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COMMON FIXED POINT RESULTS FOR FOUR MAPS IN S-METRIC SPACE WHICH INVOLVE ALTERING DISTANCE FUNCTIONS

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Abstract. In this paper, the existence and the uniqueness of common fixed point for four self-mappings satisfying some generalized contractive condition in S-metric space which involve altering distance function is established. Our theorems extend, unify and generalize some existing results in the literature.

Keywords: S-metric space; common fixed point; altering distance function; compatible mappings.

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1. Introduction and Preliminaries

Fixed point theory is one of the most fruitful and effective tools in mathematics which has enormous applications within as well as outside mathematics. In 1922, Banach established the famous fixed point theorem which called Banach contraction principle. This principle is a forceful tool in nonlinear analysis. It has many applications in solving nonlinear equations. Because of its importance, many results of fixed point in generalized metric spaces have been proved, see [1-5], for example. In [6] Sedghi et al. have introduced the notion of S-metric

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spaces as one of the generalization of metric spaces. For more details regarding this spaces and their properties we refer [6-10].

Now, we recall some notions and lemmas which will be useful later.

Definition 1.1. [6] Let X be a nonempty set. A function $S : X^3 \to [0,\infty)$ is said to be an S-metric on X, if for each $x,y,z,a \in X$,

(1) $S(x,y,z) = 0$ if and only if $x = y = z$,

(2) $S(x,y,z) \leq S(x,x,a)+S(y,y,a)+S(z,z,a)$.

The pair (X,S) is called an S-metric space.

Example 1.2. [6] *We can easily check that the following examples are S-metric spaces.*

(1) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ be a norm on X. Then $S(x,y,z) = \|y+z-2x\|+\|y-z\|$ is an S-metric on X. In general, if X is a vector space over $\mathbb{R}$ and $\|\cdot\|$ is a norm on X. Then it is easy to see that

$$S(x,y,z) = ||\alpha y+\beta z-\lambda x||+||y-z||,$$

where $\alpha+\beta = \lambda$ for every $\alpha,\beta \geq 1$, is an S-metric on X.

(2) Let X be a nonempty set and d_1,d_2 be two ordinary metric on X. Then

$$S(x,y,z) = d_1(x,z)+d_2(y,z),$$

is an S-metric on X.

Lemma 1.3. [9] *Let (X,S) be an S-metric space. Then, we have $S(x,x,y) = S(y,y,x)$, $x,y \in X$.*

Definition 1.4. [10] Let (X,S) be an S-metric space

(1) A sequence $\{x_n\}$ in X converges to x if $S(x_n,x_n,x) \to 0$ as $n \to \infty$, that is for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $S(x_n,x_n,x) < \varepsilon$. This case, we denote by $\lim_{n\to\infty} x_n = x$ and we say that x is the limit of $\{x_n\}$ in X.

(2) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \varepsilon$ for each $n,m \geq n_0$.

(3) The S-metric space (X,S) is said to be complete if every Cauchy sequence is convergent.

Definition 1.5. [7] Let (X,S) and $(\acute{X},\acute{S})$ be two S-metric spaces, and let $f:(X,S)\to(\acute{X},\acute{S})$ be a function. Then f is said to be continuous at a point $a\in X$ if and only if for every sequence x_n in X, $S(x_n,x_n,a)\to 0$ implies $\acute{S}(f(x_n),f(x_n),f(a))\to 0$. A function f is continuous at X if and only if it is continuous at all $a\in X$.

Lemma 1.6. [10] *Let (X,S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty}x_n=x$ and $\lim_{n\to\infty}y_n=y$, then $\lim_{n\to\infty}S(x_n,x_n,y_n)=S(x,x,y)$.*

Definition 1.7. Let (X,S) be an S-metric space. A pair $\{f,g\}$ is said to be compatible if and only if $\lim_{n\to\infty}S(fgx_n,fgx_n,gfx_n)=0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}fx_n=\lim_{n\to\infty}gx_n=t$ for some $t\in X$.

Lemma 1.8. [8] *Let (X,S) be an S-metric space. If there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty}S(x_n,x_n,y_n)=0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty}x_n=t$ for some $t\in X$, then $\lim_{n\to\infty}y_n=t$.*

A new category of contractive fixed point problems was addressed by Khan *et al.* [11]. In their study they introduced the notion of an altering distance function which is a control function that alters distance between two points in a metric space.

Definition 1.9. The function $\psi:[0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is continuous and nondecreasing ;
(ii) $\psi(t)=0\Leftrightarrow t=0$.

The aim of this work is to prove that there is a unique common fixed point for four self-mappings satisfying generalized contractive condition in S-metric space which involve altering distance function. These results extend and generalize several well known compatible recent results in the literature.

2. Main Results

Theorem 2.1. *Let (X,S) be a complete S-metric space. Suppose $\psi:[0,\infty)\to[0,\infty)$ is an altering distance function, and $\varphi:[0,\infty)\to[0,\infty)$ is a lower semi-continuous function with*

$\varphi(t) = 0$ if and only if $t = 0$. Moreover, suppose that f, g, R and T are self-maps with $f(X) \subseteq T(X)$, $g(X) \subseteq R(X)$ and that the pairs $\{f, R\}$ and $\{g, T\}$ are compatible and satisfying the inequality

$$\psi(S(fx, fy, gz)) \leq \psi(\lambda M(x,y,z)) - \varphi(\lambda M(x,y,z)) + LN(x,y,z) \tag{2.1}$$

where

$$\begin{aligned} M(x,y,z) &= \max\{S(Rx,Ry,Tz), S(fx,fx,Rx), S(gz,gz,Tz), \\ &\quad S(fy,fy,gz), S(Tz,Tz,fx)\}, \end{aligned}$$

and

$$\begin{aligned} N(x,y,z) &= \min\{S(Rx,Ry,Tz), S(fx,fx,Rx), S(gz,gz,Tz), \\ &\quad S(fy,fy,gz), S(Tz,Tz,fx)\}, \end{aligned}$$

with $L \geq 0$ and $0 < \lambda < 1$. Then f, g, R and T have a unique common fixed point in X provided that R and T are continuous.

Proof. Let $x_0 \in X$ be an arbitrary point.

Because of $f(X) \subseteq T(X)$, there exists $x_1 \in X$ such that $Tx_1 = fx_0$, and also as $gx_1 \in R(X)$, take $x_2 \in X$ such that $Rx_2 = gx_1$. Continue in this way, we can choose $x_{2n+1} \in X$ such that $Tx_{2n+1} = fx_{2n}$ and $x_{2n+2} \in X$ so that $Rx_{2n+2} = gx_{2n+1}$ for all $n = 0, 1, 2, \ldots$. Let

$$\begin{aligned} y_{2n} &= Tx_{2n+1} = fx_{2n}, \\ y_{2n+1} &= Rx_{2n+2} = gx_{2n+1}, \qquad n \geq 0. \end{aligned}$$

Now, we shall show that $\{y_n\}$ is a Cauchy sequence. We have

$$\begin{aligned} \psi(S(y_{2n}, y_{2n}, y_{2n+1})) &= \psi(S(fx_{2n}, fx_{2n}, gx_{2n+1})) \\ &\leq \psi(\lambda M(x_{2n}, x_{2n}, x_{2n+1})) - \varphi(\lambda M(x_{2n}, x_{2n}, x_{2n+1})) \\ &\quad + LN(x_{2n}, x_{2n}, x_{2n+1}), \end{aligned}$$

where

$$M(x_{2n},x_{2n},x_{2n+1}) = \max\{S(Rx_{2n},Rx_{2n},Tx_{2n+1}),S(fx_{2n},fx_{2n},Rx_{2n}),$$
$$S(gx_{2n+1},gx_{2n+1},Tx_{2n+1}),S(fx_{2n},fx_{2n},gx_{2n+1}),S(Tx_{2n+1},Tx_{2n+1},fx_{2n})\},$$

and

$$N(x_{2n},x_{2n},x_{2n+1}) = \min\{S(Rx_{2n},Rx_{2n},Tx_{2n+1}),S(fx_{2n},fx_{2n},Rx_{2n}),$$
$$S(gx_{2n+1},gx_{2n+1},Tx_{2n+1}),S(fx_{2n},fx_{2n},gx_{2n+1}),S(Tx_{2n+1},Tx_{2n+1},fx_{2n})\}.$$

That is

$$M(x_{2n},x_{2n},x_{2n+1}) = \max\{S(y_{2n-1},y_{2n-1},y_{2n}),S(y_{2n},y_{2n},y_{2n-1}),$$
$$S(y_{2n+1},y_{2n+1},y_{2n}),S(y_{2n},y_{2n},y_{2n+1}),S(y_{2n},y_{2n},y_{2n})\},$$

and

$$N(x_{2n},x_{2n},x_{2n+1}) = \min\{S(y_{2n-1},y_{2n-1},y_{2n}),S(y_{2n},y_{2n},y_{2n-1}),$$
$$S(y_{2n+1},y_{2n+1},y_{2n}),S(y_{2n},y_{2n},y_{2n+1}),S(y_{2n},y_{2n},y_{2n})\}.$$

Therefore,

$$\begin{aligned} M(x_{2n},x_{2n},x_{2n+1}) &= \max\{S(y_{2n-1},y_{2n-1},y_{2n}),S(y_{2n},y_{2n},y_{2n+1}),0\} \\ &= \max\{S(y_{2n-1},y_{2n-1},y_{2n}),S(y_{2n},y_{2n},y_{2n+1})\}, \end{aligned}$$

and

$$N(x_{2n},x_{2n},x_{2n+1}) = \min\{S(y_{2n-1},y_{2n-1},y_{2n}),S(y_{2n},y_{2n},y_{2n+1}),0\} = 0.$$

Now, we want to show that $S(y_{2n-1},y_{2n-1},y_{2n}) \geq S(y_{2n},y_{2n},y_{2n+1})$ for each $n \in \mathbb{N}$. Assume that $S(y_{2n-1},y_{2n-1},y_{2n}) < S(y_{2n},y_{2n},y_{2n+1})$ for some $n \in \mathbb{N}$, then from (2.1) we have

$$\begin{aligned} \psi(S(y_{2n},y_{2n},y_{2n+1})) &\leq \psi(\lambda S(y_{2n},y_{2n},y_{2n+1})) - \varphi(\lambda S(y_{2n},y_{2n},y_{2n+1})) \\ &\leq \psi(\lambda S(y_{2n},y_{2n},y_{2n+1})). \end{aligned}$$

Since ψ is increasing, we get

$$S(y_{2n},y_{2n},y_{2n+1}) \leq \lambda S(y_{2n},y_{2n},y_{2n+1}),$$

which is a contradiction because $0 < \lambda < 1$. Thus, $S(y_{2n}, y_{2n}, y_{2n+1}) \leq S(y_{2n-1}, y_{2n-1}, y_{2n})$, therefore by above inequality we get

$$S(y_{2n}, y_{2n}, y_{2n+1}) \leq \lambda S(y_{2n-1}, y_{2n-1}, y_{2n}). \tag{2.2}$$

By similar arguments we obtain

$$S(y_{2n-1}, y_{2n-1}, y_{2n}) \leq \lambda S(y_{2n-2}, y_{2n-2}, y_{2n-1}). \tag{2.3}$$

From (2.2) and (2.3) we have

$$S(y_n, y_n, y_{n-1}) \leq \lambda S(y_{n-1}, y_{n-1}, y_{n-2}), \qquad n \geq 2,$$

where $0 < \lambda < 1$.

Hence, for $n \geq 2$ it follows that

$$S(y_n, y_n, y_{n-1}) \leq \cdots \leq \lambda^{n-1} S(y_1, y_1, y_0). \tag{2.4}$$

By the triangle inequality in S-metric space, for $n > m$ we get

$$\begin{aligned} S(y_n, y_n, y_m) &\leq 2S(y_m, y_m, y_{m+1}) + 2S(y_{m+1}, y_{m+1}, y_{m+2}) + \cdots + S(y_{n-1}, y_{n-1}, y_n) \\ &< 2S(y_m, y_m, y_{m+1}) + 2S(y_{m+1}, y_{m+1}, y_{m+2}) + \cdots + 2S(y_{n-1}, y_{n-1}, y_n). \end{aligned}$$

Thus, from (2.4) and as $0 < \lambda < 1$ we obtain

$$\begin{aligned} S(y_n, y_n, y_m) &\leq 2(\lambda^m + \lambda^{m+1} + \cdots + \lambda^{n-1}) S(y_1, y_1, y_0) \\ &\leq 2\lambda^m (1 + \lambda + \lambda^2 + \cdots) S(y_1, y_1, y_0) \\ &\leq \frac{2\lambda^m}{1-\lambda} S(y_1, y_1, y_0) \to 0, \text{ as } m \to \infty. \end{aligned}$$

Therefore, $\{y_n\}$ is a Cauchy sequence. Since X is a complete S-metric space, there exists some y in X such that

$$\lim_{n\to\infty} f x_{2n} = \lim_{n\to\infty} T x_{2n+1} = \lim_{n\to\infty} g x_{2n+1} = \lim_{n\to\infty} R x_{2n+2} = y.$$

Next, we want to prove that y is a common fixed point of f, T, g and R.

Since T is continuous, we get that

$$\lim_{n\to\infty} T^2 x_{2n+1} = Ty, \qquad \lim_{n\to\infty} T g x_{2n+1} = Ty.$$

$\lim_{n\to\infty} S(gTx_{2n+1}, gTx_{2n+1}, Tgx_{2n+1}) = 0$ because g and T are compatible. Thus, by Lemma (1.8) $\lim_{n\to\infty} gTx_{2n+1} = Ty$. By take $x = y = x_{2n}$ and $z = Tx_{2n+1}$ in condition (2.1), we have

$$\begin{aligned} \psi(S(fx_{2n}, fx_{2n}, gTx_{2n+1})) &\leq \psi(\lambda M(x_{2n}, x_{2n}, Tx_{2n+1})) - \varphi(\lambda M(x_{2n}, x_{2n}, Tx_{2n+1})) \\ &\quad + LN(x_{2n}, x_{2n}, Tx_{2n+1}) \end{aligned} \tag{2.5}$$

where

$$M(x_{2n}, x_{2n}, Tx_{2n+1}) = \max\{S(Rx_{2n}, Rx_{2n}, T^2x_{2n+1}), S(fx_{2n}, fx_{2n}, Rx_{2n}),$$
$$S(gTx_{2n+1}, gTx_{2n+1}, T^2x_{2n+1}), S(fx_{2n}, fx_{2n}, gTx_{2n+1}), S(T^2x_{2n+1}, T^2x_{2n+1}, fx_{2n})\},$$

and

$$N(x_{2n}, x_{2n}, Tx_{2n+1}) = min\{S(Rx_{2n}, Rx_{2n}, T^2x_{2n+1}), S(fx_{2n}, fx_{2n}, Rx_{2n}),$$
$$S(gTx_{2n+1}, gTx_{2n+1}, T^2x_{2n+1}), S(fx_{2n}, fx_{2n}, gTx_{2n+1}), S(T^2x_{2n+1}, T^2x_{2n+1}, fx_{2n})\}.$$

By taking the upper limit when $n \to \infty$ in (2.5) we obtain

$$\begin{aligned} \psi(S(y, y, Ty)) &= \lim_{n\to\infty} \psi(S(fx_{2n}, fx_{2n}, gTx_{2n+1})) \\ &\leq \psi(\lambda \lim_{n\to\infty} M(x_{2n}, x_{2n}, Tx_{2n+1})) - \varphi(\lambda \lim_{n\to\infty} M(x_{2n}, x_{2n}, Tx_{2n+1})) \\ &\quad + L \lim_{n\to\infty} N(x_{2n}, x_{2n}, Tx_{2n+1}) \end{aligned}$$

where

$$\begin{aligned} \lim_{n\to\infty} M(x_{2n}, x_{2n}, Tx_{2n+1}) &= \max\{S(y, y, Ty), 0, 0, S(y, y, Ty), S(Ty, Ty, y)\} \\ &= S(y, y, Ty), \end{aligned}$$

and

$$\begin{aligned} \lim_{n\to\infty} N(x_{2n}, x_{2n}, Tx_{2n+1}) &= \min\{S(y, y, Ty), 0, 0, S(y, y, Ty), S(Ty, Ty, y)\} \\ &= 0. \end{aligned}$$

That is

$$\begin{aligned}\psi(S(y,y,Ty)) &\leq \psi(\lambda S(y,y,Ty)) - \varphi(\lambda S(y,y,Ty))\\ &\leq \psi(\lambda S(y,y,Ty)),\end{aligned}$$

which is impossible. Hence, $S(y,y,Ty)=0$, it follows that $Ty=y$.

In a similar way, R is continuous so we have

$$\lim_{n\to\infty} R^2 x_{2n+1} = Ry, \qquad \lim_{n\to\infty} Rfx_{2n} = Ry.$$

Since f and R are compatible, $\lim_{n\to\infty} S(fRx_{2n}, fRx_{2n}, Rfx_{2n}) = 0$. Therefore, by Lemma (1.8) $\lim_{n\to\infty} fRx_{2n} = Ry$.

Take $x=y=Rx_{2n}$ and $z=x_{2n+1}$ in condition (2.1), we obtain

$$\begin{aligned}\psi(S(fRx_{2n}, fRx_{2n}, gx_{2n+1})) &\leq \psi(\lambda M(Rx_{2n}, Rx_{2n}, x_{2n+1})) - \varphi(\lambda M(Rx_{2n}, Rx_{2n}, x_{2n+1}))\\ &\quad + LN(Rx_{2n}, Rx_{2n}, x_{2n+1}) \qquad (2.6)\end{aligned}$$

where

$$M(Rx_{2n}, Rx_{2n}, x_{2n+1}) = \max\{S(R^2x_{2n}, R^2x_{2n}, Tx_{2n+1}), S(fRx_{2n}, fRx_{2n}, R^2x_{2n}),$$
$$S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), S(fRx_{2n}, fRx_{2n}, gx_{2n+1}), S(Tx_{2n+1}, Tx_{2n+1}, fRx_{2n})\},$$

and

$$N(x_{2n}, x_{2n}, Tx_{2n+1}) = \min\{S(R^2x_{2n}, R^2x_{2n}, Tx_{2n+1}), S(fRx_{2n}, fRx_{2n}, R^2x_{2n}),$$
$$S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), S(fRx_{2n}, fRx_{2n}, gx_{2n+1}), S(Tx_{2n+1}, Tx_{2n+1}, fRx_{2n})\}.$$

Similarly by taking the upper limit when $n\to\infty$ in (2.6), we get

$$\begin{aligned}\psi(S(Ry,Ry,y)) &= \lim_{n\to\infty} \psi(S(fRx_{2n}, fRx_{2n}, gx_{2n+1}))\\ &\leq \psi(\lambda S(Ry,Ry,y)) - \varphi(\lambda S(Ry,Ry,y)) + 0\\ &\leq \psi(\lambda S(Ry,Ry,y)),\end{aligned}$$

which is impossible. Hence, $S(Ry,Ry,y)=0$, it follows that $Ry=y$. Now, we will show that also $fy=y$ and $gy=y$.

Put y instead of x and $z = x_{2n+1}$ in condition (2.1) to have

$$\begin{aligned} \psi(S(fy, fy, gx_{2n+1})) &\leq \psi(\lambda M(y,y,x_{2n+1})) - \varphi(\lambda M(y,y,x_{2n+1})) \\ &\quad + LN(y,y,x_{2n+1}) \end{aligned} \tag{2.7}$$

where

$$M(y,y,x_{2n+1}) = \max\{S(Ry,Ry,Tx_{2n+1}), S(fy,fy,Ry),$$
$$S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), S(fy,fy,gx_{2n+1}), S(Tx_{2n+1}, Tx_{2n+1}, fy)\},$$

and

$$N(y,y,x_{2n+1}) = \min\{S(Ry,Ry,Tx_{2n+1}), S(fy,fy,Ry),$$
$$S(gx_{2n+1}, gx_{2n+1}, Tx_{2n+1}), S(fy,fy,gx_{2n+1}), S(Tx_{2n+1}, Tx_{2n+1}, fy)\}.$$

And by taking the upper limit when $n \to \infty$ in (2.7), as $Ty = Ry = y$, we obtain

$$\psi(S(fy,fy,y)) \leq \psi(\lambda \max\{S(fy,fy,y), 0, S(y,y,fy)\}) -$$
$$\varphi(\lambda \max\{S(fy,fy,y), 0, S(y,y,fy)\}) + L\min\{S(fy,fy,y), 0, S(y,y,fy)\}.$$

That is

$$\begin{aligned} \psi(S(fy,fy,y)) &\leq \psi(\lambda S(fy,fy,y)) - \varphi(\lambda S(fy,fy,y)) \\ &\leq \psi(\lambda S(fy,fy,y)), \end{aligned}$$

it is impossible, it follows that $S(fy,fy,y) = 0$ and $fy = y$.

Finally, since $Ry = Ty = fy = y$ and by using condition (2.1) with replacing x and z by y, we get

$$\psi(S(fy,fy,gy)) \leq \psi(\lambda M(y,y,y)) - \varphi(\lambda M(y,y,y)) + LN(y,y,y)$$

where

$$\begin{aligned} M(y,y,y) &= \max\{S(Ry,Ry,Ty), S(fy,fy,Ry), S(gy,gy,Ty), \\ &\quad S(fy,fy,gy), S(Ty,Ty,fy)\}, \end{aligned}$$

and

$$\begin{aligned} N(x,y,z) &= \min\{S(Ry,Ry,Ty), S(fy,fy,Ry), S(gy,gy,Ty), \\ &\quad S(fy,fy,gy), S(Ty,Ty,fy)\}, \end{aligned}$$

i.e.,

$$\psi(S(y,y,gy)) \le \psi(\lambda \max\{S(gy,gy,y), 0, , S(y,y,gy)\}) -$$

$$\varphi(\lambda \max\{S(gy,gy,y), 0, S(y,y,gy)\}) + L\min\{S(gy,gy,y), 0, S(y,y,gy)\}.$$

Thus,

$$\begin{aligned} \psi(S(y,y,gy)) &\le \psi(\lambda S(y,y,gy)) - \varphi(\lambda S(y,y,gy)) \\ &\le \psi(\lambda S(y,y,gy)), \end{aligned}$$

which implies that $S(y,y,gy) = 0$ and so $gy = y$. Hence we prove that

$$Ty = Ry = fy = gy = y.$$

For the uniqueness, we suppose that y^* is another common fixed point of all T, R, f and g. Then from (2.1) we have

$$\begin{aligned} \psi(S(y^*,y^*,y)) &= \psi(S(fy^*,fy^*,gy)) \\ &\le \psi(\lambda M(y^*,y^*,y)) - \varphi(\lambda M(y^*,y^*,y)) + LN(y^*,y^*,y) \end{aligned}$$

where

$$\begin{aligned} M(y^*,y^*,y) &= \max\{S(Ry^*,Ry^*,Ty), S(fy^*,fy^*,Ry^*), S(gy,gy,Ty), \\ &\quad S(fy^*,fy^*,gy), S(Ty,Ty,fy^*)\} \\ &= \max\{S(y^*,y^*,y), S(y^*,y^*,y^*), S(y,y,y), S(y,y,y^*)\} \\ &= \max\{S(y^*,y^*,y), 0\} \\ &= S(y^*,y^*,y), \end{aligned}$$

and

$$\begin{aligned} N(y^*,y^*,y) &= \min\{S(Ry^*,Ry^*,Ty),S(fy^*,fy^*,Ry^*),S(gy,gy,Ty), \\ &\quad S(fy^*,fy^*,gy),S(Ty,Ty,fy^*)\} \\ &= \min\{S(y^*,y^*,y),S(y^*,y^*,y^*),S(y,y,y),S(y,y,y^*)\} \\ &= \min\{S(y^*,y^*,y),0\} \\ &= 0. \end{aligned}$$

Thus,

$$\begin{aligned} \psi(S(y^*,y^*,y)) &\leq \psi(\lambda S(y^*,y^*,y)) - \varphi(\lambda S(y^*,y^*,y)) \\ &\leq \psi(\lambda S(y^*,y^*,y)), \end{aligned}$$

which implies that $S(y^*,y^*,y) = 0$ and then $y = y^*$. Hence, y is a unique common fixed point of T,R,f and g. □

From Theorem 2.1, assuming $L = 0$ we deduce the following result.

Corollary 2.2. *Let (X,S) be a complete S-metric space. Suppose $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, and $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$. Moreover, suppose that f,g,R and T are self-maps with $f(X) \subseteq T(X)$, $g(X) \subseteq R(X)$ and that the pairs $\{f,R\}$ and $\{g,T\}$ are compatible and satisfying the inequality*

$$\psi(S(fx,fy,gz)) \leq \psi(\lambda M(x,y,z)) - \varphi(\lambda M(x,y,z))$$

where

$$\begin{aligned} M(x,y,z) &= \max\{S(Rx,Ry,Tz),S(fx,fx,Rx),S(gz,gz,Tz), \\ &\quad S(fy,fy,gz),S(Tz,Tz,fx)\}, \end{aligned}$$

with $0 < \lambda < 1$. Then f,g,R and T have a unique common fixed point in X provided that R and T are continuous.

By taking R and T as identity mappings on X in Theorem 2.1, we obtain the following result:

Corollary 2.3. *Let (X,S) be a complete S-metric space. Suppose $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, and $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$. Moreover, suppose that $f,g : X \to X$ are two mappings such that*

$$\psi(S(fx,fy,gz)) \leq \psi(\lambda M(x,y,z)) - \varphi(\lambda M(x,y,z)) + LN(x,y,z)$$

where

$$\begin{aligned} M(x,y,z) &= \max\{S(x,y,z), S(fx,fx,x), S(gz,gz,z), \\ &\quad S(fy,fy,gz), S(z,z,fx)\}, \end{aligned}$$

and

$$\begin{aligned} N(x,y,z) &= \min\{S(x,y,z), S(fx,fx,x), S(gz,gz,z), \\ &\quad S(fy,fy,gz), S(z,z,fx)\}, \end{aligned}$$

with $L \geq 0$ and $0 < \lambda < 1$. Then f and g have a unique common fixed point in X.

If we take $f = g$ in Corollary 2.3 we obtain the following result:

Corollary 2.4. *Let (X,S) be a complete S-metric space. Suppose $\psi : [0,\infty) \to [0,\infty)$ is an altering distance function, and $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$. Moreover, suppose that $f : X \to X$ is a mapping such that*

$$\psi(S(fx,fy,fz)) \leq \psi(\lambda M(x,y,z)) - \varphi(\lambda M(x,y,z)) + LN(x,y,z)$$

where

$$\begin{aligned} M(x,y,z) &= \max\{S(x,y,z), S(fx,fx,x), S(fz,fz,z), \\ &\quad S(fy,fy,fz), S(z,z,fx)\}, \end{aligned}$$

and

$$\begin{aligned} N(x,y,z) &= \min\{S(x,y,z), S(fx,fx,x), S(fz,fz,z), \\ &\quad S(fy,fy,fz), S(z,z,fx)\}, \end{aligned}$$

with $L \geq 0$ and $0 < \lambda < 1$. Then f has a unique fixed point in X.

By taking f and g as identity mappings on X in Theorem 2.1, we obtain the following:

Corollary 2.5. *Let (X,S) be a complete S-metric space. Suppose $\psi:[0,\infty)\to[0,\infty)$ is an altering distance function, and $\varphi:[0,\infty)\to[0,\infty)$ is a lower semi-continuous function with $\varphi(t)=0$ if and only if $t=0$. Moreover, suppose that $R,T:X\to X$ are continuous mappings onto X and satisfying the inequality*

$$\psi(S(x,y,z)) \leq \psi(\lambda M(x,y,z)) - \varphi(\lambda M(x,y,z)) + LN(x,y,z)$$

where

$$\begin{aligned} M(x,y,z) &= \max\{S(Rx,Ry,Tz),S(x,x,Rx),S(z,z,Tz),\\ &\quad S(y,y,z),S(Tz,Tz,x)\}, \end{aligned}$$

and

$$\begin{aligned} N(x,y,z) &= \min\{S(Rx,Ry,Tz),S(x,x,Rx),S(z,z,Tz),\\ &\quad S(y,y,z),S(Tz,Tz,x)\}, \end{aligned}$$

with $L\geq 0$ and $0<\lambda<1$. Then R and T have a unique common fixed point in X.

Theorem 2.6. *Let (X,S) be a complete S-metric space. Suppose $\psi:[0,\infty)\to[0,\infty)$ is an altering distance function, and $\varphi:[0,\infty)\to[0,\infty)$ is a lower semi-continuous function with $\varphi(t)=0$ if and only if $t=0$. Moreover, suppose that f,g,R and T are self-maps with $f(X)\subseteq T(X)$, $g(X)\subseteq R(X)$ and that the pairs $\{f,R\}$ and $\{g,T\}$ are compatible and satisfying the inequality*

$$\psi(S(fx,fy,gz)) \leq \psi(\lambda M(x,y,z)) - \varphi(\lambda M(x,y,z)) + LN(x,y,z)$$

where

$$\begin{aligned} M(x,y,z) &= \max\{S(Rx,Ry,Tz),S(fx,fx,Rx),S(gz,gz,Tz),\\ &\quad S(fy,fy,gz)\}, \end{aligned}$$

and

$$\begin{aligned} N(x,y,z) &= \min\{S(Rx,Ry,Tz), S(fx,fx,Rx), S(gz,gz,Tz), \\ &\quad S(fy,fy,gz), S(Tz,Tz,fx)\}, \end{aligned}$$

with $L \geq 0$ *and* $0 < \lambda < 1$. *Then* f,g,R *and* T *have a unique common fixed point in* X *provided that* R *and* T *are continuous.*

Proof. The proof is the same as the proof of Theorem 2.1 and so to avoid repetition we omit it. □

If we take $L = 0$, $\psi(t) = t$ and $\varphi(t) = (1-k)t$ with $k < 1$ in Theorem 2.6, we have the following corollary witch is the main result of Sedghi *et al.* ([8], Theorem 2.2)

Corollary 2.7. *Let* (X,S) *be a complete S-metric space. Moreover, suppose that* f,g,R *and* T *are self-maps with* $f(X) \subseteq T(X)$, $g(X) \subseteq R(X)$ *and that the pairs* $\{f,R\}$ *and* $\{g,T\}$ *are compatible and satisfying the inequality*

$$S(fx,fy,gz) \leq \lambda \max\{S(Rx,Ry,Tz), S(fx,fx,Rx), S(gz,gz,Tz), S(fy,fy,gz)\}$$

with $L \geq 0$ *and* $0 < \lambda < 1$. *Then* f,g,R *and* T *have a unique common fixed point in* X *provided that* R *and* T *are continuous.*

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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