\beta_m\gamma_m(1-a)]\|Sv_m - q\| \\
&\quad + \alpha_m a[1-\gamma_m(1-a)]\|Sv_m - q\| + \varepsilon_m \\
&= [a^2(1-\alpha_m)\{1-\beta_m\gamma_m(1-a)\} \\
&\quad + a\alpha_m\{1-\gamma_m(1-a)\}]\|Sv_m - q\| + \varepsilon_m \\
&= h\|Sv_m - q\| + \varepsilon_m \qquad (26)
\end{aligned}$$

where,

$$h = [a^2(1-\alpha_m)\{1-\beta_m\gamma_m(1-a)\} + a\alpha_m\{1-\gamma_m(1-a)\}]$$

Taking the limit of both sides of above equation as $m \to \infty$

$$\lim_{m\to\infty}\|Sv_{m+1} - q\| \leq h\lim_{m\to\infty}\|Sv_m - q\| + \lim_{m\to\infty}\varepsilon_m$$

But $\lim_{m\to\infty} \varepsilon_m = 0$. Therefore,

$$\lim_{m\to\infty} \|Sv_{m+1} - q\| \leq h \lim_{m\to\infty} \|Sv_m - q\| \tag{27}$$

Now, using $0 < A \leq \alpha_m$ and $a \in [0,1)$, we have

$$\begin{aligned} h &= a[a(1-\alpha_m)\{1-\beta_m\gamma_m(1-a)\} + \alpha_m\{1-\gamma_m(1-a)\}] \\ &\leq [a(1-\alpha_m)\{1-\beta_m\gamma_m(1-a)\} + \alpha_m\{1-\gamma_m(1-a)\}] \\ &\leq [(1-\alpha_m)\{1-\gamma_m(1-a)\} + \alpha_m\{1-\gamma_m(1-a)\}] \\ &\leq [1-\gamma_m(1-a)] < 1 \end{aligned}$$

Therefore,

$$h = [a^2(1-\alpha_m)\{1-\beta_m\gamma_m(1-a)\} + a\alpha_m\{1-\gamma_m(1-a)\}] < 1 \tag{28}$$

Using (27), (28) and lemma (2.1), we have

$$\lim_{m\to\infty} Sv_m = p.$$

Conversely,let $\lim_{m\to\infty} Sv_m = q$.
Then using contractive condition (16) and the triangle inequality, we have

$$\begin{aligned} \varepsilon_m &= \|Sv_{m+1} - (1-\alpha_m)Tb_m - \alpha_m Tc_m\| \\ &\leq \|Sv_{m+1} - q\| + (1-\alpha_m)\|q - Tb_m\| + \alpha_m\|q - Tc_m\| \\ &= \|Sv_{m+1} - q\| + (1-\alpha_m)\|Ty - Tb_m\| + \alpha_m\|Ty - Tc_m\| \\ &\leq \|Sv_{m+1} - q\| + (1-\alpha_m)e^{L\|Sy-Ty\|}\{\varphi(\|Sy-Ty\|) + a\|Sy - Sb_m\|\} \\ &\quad + \alpha_m e^{L\|Sy-Ty\|}\{\varphi(\|Sy-Ty\|) + a\|Sy - Sc_m\|\} \\ &= \|Sv_{m+1} - q\| + (1-\alpha_m)a\|Sy - Sb_m\| + \alpha_m a\|Sy - Sc_m\| \\ &= \|Sv_{m+1} - q\| + (1-\alpha_m)a\|Sb_m - q\| + \alpha_m a\|Sc_m - q\| \end{aligned} \tag{29}$$

Putting the value of equations (25) and (24) in equation (29), we have

$$\begin{aligned}
\varepsilon_m &\leq \|Sv_{m+1} - q\| + a^2(1-\alpha_m)[1-\beta_m\gamma_m(1-a)]\|Sv_m - q\| \\
&\quad + \alpha_m a[1-\gamma_m(1-a)]\|Sv_m - q\| \\
&= \|Sv_{m+1} - q\| + [a^2(1-\alpha_m)\{1-\beta_m\gamma_m(1-a)\} \\
&\quad + a\alpha_m\{1-\gamma_m(1-a)\}]\|Sv_m - q\| \\
&= \|Sv_{m+1} - q\| + \delta\|Sv_m - q\| \qquad (30)
\end{aligned}$$

where,

$$\delta = [a^2(1-\alpha_m)\{1-\beta_m\gamma_m(1-a)\} + a\alpha_m\{1-\gamma_m(1-a)\}]$$

Taking the limit of both sides of above equation as $m \to \infty$

$$\lim_{m\to\infty} \varepsilon_m \leq \lim_{m\to\infty} \|Sv_{m+1} - q\| + h \lim_{m\to\infty} \|Sv_m - q\|$$

But $\lim_{m\to\infty} Sv_m = q$. Therefore, $\lim_{m\to\infty} \varepsilon_m = 0$.
Hence the Jungck-Abbas iterative scheme is $(S,T)-$stable.

□

3.3. Experimentsle.

We solve the following equation

$$x^2 - 7x - 10 = 0 \qquad (31)$$

and prove that Jungck-Abbas iteration converges faster than Jungck-Noor, jungck-Ishikawa and Jungck-Mann iterative schemes. We solve the equation by rewriting as

$$Sx = 7x \ \& \ Tx = x^2 - 10 \qquad (32)$$

To solve equation (4.1) using simple iterative schemes (Abbas, Noor, Ishikawa and Mann), we write this equation as follows:

$$x = Tx = (x^2 - 10)/7$$

Then the coincidence point of S & T in (32) leads to the solution of equation (31). We show our output in the table 1 and table 2 by taking initial approximation $x_0 = 1$ and $\alpha_n = \beta_n = \gamma_n = 0.9$.

3.4. Conclution. Following are the convergence events of Jungck type iterative processes in table-1: Jungck-Mann, Jungck-Ishikawa, Jungck-Noor and Jungck-Abbas iterative schemes while table-2 follows the convergence events of simple iteration schemes Mann, Ishikawa, Noor and Abbas. After looking at both the tables, we come to the conclusion that the result of table-2 is table-1 changes faster than results.

TABLE 1.

Nu.	Jungck-Mann Iterative Scheme			Jungck-Ishikawa Iterative Scheme			Jungck-Noor Iterative Scheme			Jungck-Abbas Iterative Scheme		
n	Su_{m+1}	Tu_m	u_{m+1}	Su_{m+1}	Tu_m	u_{m+1}	Su_{m+1}	Tu_m	u_{m+1}	Su_{m+1}	Tu_m	u_{m+1}
0	-7.4	-9	-1.05714	-7.2942	-9	-1.04203	-7.32276	-9	-1.04611	-8.83276	-9	-1.26182
1	-8.7342	-8.88245	-1.24774	-8.32247	-8.91418	-1.18892	-8.45874	-8.90566	-1.20839	-8.53997	-8.4078	-1.22
2	-8.47224	-8.44314	-1.21032	-8.48639	-8.58646	-1.21234	-8.51605	-8.53979	-1.21658	-8.52031	-8.51161	-1.21719
3	-8.52884	-8.53512	-1.21841	-8.51352	-8.53023	-1.21622	-8.5188	-8.51994	-1.21697	-8.51902	-8.51845	-1.217
4	-8.51682	-8.51549	-1.21699	-8.51803	-8.52082	-1.21686	-8.51893	-8.51898	-1.21699	-8.51894	-8.5189	-1.21699
5	-8.51938	-8.51967	-1.21705	-8.51878	-8.51925	-1.21697	-8.51893	-8.51894	-1.21699	**-8.51893**	**-8.51893**	**-1.21699**
6	-8.51884	-8.51878	-1.21698	-8.51891	-8.51899	-1.21699	**-8.51893**	**-8.51893**	**-1.21699**	**-8.51893**	**-8.51893**	**-1.21699**
7	-8.51895	-8.51897	-1.21699	-8.51893	-8.51894	-1.21699	**-8.51893**	**-8.51893**	**-1.21699**	**-8.51893**	**-8.51893**	**-1.21699**
8	**-8.51893**	**-8.51893**	**-1.21699**	-8.51893	-8.51894	-1.21699	**-8.51893**	**-8.51893**	**-1.21699**	**-8.51893**	**-8.51893**	**-1.21699**
9	**-8.51893**	**-8.51893**	**-1.21699**	**-8.51893**	**-8.51893**	**-1.21699**	**-8.51893**	**-8.51893**	**-1.21699**	**-8.51893**	**-8.51893**	**-1.21699**
10	**-8.51893**	**-8.51893**	**-1.21699**	**-8.51893**	**-8.51893**	**-1.21699**	**-8.51893**	**-8.51893**	**-1.21699**	**-8.51893**	**-8.51893**	**-1.21699**

TABLE 2.

Nu.	Mann Iterative Scheme		Ishikawa Iterative Scheme		Noor Iterative Scheme		Abbas Iterative Scheme	
n	Tu_m	u_{m+1}	Tu_m	u_{m+1}	Tu_m	u_{m+1}	Tu_m	u_{m+1}
0	-1.28571	-1.05714	-1.28571	-1.04203	-1.28571	-1.04611	-1.28571	-1.26182
1	-1.26892	-1.24774	-1.27345	-1.18892	-1.27224	-1.20839	-1.20111	-1.22
2	-1.20616	-1.21032	-1.22664	-1.21234	-1.21997	-1.21658	-1.21594	-1.21719
3	-1.2193	-1.21841	-1.2186	-1.21622	-1.21713	-1.21697	-1.21692	-1.217
4	-1.2165	-1.21669	-1.21726	-1.21686	-1.217	-1.21699	-1.21699	-1.21699
5	-1.2171	-1.21705	-1.21704	-1.21697	-1.21699	-1.21699	-1.21699	-1.21699
6	-1.21697	-1.21698	-1.217	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699
7	-1.217	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699
8	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699
9	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699
10	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699	-1.21699

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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