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ON SOME KKM THEORETIC RESULTS ON CAT(0) SPACES AND OTHERS

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Abstract. Our typical KKM theoretic results on abstract convex spaces can be applied to the CAT(0) spaces and others. In fact, the KKM theoretic results on the *complete continuous midpoint metric spaces* due to Horvath in 2009 and the *metric spaces with global nonpositive curvature* (NPC) *and convex hull finite property* (CHFP) due to Niculescu–Rovenţa in the same year are consequences of our earlier works. Some later related works are also discussed.

Keywords: abstract convex space; KKM theorem; KKM space; mapping classes $\mathfrak{KC}$, $\mathfrak{KO}$; CAT(0) space.

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1. Introduction

The KKM theory, first called by the author in 1992, is the study on applications of equivalent formulations or generalizations of the KKM theorem due to Knaster, Kuratowski, and Mazurkiewicz in 1929. The KKM theorem is one of the most well-known and important existence principles and provides the foundations for many of the modern essential results in diverse areas of mathematical sciences.

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The KKM theory was first devoted to convex subsets of topological vector spaces mainly by Ky Fan and Granas, and later to the so-called convex spaces by Lassonde, to c-spaces (or H-spaces) by Horvath and others, to generalized convex (G-convex) spaces mainly by the present author. Since 2006, we proposed new concepts of *abstract convex spaces and partial KKM spaces* which are proper generalizations of G-convex spaces and adequate to establish the KKM theory. Now the KKM theory becomes the study of abstract convex spaces, and we obtained a large number of new results in such frame in 2006–2011. For the history of the KKM theory, see our recent article [12].

Even after we established the general theory on abstract convex spaces, some works on very particular spaces such as CAT(0) spaces and some others also appeared. A metric space is a CAT(0) space if it is geodesically connected and if every geodesic triangle in this space is at least as thin as its comparison triangle in Euclidean plane.

In this article, we show that the typical KKM theoretic results on abstract convex spaces appeared in our previous works can be applied to the CAT(0) spaces and others. In fact, the KKM theoretic results on the *complete continuous midpoint metric spaces* due to Horvath [2] in 2009 and the *metric spaces with global nonpositive curvature* (NPC) *and convex hull finite property* (CHFP) due to Niculescu-Rovenţa [5] in the same year are consequences of our theory. Some later works on CAT(0) spaces and others also follow them.

This paper is organized as follows: In Section 2, we introduce some basic things on our abstract convex spaces as a preliminary. In Section 3, we introduce the *complete continuous midpoint metric spaces* due to Horvath [2] in 2009 and the *metric spaces with global nonpositive curvature* (NPC) *and convex hull finite property* (CHFP) due to Niculescu-Rovenţa [5] in the same year. In Section 4, we show that the KKM theorem on CAT(0) spaces due to Shabanian-Vaezpour [18] in 2011 follows from ours in 2006. Section 5 deals with some related results on generalized KKM maps in the sense of Kassay-Kolumbán in 1990 and Chang-Zhang in 1991. In Section 6, the minimax inequality due to Niculescu-Rovenţa [5] is shown to be a consequence of our previous works. Finally, in Section 7, we consider several general best approximation theorems and their corollaries.

2. Abstract Convex Spaces

In order to upgrade the KKM theory, in 2006-11, we established new theory of abstract convex spaces and the KKM spaces which are proper generalizations of various known types of particular spaces and adequate to establish the theory.

Recall the following in [7–11]:

Definition 2.1. An *abstract convex space* $(E,D;\Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D\rangle$, where $\langle D\rangle$ is the set of all nonempty finite subsets of D, such that, for any $D' \subset D$, the Γ-*convex hull* of D' is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup\{\Gamma_A \mid A \in \langle D'\rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E,D;\Gamma)$ relative to D' if for any $N \in \langle D'\rangle$, we have $\Gamma_N \subset X$, that is, $\mathrm{co}_\Gamma D' \subset X$.

When $D \subset E$, a subset X of E is said to be Γ-*convex* if $\mathrm{co}_\Gamma(X\cap D) \subset X$; in other words, X is Γ-convex relative to $D' := X\cap D$.

In case $E = D$, let $(E;\Gamma) := (E,E;\Gamma)$.

Definition 2.2. Let $(E,D;\Gamma)$ be an abstract convex space and Z be a set. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_N) \subset G(N) := \bigcup_{y\in N} G(y) \qquad \text{for all } N \in \langle D\rangle,$$

then G is called a *KKM map* with respect to F. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E.

Definition 2.3. A multimap $F : E \multimap Z$ to a set Z is called *a* $\mathfrak{K}$-*map* if, for a KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. We denote

$$\mathfrak{K}(E,Z) := \{F : E \multimap Z \mid F \text{ is a } \mathfrak{K}\text{−map}\}.$$

Similarly, when Z is a topological space, a $\mathfrak{KC}$-*map* is defined for closed-valued maps G, and a $\mathfrak{KO}$-*map* for open-valued maps G. In this case, we denote $F \in \mathfrak{KC}(E,Z)$ [resp. $F \in \mathfrak{KO}(E,Z)$].

Definition 2.4. The *partial KKM principle* for an abstract convex space $(E,D;\Gamma)$ is the statement $1_E \in \mathfrak{KC}(E,E)$; that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y\in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E,E) \cap \mathfrak{KO}(E,E)$; that is, the same property also holds for any open-valued KKM map.

An abstract convex space is called a $($*partial*$)$ *KKM space* if it satisfies the (partial) KKM principle, respectively.

Our KKM theory concerns mainly with the study of partial KKM spaces and their applications.

Remark 2.5. Motivated our earlier works, Chang and Yen in 1996 called a *generalized* KKM map with respect to F instead of our KKM map with respect to F, and called the KKM family for our $\mathfrak{KC}$. Many authors adopted such obsolete terminology of Chang and Yen without recognizing the existence of $\mathfrak{KO}$.

A nonempty topological space X is *homotopically trivial* if for any natural number n and any continuous function $f : \mathrm{Bd}\,\Delta_n \to X$, defined on the boundary of the standard n-dimensional simplex Δ_n, there exists its continuous extension $g : \Delta_n \to X$.

Definition 2.6. A triple $(X \supset D;\Gamma)$ is called an H-*space* if X is a topological space and $\Gamma = \{\Gamma_A\}$ a family of contractible subsets of X indexed by $A \in \langle D\rangle$ such that $\Gamma_A \subset \Gamma_B$ whenever $A \subset B \in \langle D\rangle$. If $D = X$, $(X;\Gamma) := (X,X;\Gamma)$ is called a c-space by Horvath in 1991.

In case Γ is a family of homotopically trivial sets, then $(X \supset D;\Gamma)$ will be called a *Horvath space* which is more general than H-spaces and satisfies clearly the following well-known diagram for subclasses of abstract convex spaces $(E,D;\Gamma)$. See [13, 14]:

Simplex $\Longrightarrow$ Convex subset of a t.v.s. $\Longrightarrow$ Lassonde's convex space

$\Longrightarrow$ Horvath space $\Longrightarrow$ G-convex space $\Longrightarrow$ ϕ_A-space

$\Longrightarrow$ KKM space $\Longrightarrow$ Partial KKM space

$\Longrightarrow$ Abstract convex space.

3. Horvath Spaces and Global NPC Spaces

In the last decade the KKM theoretic results on the Hadamard manifolds have been extended to various types of H-spaces or Horvath spaces or partial KKM spaces. Such types include the *complete continuous midpoint metric spaces* due to Horvath [2] in 2009 and the *metric spaces with global nonpositive curvature* (NPC) *and convex hull finite property* (CHFP) due to Niculescu-Rovenţa [5] in the same year.

Based on our KKM theory on abstract convex spaces, in 2019 [15], we showed that our method can be applied to various types of new spaces. Such results are the KKM theorem, the Fan type minimax inequality, the Fan-Browder fixed point theorem, variational inequalities, von Neumann minimax theorem, Nash equilibrium theorem, etc. Historical remarks are added in [15] to the literature on the KKM type results or others on such new spaces.

In 2009, Horvath [2] introduced the continuous midpoint spaces as a generalization of various types of metric spaces as follows:

Definition 3.1. (Horvath [2]) A *continuous midpoint map* on a metric space (X,d) is a continuous map $\mu : X\times X \to X$ such that, for all $(a,b)\in X\times X,\ d(a,\mu(a,b)) = (1/2)d(a,b) = d(b,\mu(a,b))$. If μ is a continuous midpoint map then $\check{\mu}(a,b) = \mu(b,a)$ is also a continuous midpoint map. The triple (X,d,μ) is called *a continuous midpoint space*. Given a continuous midpoint space (X,d,μ) it is natural to say that a closed subset C of X is convex if, for all $(a,b)\in C\times C,\ \mu(a,b)\in C$.

Example 3.2. Horvath [2] gave a large number of examples of continuous midpoint spaces. We give some of them which are complete as follows:

(1) Closed convex subsets of Banach spaces.

(2) Hyperconvex metric spaces due to Aronszajn-Panitchpackdi.

(3) Hilbert spaces.

(4) Completion of Bruhat-Tits spaces [= Hadamard spaces, that is, complete and simply connected metric spaces of nonpositive curvature (= complete CAT(0) spaces)].

(5) Complete $\mathbb{R}$-trees [= hyperconvex metric spaces with unique metric segments].

(6) Buseman midpoint spaces [includes hyperbolic metric spaces in the sense of Kirk and Reich-Shafrir (see [14])].

For the definitions and the meaning of 'cat' and of CAT(0), see [1].

Recently we defined the following in [13]:

Definition 3.3. An abstract convex space $(X,D;\Gamma)$ is called a *Horvath midpoint space* whenever X is a complete continuous midpoint metric space, D is a nonempty subset of X, and $\Gamma : \langle D \rangle \multimap X$ is a multimap such that $\Gamma(A) = \Gamma_A$ is a geodesically convex subset containing A and $\Gamma_A \subset \Gamma_B$ if $A \subset B \in \langle D \rangle$.

Proposition 3.4. ([13]) *Any Horvath midpoint space is a Horvath space and hence a KKM space.*

Hence many results in [2] are consequences of our KKM theory on abstract convex spaces.

In 2009, Niculescu-Rovenţa [5] extended Fan's minimax inequality to the context of metric spaces with global nonpositive curvature (NPC). As a consequence, a general result on the existence of a Nash equilibrium is obtained by them:

Definition 3.5. (Niculescu-Rovenţa [5]) A *global NPC space* is a complete metric space $E = (E,d)$ such that, for each pair of points $x_0,\ x_1 \in E$, there exists a point $y \in E$ such that

$$d(x_0,y) = d(y,x_1) = \frac{1}{2}d(x_0,x_1).$$

The authors stated that the space is also known as the Cat(0) space. In a global NPC space E each pair of points $x_0,\ x_1 \in E$ can be connected by a *geodesic* (that is, by a rectifiable curve $\gamma : [0,1] \to E$ such that the length of $\gamma|_{[s,t]}$ is $d(\gamma(s),\gamma(t))$ for all $0 \le s \le t \le 1$). Moreover, this geodesic is unique.

Definition 3.6. (Niculescu-Rovenţa [5]) A set $C \subset E$ is called *convex* if $\gamma([0,1]) \subset C$ for each geodesic $\gamma : [0,1] \to C$ joining $\gamma(0),\ \gamma(1) \in C$.

Example 3.7. Some of global NPC spaces E are given in [5]:

(1) Every Hilbert space.

(2) A complete, simply connected Riemannian manifold (M,g) with a nonpositive sectional curvature.

(3) The Bruhat-Tits building (in particular, trees).

(4) All closed convex subset of a global NPC space. etc.

They proved the following [5, Lemma 2]:

Lemma 3.8. *The KKM Lemma extends to any global NPC space E, provided that the closed convex hull of every nonempty finite family of points of E has the fixed point property.*

Let the abstract convex space $(E \supset D;\Gamma)$ consist of a global NPC space E, provided that the closed convex hull $\Gamma(A)$ of every nonempty finite subset A of D has the fixed point property. Then Lemma 3.8 implies

Theorem 3.9. $(E \supset D;\Gamma)$ *is a partial KKM space.*

In 2011, Shabanian-Vaezpour [18, Definition 2.2] defined the following:

Definition 3.10. We say that a CAT(0) space X has *the convex hull finite property* (CHFP) if the closed convex hull of every nonempty finite family of points of X has the fixed point property.

4. The KKM Theorems

In 2006, we obtained the following KKM theorem [6, Proposition 5] on abstract convex spaces:

Theorem 4.1. *Let $(E,D;\Gamma)$ be an abstract convex space, the identity map $1_E \in \mathfrak{KC}(E,E)$ [resp. $1_E \in \mathfrak{KO}(E,E)$], and $G : D \multimap E$ a multimap satisfying*

(1) G has closed [resp. open] values,

(2) G is a KKM map, and

(3) $\bigcap_{z\in M} \overline{G}(z)$ is compact for some $M \in \langle D\rangle$.

Then we have $\bigcap_{y\in D} \overline{G}(y) \neq \emptyset$.

This is the first KKM type theorem for abstract convex spaces including scores of particular cases.

Later we defined that $(E,D;\Gamma)$ is a *partial KKM space* if $1_E \in \mathfrak{KC}(E,E)$; and a *KKM space* if $1_E \in \mathfrak{KC}(E,E) \cap \mathfrak{KO}(E,E)$.

Moreover, in 2011, we had the following [9, Theorem 2.1]:

Theorem 4.2. *Let $(E,D;\Gamma)$ be a partial KKM space and $G : D \multimap E$ a map such that:*

(1) G is closed-valued;

(2) G is a KKM map; and

(3) there exists a nonempty compact subset K of E such that one of the following holds:

(i) K = E;

(ii) $K = \bigcap\{G(z) : z \in M\}$ for some $M \in \langle D \rangle$: or

(iii) for each $N \in \langle D \rangle$, there exists a compact Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$ and

$$L_N \cap \bigcap_{z \in D'} G(z) \subset K.$$

Then $K \cap \bigcap\{G(z) : z \in D\} \neq \emptyset$.

By adopting the CHFP, we have the following extension of Lemma 3.8 as a Corollary of Theorem 4.2 as follows:

Corollary 4.3. *Theorem 4.2 holds for any global NPC space E having the CHFP.*

PROOF. Any global NPC space E having the CHFP is a partial KKM space by Lemma 3.8. Hence Theorem 4.2 works, $\square$

In 2011, Shabanian-Vaezpour [18, Theorem 2.1] restated Niculescu-Rovenţa [5, Lemma 2] as follows:

Lemma 4.4. (KKM mapping principle) *Suppose that E is a complete CAT(0) space with the CHFP and X is a nonempty subset of E. Furthermore, suppose $M : X \to 2^X$ is a KKM mapping with closed values. Then, if M(z) is compact for some $z \in X$, then $\bigcap_{x \in X} M(x) \neq \emptyset$.*

Now, we can improve this to a Corollary of Theorem 4.2 as follows:

Corollary 4.5. *Let $(E,X;\Gamma)$ be an abstract convex space such that E is a complete CAT(0) space with the CHFP, X is a nonempty subset of E, and $\Gamma : \langle X \rangle \multimap E$ be such that $\Gamma(N)$ is the Γ-convex hull of $N \in \langle X \rangle$. Then $(E,X;\Gamma)$ is a partial KKM space and hence Theorem 4.2 holds.*

Shabanian-Vaezpour [18] applied Lemma 4.4 to some fixed point theorems and best approximation theorems in CAT(0) spaces.

Note that, as shown in [8, 9], any KKM type theorem has scores of equivalent statements and applications. However, the authors mentioned in this section gave only a small number of such statements from their KKM type theorems.

5. Generalized KKM Maps

Inspired by recent works on generalized KKM maps in the sense of Kassay-Kolumbán in 1990 and Chang-Zhang in 1991, we introduce the following definition [11]:

Definition 5.1. Let $(X,D;\Gamma)$ be an abstract convex space and Y be a nonempty set such that, for each $A \in \langle Y \rangle$, there exists a function $\sigma_A : A \to D$. Then a new abstract convex space $(X,A;\Lambda_A)$ induced by Γ and A is defined by the following

$$\Lambda_A(J) := \Gamma(\sigma_A(J)) \text{ for each } J \subset A.$$

Moreover, a multimap $T : Y \multimap X$ (called a *generalized KKM map*) reduces to a KKM map on $(X,A;\Lambda_A)$ for each $A \in \langle Y \rangle$ satisfying $\Lambda_A(J) \subset T(J)$ for each $J \subset A$.

The following characterization of generalized KKM maps extends of Park and Lee [17, Theorem 2], which was stated for G-convex spaces:

Theorem 5.2. ([11]) *Let $(E,D;\Gamma)$ be a partial KKM space [resp. KKM space], X a nonempty set, and $T : X \multimap E$ a map with closed [resp. open] values.*

(i) If T is a generalized KKM map, then the family of its values has the finite intersection property.

(ii) The converse holds whenever $E = D$ and $\Gamma_{\{x\}} = \{x\}$ for all $x \in E$.

In 2016, Shabanian-Vaezpour [19] characterized generalized KKM map in modular spaces $(Y_\rho;\text{co})$ and obtained the following [19, Corollary 3.4]:

Lemma 5.3. ([19]) *Let ρ be a modular on Y, X be a nonempty set, and $G : X \multimap Y_\rho$ be a closed-valued map. If G is KKM, then the family $\{G(x) : x \in X\}$ has the finite intersection property.*

This means that $(Y_\rho, X; \mathrm{co})$ is a partial KKM space and, hence, satisfies a large number of KKM theoretic statements. Especially, the following characterization [19, Theorem 3.3] for generalized KKM map in modular spaces $(Y_\rho; \mathrm{co})$ holds:

Corollary 5.4. ([19]) *Let X be a nonempty set, ρ be a modular on Y, and $F : X \multimap Y_\rho$ be a multimap with closed values. Then the family $\{F(x) : x \in X\}$ has the finite intersection property if and only if the mapping F is a generalized KKM mapping.*

PROOF. Note that $(Y_\rho; \mathrm{co})$ is a partial KKM space by Lemma 5.3. Therefore Theorem 5.2 for $(E;\Gamma) = (Y_\rho; \mathrm{co})$ holds since we assumed $E = D$. □

Note that Corollary 5.4 also follows from Park and Lee [17, Theorem 2] in earlier 2001 for G-convex spaces.

After giving Corollary 5.4, as an application, the authors of [19] give some sufficient conditions which guarantee existence of solutions of minimax problems in which they get Fan's minimax inequality in modular spaces. However, we will not follow them.

6. The Minimax Inequalities

Recall that, in 2008 [7], from a basic KKM type theorem for a $\mathfrak{K}$-map defined on an abstract convex space without any topology, we deduce ten equivalent formulations of the theorem. As applications of the equivalents, in the frame of abstract convex topological spaces, we obtain Fan-Browder type fixed point theorems, almost fixed point theorems for multimaps, mutual relations between the map classes $\mathfrak{K}$ and the better admissible class $\mathfrak{B}$, variational inequalities, the von Neumann type minimax theorems, and the Nash equilibrium theorems.

In fact, we had the following analytic alternative [7, Theorem 6], which is a basis of various equilibrium problems:

Theorem 6.1. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a set, $F \in \mathfrak{K}(E, Z)$, α, $\beta \in \mathbb{R}$, and $f : E \times Z \to \overline{\mathbb{R}}$, $g : D \times Z \to \overline{\mathbb{R}}$ extended real-valued functions. Suppose that for each $z \in F(E)$*

and $M \in \langle \{y \in D : g(y,z) > \alpha\} \rangle$*, we have*

$$\Gamma_M \subset \{x \in E : f(x,z) > \beta\}.$$

Then either

(a) for each $N \in \langle D \rangle$*, there exists a* $z_N \in F(E)$ *such that* $g(y,z_N) \leq \alpha$ *for all* $y \in N$*; or*

(b) there exists an $(\hat{x},\hat{z}) \in F$ *such that* $f(\hat{x},\hat{z}) > \beta$*.*

From Theorem 6.1, we clearly have the following generalized form [7, Theorem 7] of the Ky Fan minimax inequality:

Theorem 6.2. *Under the hypothesis of Theorem 6.1, if*

$$\alpha = \beta = \sup\{f(x,z) : (x,z) \in F\},$$

then for each $N \in \langle D \rangle$*,*

(c) there exists a $z_N \in F(E)$ *such that*

$$g(y,z_N) \leq \sup_{(x,z)\in F} f(x,z) \;\; for\ all\ y \in N; \text{ and}$$

(d) we have the following minimax inequality

$$\inf_{z\in F(E)} \sup_{y\in N} g(y,z) \leq \sup_{(x,z)\in F} f(x,z).$$

The following is [7, Theorem 18]:

Theorem 6.3. *Let* $(X;\Gamma)$ *be a compact partial KKM space and* $f,g : X \times X \to \mathbb{R}\bigcup\{+\infty\}$ *be functions such that*

(1) $f(x,y) \leq g(x,y)$ *for each* $(x,y) \in X \times X$*,*

(2) for each $x \in X$*,* $g(x,\cdot)$ *is quasiconcave on X; and*

(3) for each $y \in X$*,* $f(\cdot,y)$ *is l.s.c. on X.*

Then we have

$$\min_{y\in X} \sup_{x\in X} f(x,y) \leq \sup_{x\in X} g(x,x).$$

In 2009, Fan's minimax inequality is extended to the context of metric spaces with global nonpositive curvature by Niculescu-Rovenţa [5, Theorem 1] as follows:

Corollary 6.4. ([5]) *Let C be a compact convex subset of a global NPC space E with the CHFP. If* $f : C \times C \to \mathbb{R}^+$ *is quasi-concave in the first variable and lower semicontinuous in the second variable, then*

$$\min_{y\in C} \sup_{x\in C} f(x,y) \le \sup_{z\in C} f(z,z).$$

PROOF. Note that C is also a partial KKM space as in Theorem 3.9. Hence Theorem 6.3 works on $X = C$. □

Niculescu-Rovenţa [5] insisted that an important application of Corollary 6.4 is the existence of a g-equilibrium, a fact that generalizes the well known result on the Nash equilibrium. On this matter, recall our previous articles [8, 10].

7. THE BEST APPROXIMATION THEOREMS

In 2011, Shabanian-Vaezpour [18] claimed to obtain a CAT(0) version of Fan's minimax inequality and, as its application, they also obtained some fixed point theorems and best approximation theorems in CAT(0) spaces. In this section, we extend their best approximation theorem and some related results.

Recall that we obtained the following [7, Theorem 20] in 2008:

Theorem 7.1. *Let* $(X;\Gamma)$ *be a compact partial KKM space and* $p,\ q : X \times X \to \mathbb{R}$ *functions such that*

(1) $p \le q$ *on the diagonal* $\Delta := \{(x,x) : x \in X\}$ *and* $q \le p$ *on* $(X \times X) \setminus \Delta$*;*

(2) for each $x \in X,\ y \mapsto q(y,y) - q(x,y)$ *is quasiconcave on X; and*

(3) for each $y \in X,\ x \mapsto p(x,y)$ *is u.s.c. on X.*

Then there exists a $y_0 \in X$ *such that* $p(y_0,y_0) \le p(x,y_0)$ *for all* $x \in X$.

We introduce the following definitions:

Definition 7.2. An *abstract convex metric space* $(E,D;\Gamma,d)$ consists of a metric space (E,d), a nonempty set D, and a multimap $\Gamma : \langle D\rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for each

$A \in \langle D \rangle$, such that the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

Definition 7.3. Let (M,d) be a metric space. Define a multimap $\Gamma : \langle M \rangle \multimap M$ by

$$\Gamma_A = \Gamma(A) = \mathrm{BI}(A) := \bigcap \{B \mid B \text{ is a closed ball containing } A\}$$

for each $A \in \langle M \rangle$.

A subset $X \subset M$ is said to be *subadmissible* or Γ-*convex* if for each $N \in \langle X \rangle$, we have $\Gamma_N = \Gamma(N) \subset X$.

Definition 7.4. Let (X,d) be a metric space, $r \in \mathbb{R}^+ \cup \{0\}$ and $\emptyset \neq S \subset X$. We denote the *r-parallel set* of S by

$$S + r = \bigcup \{B(s,r) : s \in S\},$$

where $B(s,r) = \{t \in X : d(s,t) \leq r\}$.

Recently, we obtained the following best approximation theorem for multimaps in [16, Theorem 4.1]:

Theorem 7.5. ([16]) *Let $(X,S;\Gamma,d)$ be a partial KKM metric space, S a nonempty Γ-convex set contained in a compact subset K of X, $\Phi : S \multimap X$ is a continuous multimap with compact values such that the condition*

$$(1) \;\; \Phi(x) + r \; is \; \Gamma - convex \; for \; all \; x \in S, \; r \geq 0$$

is satisfied. Then there exists $v_0 \in S$ such that

$$d(v_0, \Phi(v_0)) = \inf_{x \in S} d(x, \Phi(v_0)).$$

(In particular, if $\Phi(S) \subset S$, then v_0 is a fixed point of Φ.)

If K is metrically convex and $v_0 \notin \Phi(v_0)$, then $v_0 \in \mathrm{Bd}(S)$.

Note that this is a quite rare best approximation theorem for multimaps and that this reduces to the following in case $\Phi = f$ is a single valued map:

Theorem 7.6. *Let $(X,S;\Gamma,d)$ be a partial KKM metric space, S be a nonempty Γ-convex set contained in a compact subset K of X, and $f : S \to X$ be a continuous map.*

Then there exists $v_0 \in S$ such that

$$d(v_0, f(v_0)) = \inf_{x \in S} d(x, f(v_0)).$$

(In particular, if $f(S) \subset S$, then v_0 is a fixed point of f.)

PROOF 1. If the condition

(1) $B(f(x), r)$ is Γ-convex for all $x \in S$ and $r \geq 0$

is satisfied, then clearly Theorem 7.6 follows from Theorem 7.5. However, the condition (1) simply tells that

(2) each $B(f(x), r)$ is subadmissible for all $x \in S$ and $r \geq 0$,

and this holds for any metric spaces. Hence we have done the proof. □

We give another proof:

PROOF 2. Consider the map $G : S \multimap K$ defined by

$$G(x) := \{y \in S : d(y, f(y)) \leq d(x, f(y))\} \subset K$$

for each $x \in S$. Since f is continuous, $G(x)$ is closed in K for each $x \in S$. We claim that G is a KKM map on the partial KKM metric space $(X, S; \Gamma, d)$. Indeed, assume not. Then there exist $A = \{x_1, \ldots, x_n\} \in \langle S \rangle$ and $y \in \mathrm{co}_\Gamma(A) = \mathrm{BI}(A)$ such that $y \notin G(A)$. This clearly implies

$$d(x_i, f(y)) < d(y, f(y)) \quad \text{for } i = 1, \ldots, n.$$

Let $\varepsilon > 0$ such that $d(x_i, f(y)) \leq d(y, f(y)) - \varepsilon$ for each i. Hence $x_i \in B(f(y), d(y, f(y)) - \varepsilon)$ for each i. Therefore, we have $\mathrm{BI}(A) \subset B(f(y), d(y, f(y)) - \varepsilon)$, which implies $y \in B(f(y), d(y, f(y)) - \varepsilon)$. Clearly this gets us our contradiction which completes the proof of our claim. By the compactness of K and $G(x) \subset S \subset K$, we deduce that $G(x)$ is compact for any $x \in S$. Therefore, there exists $y_0 \in \bigcap_{x \in S} G(x)$. This clearly implies $d(y_0, f(y_0)) \leq d(x, f(y_0))$ for any $x \in S$ which implies $d(y_0, f(y_0)) = \min_{x \in S} d(x, f(y_0))$ and the proof is complete. □

In case $S = K$, Theorem 7.6 reduces to the following [16, Theorem 5.2] known in 2017.

Corollary 7.7. ([16]) *Let $(M,X;\Gamma,d)$ be a partial KKM metric space where $X \subset M$ is a compact Γ-convex subset. Let $f : X \to M$ be continuous. Then there exists $y_0 \in X$ such that*

$$d(y_0, f(y_0)) = \min_{x\in X} d(x, f(y_0)).$$

Corollary 7.8. *Suppose X is a compact subadmissible subset of a complete CAT(0) space E with convex hull finite property and $f : X \to E$ is continuous. Then, there exists $y_0 \in X$ such that*

$$d(y_0, f(y_0)) = \inf_{x\in X} d(x, f(y_0)).$$

Note that the main theorem of Shabanian-Vaezpour [18, Theorem 3.1] in 2011 is Corollary 7.8 without assuming the subadmissibility of X. However their proof is incorrect. Therefore some other results in [18] also may be incorrect.

Moreover we have the following generalized form of Khamsi [3. Lemma]:

Corollary 7.9. *Let H be a hyperconvex metric space and X an subadmissible compact subset of H. Let $f : X \to H$ be continuous. Then there exists $y \in X$ such that*

$$d(y, fy) \leq \inf_{x\in X} d(x, fy).$$

Recall that the following is the celebrated 1961 best approximation theorem of Ky Fan:

Corollary 7.10. (Ky Fan) *Let X be a nonempty compact convex set in a normed vector space E. For any continuous map $f : X \to E$, there exists a point $y_0 \in X$ such that*

$$||y_0 - f(y_0)|| = \min_{x\in X} ||x - f(y_0)||.$$

(In particular, if $f(X) \subset X$, then y_0 is a fixed point of f.)

In this well-known Theorem, according to our Theorem 7.6, the convexity of X can be replaced by the subadmissibility.

Conflict of Interests

The author declares that there is no conflict of interests.

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