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A UNIFIED GENERALIZATION OF THE BROUWER FIXED POINT THEOREM

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Abstract. In 1991, we obtained a generalization of the Brouwer fixed point theorem based on the Fan-Browder fixed point theorem. In the present article, we generalize the 1991 theorem for the class of half-continuous multifunctions due to Bich in 2006 and show that this new result implies and improves fixed point results of Termwuttipong and Kaewtem in 2010 and ourselves in 1992. Consequently, a large number of generalizations or equivalents of the Brouwer theorem are unified.

Keywords: fixed point; Fan-Browder fixed point theorem; half-continuous multifunction.

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1. Introduction

The well-known Brouwer fixed point theorem states that a continuous function f from a nonempty compact convex subset of a Euclidean space into itself has a fixed point. The Brouwer theorem has numerous generalizations and applications related to compact convex subsets in various types of topological vector spaces; see [7].

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In our 1991 paper [5], we gave a generalization of the Brouwer theorem for a broader class of functions $f : X \to E$, where X is a nonempty compact convex subset of a topological vector space E on which E^* separates points. Moreover, in our 1992 paper [6], we obtained coincidence, fixed point, and surjectivity theorems, and existence theorems on critical points for a larger class of multifunctions than upper hemicontinuous ones defined on convex sets. In [6], we also dealt with a larger class of multifunctions having more general boundary conditions than weakly inward (weakly outward, resp.) ones. Moreover, the main theorem of [6] was also appeared in [7-9].

Later in 2006, Bich [1] defined a new class of *half-continuous* multifunctions on Banach spaces and obtained fixed point theorems for such class. In 2008, Hering et al. [3] obtained a generalization of the Brouwer fixed point theorem for possibly discontinuous functions defined on polytopes in $\mathbb{R}^n$. Moreover, in the same year, Bich [2] showed that his half-continuous functions include functions in [3]. Furthermore, in 2010, Termwuttipong and Kaewtem [11] showed that some results due to Bich and many other results were also valid for half-continuous multifunctions in Hausdorff topological vector spaces which are locally convex or have sufficiently many linear functionals.

In the present article, we obtain a generalization of the Brouwer fixed point theorem on half-continuous multifunctions generalizing the main theorem of [5], and show that this new result implies and improves many known generalizations of the Brouwer theorem appeared in our previous works [5-9], Bich [1] and Termwuttipong and Kaewtem [11].

This article is organized as follows: In the Preliminary Section 2, we introduce the Fan-Browder fixed point theorem as the base of this article. Section 3 deals with the main theorem generalizing the corresponding one in [5] to half-continuous functions. In Section 4, we show that the results of 1991 paper [5] follow from the main theorem. Section 5 deals with improvements of the results of Bich [1] and Termwuttipong and Kaewtem [11].

In Section 6, we show that our 1992 definition of generalized upper hemicontinuous (g.u.h.c.) multifunctions are half-continuous under certain restrictions, and obtain a fixed point theorem. Section 7 deals with our 1992 unified fixed point theorem appeared in [6-9], It is shown that two important particular cases of the 1992 theorem can be obtained from our theorems on half-continuous functions. Finally, in Section 8, an extended version of the Schauder conjecture is raised as a problem.

In this article, we restrict ourselves within Analytical Fixed Point Theory, not in Metric Fixed Point Theory, Topological Fixed Point Theory, nor Order Theoretic Fixed Point Theory. Also we adopt the classical logic, not the intuitionism logic, as Brouwer used to do for his theorem.

2. PRELIMINARIES

In 1912, the Brouwer fixed point theorem appeared:

Theorem 2.1. (Brouwer) *A continuous map from an n-simplex to itself has a fixed point.*

It is clear that, in this theorem, the n-simplex can be replaced by the unit ball $\mathbb{B}^n$ or any compact convex subset of $\mathbb{R}^n$.

Our proof of the main theorem in the present article is based on the partition of unity argument and the following Fan-Browder fixed point theorem [7]:

Theorem 2.2. (Fan-Browder) *Let X be a nonempty compact convex subset of a topological vector space E and T a multifunction on X such that, for each $x \in X$, Tx is a nonempty convex subset of X and, for each $y \in X$, $T^{-1}y = \{x \in X : y \in Tx\}$ is open in X. Then there is an $x_0 \in X$ such that $x_0 \in Tx_0$.*

It is well-known that Theorem 2.2 is equivalent to the Brouwer theorem, the weak Sperner lemma, the Knaster-Kuratowski-Mazurkiewicz theorem, and many others; see [7]. Originally, Browder assumed the Hausdorffness of E in Theorem 2.2, but it is known to be redundant later.

For a (real or complex) topological vector space E, let E^* denote its dual space, that is, the vector space of all continuous linear functionals defined on E.

For any $X \subset E$, let $\mathrm{Bd}\,A$, $\mathrm{Int}\,A$, and $\overline{A}$ denote the boundary, interior, and closure, resp., of a set $A \subset X$ with respect to E. For any $x \in E$, the *inward and outward sets* of X at x, $I_X(x)$ and $O_X(x)$, resp., are defined by Halpern as follows:

$$I_X(x) = x + \bigcup_{\lambda>0} \lambda(X - x) \text{ and } O_X(x) = x + \bigcup_{\lambda<0} \lambda(X - x).$$

A multifunction $F : X \multimap E$ is said to be *inward* (*outward*, resp.) if

$$F(x) \cap I_X(x) \neq \emptyset \ \ [F(x) \cap O_X(x) \neq \emptyset, \text{resp.}]$$

for each $x \in \operatorname{Bd} X \setminus F(x)$; see [6].

More generally, a multifunction $F : X \multimap E$ is said to be *weakly inward* (*weakly outward*, resp.) if

$$F(x) \cap \overline{I_X(x)} \neq \emptyset \ \ [F(x) \cap \overline{O_X(x)} \neq \emptyset, \text{resp.}]$$

for each $x \in \operatorname{Bd} X \setminus F(x)$; see [6].

From now on, Hausdorff topological vector spaces are abbreviated as t.v.s.

3. Main Result

The following definition is motivated by [1, 11]:

Definition 3.1. Let X be a subset of a t.v.s. E. A function $f : X \to E$ is said to be *half-continuous* if for each $x \in X$ with $x \neq f(x)$ there exist $p \in E^*$ and an open neighborhood W of x in X such that

$$\operatorname{Re} p(f(y)) > \operatorname{Re} p(y)$$

for all $y \in W$ with $y \neq f(y)$.

The following is our main result:

Theorem 3.2. *Let X be a nonempty compact convex subset of a t.v.s. E and $f : X \to E$ a weakly inward (weakly outward, resp.) half-continuous function. Then f has a fixed point.*

Proof. Suppose that $x \neq f(x)$ for each $x \in X$. Since f is half-continuous, there exists a $p_x \in E^*$ and an open neighborhood W_x of x in X such that

$$\operatorname{Re} p_x(f(y)) > \operatorname{Re} p_x(y)$$

for all $y \in W_x$ with $y \neq f(y)$. Since X is compact and $\{W_x\}_{x \in X}$ is an open cover of X, there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of X such that $\{W_{x_i}\}$ covers X. Note that X is normal since it is Hausdorff and compact. Therefore, we may find a partition of unity corresponding this finite subcover of X, that is, a family of continuous functions $\alpha_1, \alpha_2, \ldots, \alpha_n : X \to [0,1]$ with the support of each α_i lying in W_{x_i} such that $\Sigma\alpha_i = 1$ on X. We form a function ϕ from X into the vector space of all continuous real functions on E by setting

$$\phi(x) = \sum_{i=1}^{n} \alpha_i(x)(\mathrm{Re}\, p_{x_i}) \text{ for each } x \in X.$$

If $\alpha_i(x) \neq 0$, then $x \in \mathrm{Supp}\, \alpha_i \subset W_{x_i}$ and hence $\mathrm{Re}\, p_{x_i}(x) < \mathrm{Re}\, p_{x_i}(f(x))$. As $\alpha_i(x) \neq 0$ for at least one i, we have

$$\phi(x)(x) = \sum \alpha_i(x)(\mathrm{Re}\, p_{x_i}(x)) < \sum \alpha_i(x)(\mathrm{Re}\, p_{x_i}(f(x)) = \phi(x)(f(x)). \tag{1}$$

Now, define a multifunction $T : X \multimap X$ by

$$Tx = \{y \in X : \phi(x)(y-x) > 0\} \text{ for each } x \in X.$$

Since $\phi(x)$ is $\mathbb{R}$-linear, that is, $\phi(x)(ry+sz) = r\phi(x)(y) + s\phi(x)(z)$ for all $y, z \in E$ and $r, s \in \mathbb{R}$, Tx is convex for each $x \in X$. Since $\alpha_i(x)$ and p_{x_i} are continuous, for each $y \in X$, $x \mapsto \phi(x)(y-x) = \Sigma\alpha_i(x)(\mathrm{Re}\, p_{x_i}(y-x))$ is continuous. Therefore, $T^{-1}y = \{x \in X : \phi(x)(y-x) > 0\}$ is open. Note that $x \notin Tx$ for each $x \in X$. Hence, by Theorem 2.2, there exists an $x_0 \in X$ such that Tx_0 is empty, that is,

$$\phi(x_0)(y) \leq \phi(x_0)(x_0) \quad \text{for all } y \in X. \tag{2}$$

However, since $\phi(x_0) : X \to \mathbb{R}$ is continuous and $\mathbb{R}$-linear, (2) holds for all $y \in \overline{I_X(x_0)}$. If $x_0 \in \mathrm{Int}\, X$, then $fx_0 \in E = \overline{I_X(x_0)}$, and hence (2) holds for $y = f(x_0)$, which contradicts (1). If $x_0 \in \mathrm{Bd}\, X$, then we have the same contradiction. Therefore, f must have a fixed point.

For the outward case, if f is weakly outward, then the function $g : X \to E$ defined by $g(x) = 2x - f(x)$ is weakly inward and has the same fixed point with f. Note that $\{x \in X : \mathrm{Re}\, p(x) < \mathrm{Re}\, p(g(x))\} = \{x \in X : \mathrm{Re}\,(-p)(x) < \mathrm{Re}\,(-p)(f(x))\}$ for all $p \in E^*$. $\square$

Remark 3.3. Instead of Theorem 2.2, we can apply the Brouwer Theorem 2.1 in the above proof.

4. The 1991 Generalization of the Brouwer Theorem

In our 1991 paper [5], we gave a generalization of the Brouwer theorem for a broader class of functions $f : X \to E$, where X is a nonempty compact convex subset of a t.v.s. E on which E^* separates points.

Corollary 4.1. ([5]) *Let X be a nonempty compact convex subset of a t.v.s. E on which E^* separates points, and $f : X \to E$ a weakly inward (weakly outward, resp.) function such that*

$$\{x \in X : \operatorname{Re} p(x) < \operatorname{Re} p(f(x))\}$$

is open for all $p \in E^*$. Then f has a fixed point.

We show that this corollary follows from Theorem 3.2.

Proof. Suppose that $x \neq f(x)$ for each $x \in X$. Since E^* separates points of E, there exists a $p_x \in E^*$ such that $\operatorname{Re} p_x(x) < \operatorname{Re} p_x(f(x))$. Since $\{y \in X : \operatorname{Re} p_x(y) < \operatorname{Re} p_x(f(y))\}$ is open, there exists an open neighborhood W_x of x such that

$$\operatorname{Re} p_x(y) < \operatorname{Re} p_x(f(y)) \text{ for all } y \in W_x.$$

Therefore $f : X \to E$ is a half-continuous function and our main Theorem 3.2 works. □

In [5], we added some remarks and examples on Corollary 4.1.

From the above proof, we have the following:

Corollary 4.2. *Let X be a nonempty compact convex subset of a t.v.s. E on which E^* separates points, and $f : X \to E$ a function such that*

$$\{x \in X : \operatorname{Re} p(x) < \operatorname{Re} p(f(x))\}$$

is open for all $p \in E^*$. Then f is half-continuous.

Note that, according to Research Gate, March 19, 2020, the paper [5] has 1,015 Reads and *only* 4 Citations.

From now on in this paper, we are mainly concerned with *real* t.v.s. for simplicity.

5. The 2010 Generalizations of the Brouwer Theorem

In this section, we consider mainly the generalizations of the Brouwer fixed point theorem due to Termwuttipong and Kaewtem [11] in 2010 for a real t.v.s. Recall that Section 3 of [11] contains various useful properties of half continuous functions. There it begins with the following:

Definition 5.1. ([11]) Let X be a subset of a real t.v.s. E. A function $f : X \to E$ is said to be *half-continuous* if for each $x \in X$ with $x \neq f(x)$ there exist $p \in E^*$ and a neighborhood W of x in X such that

$$\langle p, f(y) - y \rangle > 0$$

for all $y \in W$ with $y \neq f(y)$.

This is first introduced for Banach spaces by Bich [1] with many examples. Moreover, Bich [2] showed that a "locally gross direction preserving function" in [3] is half-continuous. Note that Definition 5.1 is nothing else than Definition 3.1 when E is real.

From our Theorem 3.2, we immediately have the following:

Theorem 5.2. *Let X be a nonempty compact convex subset of a t.v.s. E. If $f : X \to X$ is half-continuous, then f has a fixed point.*

This was given by Bich [4, Theorem 3.1] for Banach spaces, and by [11, Corollary 3.10] for a locally convex t.v.s.

Lemma 5.3. ([11]) *Let E be a t.v.s. on which E^* separates points and X a nonempty subset of E. Then every continuous function $f : X \to E$ is half-continuous.*

This is [11, Proposition 3.2] with some examples showing that continuous functions are not comparable to half-continuous functions in general. Moreover, there are examples [11, Proposition 3.6 and 3.7] of half-continuous functions and a theorem [11, Theorem 3.9] on coincidence sets of two functions.

Note that, in view of Lemma 5.3, Theorem 5.2 generalizes theorems due to Brouwer, Schauder, Tychonoff, and others; see [5-9, 11].

For multifunctions, the half-continuity is extended as follows:

Definition 5.4. ([1]) Let X be a subset of a t.v.s. E. A multifunction $F : X \multimap E$ is said to be *half-continuous* if for each $x \in X$ with $x \notin F(x)$ there exist $p \in E^*$ and a neighborhood W of x in X such that

$$\forall y \in W,\ y \notin F(y) \Longrightarrow \forall z \in F(y),\ \langle p, z - y \rangle > 0.$$

With this definition, it is known that

Lemma 5.5. ([11]) *Let X be a nonempty subset of a t.v.s. E and $F : X \multimap E$. If F is half-continuous, then it has a half-continuous selection.*

This can be strengthened as follows:

Lemma 5.6. *Let X be a nonempty subset of a t.v.s. E and $F : X \multimap E$. If F is a weakly inward (weakly outward, resp.) half-continuous multifunction, then it has a weakly inward (weakly outward, resp.) half-continuous selection.*

Proof. Assume F is half-continuous and f any selection of F such that $f(x) \in F(x) \cap \overline{I_X(x)}$ [$f(x) \in F(x) \cap \overline{O_X(x)}$, resp.] for each $x \in \mathrm{Bd}\, X \setminus F(x)$. Define $\tilde{f} : X \to E$ by

$$\tilde{f}(x) = \begin{cases} x, & \text{if } x \in F(x); \\ f(x), & \text{if } x \notin F(x). \end{cases}$$

Clearly $\tilde{f}$ is a weakly inward (weakly outward, resp.) selection of F.

We claim that $\tilde{f}$ is half-continuous. Let $x \in X$ be such that $x \neq \tilde{f}(x)$. Then $x \notin F(x)$ and hence there exist $p \in E^*$ and a neighborhood W of x in X such that

$$\forall y \in W,\ y \notin F(y) \implies \forall z \in F(y),\ \langle p, z - y\rangle > 0.$$

Therefore $\langle p, \tilde{f}(y) - y\rangle = \langle p, f(y) - y\rangle > 0$ for every $y \in W$ with $y \neq \tilde{f}(y)$. □

From Lemma 5.6 and our main Theorem 3.2, we have the following extension of Theorem 3.2:

Theorem 5.7. *Let X be a nonempty compact convex subset of a t.v.s. E and $F : X \multimap E$ a weakly inward (weakly outward, resp.) half-continuous multifunction. Then F has a fixed point.*

Note that this is actually equivalent to Theorem 3.2 and includes several statements in [11, Corollaries 3.10 and 3.12, Theorems 4.6, 5.5, and 5.8] as particular cases when E is locally convex.

Corollary 5.8. *Let X be a nonempty compact convex subset of a t.v.s. E. Suppose that $f : C \to E$ is half-continuous and for each $x \in C$ with $x \neq f(x)$ there exists $\lambda < 1$ such that $\lambda x + (1 - \lambda) f(x) \in C$. Then f has a fixed point.*

Corollary 5.9. *Let X be a nonempty compact convex subset of a t.v.s. E. Then every inward (or outward) half-continuous multifunction $F : X \multimap E$ has a fixed point.*

Note that [11, Theorems 5.1 and 5.8] are Corollaries 5.8 and 5.9, resp., when E is locally convex.

Corollary 5.10. *Let X be a nonempty compact convex subset of a t.v.s. E. Then every half-continuous multifunction $F : X \multimap X$ has a fixed point.*

This was first given by Bich [1, Theorem 3.2] for Banach spaces and by [11, Corollary 3.10] for locally convex spaces. From these results, Corollary 5.10 includes theorems due to Kakutani-Fan-Glicksberg; see [5, 6, 11].

6. For Generalized u.h.c. Multifunctions

Recall that u.s.c. multifunctions are extended as follows; see [6]. Given a subset X of a real t.v.s. E, a multifunction $F : X \multimap E$ is called

(i) *upper demicontinuous* (u.d.c.) if for each $x \in X$ and open half-space H in E containing $F(x)$, there exists an open neighborhood N of x in X such that $F(N) \subset H$.

(ii) *upper hemicontinous* (u.h.c.) if for each $p \in E^*$ and for any real α, the set $\{x \in X : \sup p(F(x)) < \alpha\}$ is open in X.

(iii) *generalized u.h.c.* (g.u.h.c.) if for each $p \in E^*$, the set $\{x \in X : \sup p(F(x)) \geq p(x)\}$ is closed in X.

Note that

$$\text{continuous} \Longrightarrow \text{u.s.c.} \Longrightarrow \text{u.d.c.} \Longrightarrow \text{u.h.c.} \Longrightarrow \text{g.u.h.c.}$$

For details and applications of these types of multifunctions, see the literature in [7].

We show that g.u.h.c. implies half-continuous under some restrictions:

Lemma 6.1. *Let X be a nonempty subset of a locally convex t.v.s. E. If $F : X \multimap E$ is a generalized u.h.c. multifunction with nonempty closed convex values, then F is half continuous.*

Proof. Consider any $x \in X$ with $x \notin F(x)$. Since $\{x\}$ is compact, $F(x)$ is closed, and both are convex, by the standard separation theorem [10, p.58], there exist $p_x \in E^*$ and $r_1,\ r_2 \in \mathbb{R}$ such that

$$p_x(x) < r_1 < r_2 < p_x(z) \text{ for every } z \in F(x).$$

We can choose a set

$$U_x = \{y \in X : p_x(y) < p_x(z) \text{ for every } z \in F(y)\}$$

$$= \{y \in X : (-p_x)(z) < (-p_x)(y) \text{ for every } z \in F(y)\}$$

Now, we claim that U_x contains an open neighborhood W_x of x. Since F is g.u.h.c., its complement

$$(U_x)^c = \{y \in X : (-p_x)(z) \geq (-p_x)(y) \text{ for some } z \in F(y)\}$$

is contained in a closed set

$$\{y \in X : \sup(-p_x)(F(y)) \geq (-p_x)(y)\}.$$

Let W_x be the complement of this set. Then for any $y \in W_x$, we have $y \notin F(y)$ and, moreover, for all $z \in F(y)$, we have $\langle p_x, z - y\rangle > 0$. Consequently, F is half continuous. □

Another standard separation theorem [10, p.70] also works for the similar proof of the following:

Lemma 6.2. *Let X be a nonempty subset of a t.v.s. E on which E^* separates points. If $F : X \multimap E$ is a generalized u.h.c. multifunction with nonempty compact convex values, then F is half continuous.*

Proof. Consider any $x \in X$ with $x \notin F(x)$. Since $\{x\}$ and $F(x)$ are disjoint nonempty compact convex subsets in E, by the standard separation theorem [10, p.70], there exist $p_x \in E^*$ such that

$$\sup p_x(F(x)) < p_x(x).$$

We can choose a set

$$U_x = \{y \in X : p_x(z) < p_x(y) \text{ for every } z \in F(y)\}.$$

Now, we claim that U_x contains an open neighborhood W_x of x. Since F is g.u.h.c. and compact-valued, its complement

$$(U_x)^c = \{y \in X : p_x(z) \geq p_x(y) \text{ for some } z \in F(y)\}$$

is contained in a closed set

$$\{y \in X : \sup p_x(F(y)) \geq p_x(y)\}.$$

Let W_x be the complement of this set. Then for any $y \in W_x$, we have $y \notin F(y)$ and, moreover, for all $z \in F(y)$, we have $\langle p_x, y - z \rangle > 0$, that is, $\langle -p_x, z - y \rangle > 0$. Hence, by considering $-p_x$ instead of p_x, we conclude that F is half continuous. □

Note that Lemma 6.2 reduces to Corollary 4.2 for single-valued functions and that Lemmas 6.1 and 6.2 reduce [11, Propositions 4.2 and 4.4], resp., when F is u.d.c. Moreover, Lemma 6.2 implies [11, Proposition 3.2] when $F = f$ is continuous. Furthermore, in view of Lemma 6.2, Corollary 4.1 follows from Theorem 3.2 and can be restated as follows:

Theorem 6.3. *Let X be a nonempty compact convex subset of a t.v.s. E on which E^* separates points, and $f : X \to E$ a weakly inward (weakly outward, resp.) generalized u.h.c. function. Then f has a fixed point.*

Note that f is half-continuous by Lemma 6.2.

7. THE 1992 GENERALIZATIONS OF THE BROUWER THEOREM

In our 1992 paper [6], we obtained coincidence, fixed point, and surjectivity theorems, and existence theorems on critical points for a larger class of multifunctions than upper hemicontinuous ones defined on convex sets. Moreover, we also deal with a larger class of multifunctions having more general boundary conditions than weakly inward (weakly outward, resp.) ones.

In [6], we unified a large number of generalizations of the Kakutani theorem to maps of the above-mentioned types.

Let $cc(E)$ denote the set of nonempty closed convex subsets of a t.v.s. E and $kc(E)$ the set of nonempty compact convex subsets of E.

Let X be a nonempty convex subset of a vector space E. The *algebraic boundary* $\delta_E(X)$ of X in E is the set of all $x \in X$ for which there exists $y \in E$ such that $x + ry \notin X$ for all $r > 0$. If E is a t.v.s., the *topological boundary* Bd $X = \mathrm{Bd}_E X$ of X is the complement of $\mathrm{Int}_E X$ in the closure $\overline{X}$. It is known that $\delta_E(X) \subset \mathrm{Bd}\, X$ and in general $\delta_E(X) \neq \mathrm{Bd}\, X$.

For $p \in E^*$ and $U, V \subset E$, let

$$d_p(U,V) = \inf\{|p(u-v)| \ : \ u \in U, \ v \in V\}.$$

According to Lassonde [4], a *convex space* X is a nonempty convex set with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A nonempty subset L of a convex space X is called a *c-compact set* if for each finite set $N \subset X$ there is a compact convex set $L_N \subset X$ such that $L \cup N \subset L_N$.

The following is one of the main theorems in [6]:

Theorem 7.1. ([6]) *Let X be a convex space, L a c-compact subset of X, K a nonempty compact subset of X, E a t.v.s. containing X as a subset, and F a multifunction satisfying either*

(A) *E^* separates points of E and $F : X \to kc(E)$, or*

(B) *E is locally convex and $F : X \to cc(E)$.*

(I) *Suppose that for each $p \in E^*$,*

(0) *$p|_X$ is continuous on X;*

(1) *$X_p := \{x \in X : \inf pF(x) \leq p(x)\}$ is closed in X;*

(2) *$d_p(F(x), \overline{I_X(x)}) = 0$ for every $x \in K \cap \delta_E(X)$; and*

(3) *$d_p(F(x), \overline{I_L(x)}) = 0$ for every $x \in X \setminus K$.*

Then there exists an $x \in X$ such that $x \in F(x)$.

(II) *Suppose that for each $p \in E^*$,*

$(0)'$ *$p|_X$ is continuous on X;*

$(1)'$ *$X_p := \{x \in X : \sup pF(x) \geq p(x)\}$ is closed in X;*

$(2)'$ *$d_p(F(x), \overline{O_X(x)}) = 0$ for every $x \in K \cap \delta_E(X)$; and*

$(3)'$ *$d_p(F(x), \overline{O_L(x)}) = 0$ for every $x \in X \setminus K$.*

Then there exists an $x \in X$ such that $x \in F(x)$. Further, if F is u.h.c., then $F(X) \supset X$.

E	$f : K \longrightarrow K$		$F : K \multimap K$	
I	Brouwer	1912	Kakutani	1941
II	Schauder	1927, 1930	Bohnenblust and Karlin	 1950
III	Tychonoff	1935	Fan Glicksberg	1952 1952
IV	Fan	1964	Granas and Liu	1986

	$f : K \longrightarrow E$		$F : K \multimap E$	
I	Bohl Knaster, Kuratowski and Mazurkiewicz	1904 1929		
II	Rothe	1938		
III	Halpern Fan Reich Sehgal and Singh	1965 1969 1972 1983	Browder Fan Glebov Halpern Cellina Reich Cornet Lasry and Robert Simons	1968 1969 1969 1970 1970 1972, 1978 1975 1975 1986
IV	Halpern and Bergman Kaczynski Roux and Singh Sehgal, Singh and Whitfield	1968 1983 1989 1990	Granas and Liu Park	1986 1988, 1991

			$F : X \multimap E$	
II			Ding and Tan	1992
III			Fan Shih and Tan Jiang	1984 1987, 1988 1988
IV			Park Yuan, Smith, and Lou	1992, 1993 1998

Remark 7.2. The major particular forms of Theorem 7.1 can be adequately summarized by the preceding diagram previously given in [6-9].

In the diagram, the class I stands for that of Euclidean spaces, II for normed vector spaces, III for locally convex t.v.s., and IV for t.v.s. having sufficiently many linear functionals (that is, E^* separates points). Moreover, f stands for single-valued functions and F for multifunctions; and K stands for a nonempty compact convex subset of a space E, and X for a nonempty convex subset of E satisfying certain coercivity conditions with respect to $F : X \multimap E$ with certain boundary conditions.

In fact, Theorem 7.1 implies all of the fixed point theorems in the diagram. Note that, in the diagram, Bohl's theorem in 1904 was well-known to be equivalent to the Brouwer fixed point theorem in 1912.

The following consequence of Theorem 7.1 extends Theorem 6.3 for multifunctions and can be regarded a 2020 version of the Fan-Glicksberg fixed point theorem; see [7-9].

Theorem 7.3. *Let K be a nonempty compact convex subset of a t.v.s. E on which E^* separates points and $F : K \multimap E$ a weakly inward (weakly outward, resp.) g.u.h.c. multifunction with nonempty compact convex values. Then F has a fixed point.*

Proof from Theorem 7.1. For the case (A) and (I) of Theorem 7.1, conditions (0) and (3) hold trivially. Since F is g.u.h.c., condition (1) holds. Moreover, F is weakly inward, that is, $Fx \cap \overline{I_K(x)} \neq \emptyset$ for every $x \in \mathrm{Bd}\, K$. Since $\delta_E(K) \subset \mathrm{Bd}\, K$, condition (2) holds. Therefore F has a fixed point by Theorem 7.1, Case (I).

Similarly, the case (A) and (II) holds. □

However, we can show that Theorem 7.3 follows from Theorem 5.7, a consequence of our main theorem.

Proof from Theorem 5.7. By Lemma 6.2, F is half-continuous. Then, by Theorem 5.7, F has a fixed point. □

Similarly to Theorem 7.3, the case (B) of Theorem 7.1 implies

Theorem 7.4. *Let K be a nonempty compact convex subset of a locally convex t.v.s. E and $F : K \multimap E$ a weakly inward (weakly outward, resp.) g.u.h.c. multifunction with nonempty closed convex values. Then F has a fixed point.*

Note that Theorems 7.3 and 7.4 cover all fixed point theorems for the domain K in the above diagram.

Corollary 7.5. *Let X be a nonempty compact convex subset of a t.v.s. E on which E^* separates points and $F : X \multimap X$ a weakly inward (weakly outward, resp.) g.u.h.c. multifunction. Then $G := \overline{\mathrm{co}\, F} : K \multimap K$ has a fixed point.*

In fact, G is easily seen to be g.u.h.c. and hence half-continuous by Lemma 6.1. Therefore, Corollary 7.5 follows from Theorem 7.3.

Problem 7.6. Is there any generalization of our main theorem which can cover all fixed point theorems in the diagram? Moreover, is there any simple proof of Theorem 7.1 as the above one?

Remark 7.7. We derived our main result from the Fan-Browder fixed point theorem which is equivalent to the Brouwer theorem, and so are its generalizations in this article. In fact, we showed the following in this article:

$$\text{Brouwer} \implies \text{Fan-Browder} \implies \text{(Main) Theorem 3.2}$$

$$\implies \text{Theorem 5.7} \implies \text{Theorems 7.3 and 7.4}$$

$$\implies \text{Their consequences in the table} \implies \text{Brouwer}$$

Moreover, we gave a large number of equivalents of the Brouwer theorem in [7] and others. That is why we used to claim that the Brouwer theorem has nearly one hundred equivalent formulations.

8. Remarks Related to the Schauder Conjecture

Recall the following long-standing *Schauder conjecture*:

Conjecture 1. *Any continuous self function f on a compact convex subset of a Hausdorff t.v.s. has a fixed point.*

This is still open.

Recall that Theorem 5.2 tells that the class of the half-continuous functions is a self function class on a compact convex subset of a Hausdorff t.v.s. having the fixed point property. However, note that a continuous function is not always half-continuous; see Examples 3.3 and 3.4 in [11]. So we have the following:

Problem 2. *Is there any self function class of a compact convex subset of a Hausdorff t.v.s. having the fixed point property and containing continuous function class and half-continuous function class, simultaneously?*

An affirmative solution of Problem 2 would be a resolution of the Schauder conjecture.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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