Available online at http://jfpt.scik.org
J. Fixed Point Theory, 2021, 2021:2
ISSN: 2052-5338

SOME OF A FAMILY OF CONTRACTIVE MAPS AND APPROXIMATE COMMON FIXED POINT

S. A. M. MOHSENIALHOSSEINI[1,*], M. SAHELI[2]

[1]Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran

[2]Department of Mathematics, Vali-e-Asr University of Rafsanjan, Rafsanjan, Iran

Abstract. In this paper, we introduce the approximately common fixed point property and the concepts of a family of contractive maps in the metric spaces. Also, Using of two general lemmas were give by Mădălina Berinde regarding $\varepsilon-$fixed points of operators we prove qualitative and quantitative theorems for a family of contractive maps.

Keywords: common fixed points; approximate common fixed points; Mohseni-saheli operator; Mohsenialhosseini-saheli operator; diameter approximate fixed point.

2010 AMS Subject Classification: 47H10, 54H25, 46B20.

1. Introduction

One of the most preponderant results used in the nonlinear analysis is the well-known Banach's contraction principle that is useful in solving existing problems in many branches of Mathematical Analysis and its applications. Nowadays, there are plenty of problems in applied mathematics which can be solved using fixed point theory such as physics, chemistry, economics. In physics and engineering, the fixed point technique has been used in areas like

*Corresponding author

E-mail addresses: amah@vru.ac.ir, mohsenialhosseini@gmail.com

Received October 12, 2020

image retrieval, signal processing, and the existence and uniqueness of solutions for a class of nonlinear integral equations were studied. Still, practice proves that in many real situations an approximate solution is more than sufficient, so the existence of fixed points is not strictly required, but that of ”nearly” fixed points.

In 1974, Lj. B. Ćirić [2], investigated a family of maps which satisfy a common fixed point and which are not necessarily continuous and commuting.

In 2003, Tijs et al [10], introduced the approximate fixed point theorems. In 2006, M̆ad̆alina Berinde [1], introduced two general lemmas regarding $\varepsilon-$fixed points which will be used for prove the existence of $\varepsilon-$fixed points for various types of operators.

In 2011, Mohsenalhosseini et al [4], introduced the approximate best proximity pairs and proved the property of approximate best proximity pairs. Also, In 2012 , Mohsenalhosseini et al [5], introduced the approximate fixed point for complete norm spaces and map T_α and proved the property of approximate fixed point. In 2014 , Mohsenalhosseini [6] introduced the Approximate best proximity pairs on metric space for contraction maps. Recently, in 2017 Mohsenialhosseini [8] introduced the approximate fixed points of operators on G-metric spaces. Now we give preliminaries and basic definitions that are used throughout the paper. Also, we study the concepts of a family of contractive maps in the metric spaces. Also, we use to of two general lemmas were give by M̆ad̆alina Berinde regarding Approximate common fixed points of operators. Moreover, Using these results we prove qualitative and quantitative theorems for a family of contractive maps.

2. Preliminaries

This section recalls the following notations and the ones that will be used in what follows.

Definition 2.1. [1] *Let* $T : X \to X$, $\varepsilon > 0$, $x_0 \in X$. *Then* $x_0 \in X$ *is an* $\varepsilon-$*fixed point for* T *if* $d(Tx_0 - x_0) < \varepsilon$.

Remark 2.2. [1] *In this paper we will denote the set of all* $\varepsilon-$ *fixed points of* T, *for a given* ε, *by :*

$$F_\varepsilon(T) = \{x \in X \mid x \text{ is an } \varepsilon - fixed\ point\ of\ T\}.$$

Definition 2.3. [1] *Let $T : X \to X$. Then T has the approximate fixed point property (a.f.p.p) if*

$$\forall \varepsilon > 0,\ F_\varepsilon(T) \neq \varnothing.$$

Lemma 2.4. [1] *Let (X,d) be a metric space, $T : X \to X$ such that T is asymptotic regular, i.e.,*

$$lim_{n\to\infty} d(T^n(x), T^{n+1}(x)) = 0,$$

for every $x \in X$. Then T has the approximate fixed point property.

Definition 2.5. *Let (X,d) be a metric space, $T : X \to X$ a operator and $\varepsilon > 0$. We define diameter of the set $F_\varepsilon(T)$, i.e.,*

$$\delta(F_\varepsilon(T)) = \sup\{d(x,y) :\ x,y \in F_\varepsilon(T)\}.$$

Lemma 2.6. [1] *Let (X,d) be a metric space, $T : X \to X$ a operator and $\varepsilon > 0$. We assume that:*

(i) $F_\varepsilon(T) \neq \emptyset$;

(ii) $\forall \theta > 0,\ \exists \phi(\theta) > 0$ *such that*;

$$d(x,y) - d(Tx,Ty) \quad \leq \quad \theta \Rightarrow d(x,y) \leq \phi(\theta),\ \forall x,y \in F_\varepsilon(T) \neq \emptyset.$$

Then:

$$\delta(F_\varepsilon(T)) \leq \phi(2\varepsilon).$$

Theorem 2.7. [5] *Let $(X, ||.||)$ be a complete mertic space, $T : X \to X$, $x_0 \in X$ and $\varepsilon > 0$. If $||(T^n(x_0), T^{n+k}(x_0)|| \to 0$ as $n \to \infty$ for some $k > 0$, then T^k has an $\varepsilon-$ fixed point.*

3. Approximate Common Fixed Point a Family of Contractive Maps

The section begins with three lemmas which will be used in order to prove all the results given in the second and in the third sections. Also, we a series of qualitative and quantitative results will be obtained regarding the properties of approximate fixed point for a family of contractive maps.

Let (M,d) be a metric space and let $\mathscr{F} = \{T_\lambda : \lambda \in (\lambda)\}$ be a family of maps which map M into itself.

Definition 3.1. [2] *Let $\mathscr{F} = \{T_\lambda : \lambda \in (\lambda)\}$ be a family of maps which map M into itself. Then $x_0 \in M$ is a common fixed point for $\mathscr{F}$ iff $u = T_\lambda u$* for each $T_\lambda \in \mathscr{F}$.

Definition 3.2. *Let $T_\lambda : M \to M$ be a sequence of maps on a complete metric space (M,d), $\varepsilon > 0$, and $x_0 \in X$. Then $x_0 \in M$ is an $\varepsilon-$common fixed point for T_λ if $d(T_\lambda x_0, x_0) < \varepsilon$.*

Remark 3.3. *In this paper we will denote the set of all $\varepsilon-$ common fixed points of T_λ , for a given ε, by :*

$$F_\varepsilon(T_\lambda) = \{x \in M \mid x \text{ is an } \varepsilon - common fixed\ point\ of\ T_\lambda\}.$$

Definition 3.4. *Let $T_\lambda : M \to M$ be a sequence of maps on a complete metric space (M,d). Then T_λ has the approximate common fixed point property (a.f.p.p) if*

$$\forall \varepsilon > 0,\ F_\varepsilon(T_\lambda) \neq \varnothing.$$

Lemma 3.5. *Let $T_\lambda : M \to M$ be a sequence of maps on a complete metric space (M,d), such that T_λ is asymptotic regular, i.e.,*

$$lim_{k\to\infty} d(T_\lambda^k(x), T_\lambda^{k+1}(x)) = 0,$$

for every $x \in M$. Then T_λ has the approximate common fixed point property.

Proof: *Let $\varepsilon > 0$ be given and $x_0 \in M$ such that $\lim_{k\to\infty} d(T_\lambda^k x_0, T_\lambda^{k+1} x_0) = 0$, then there exists $K_0 > 0$ such that*

$$\forall k \geq K_0 : d(T_\lambda^k x_0, T_\lambda^{k+1} x_0) < \varepsilon.$$

If $k = K_0$, then $d(T_\lambda^{K_0}(x_0), T_\lambda(T_\lambda^{K_0}(x_0))) < \varepsilon$, and $T_n^{K_0}(x_0) \in F_\varepsilon(T_\lambda)$. So for each $\varepsilon > 0$ there exists an $\varepsilon-$common fixed point of T_λ in M. $\square$

Definition 3.6. *Let $T_\lambda : M \to M$ be a sequence of maps on a complete metric space (M,d), and $\varepsilon > 0$. We define diameter of the set $F_\varepsilon(T_\lambda)$, i.e.,*

$$\delta(F_\varepsilon(T_\lambda)) = \sup\{d(x,y) :\ x,y \in F_\varepsilon(T_\lambda)\}.$$

Lemma 3.7. *Let $T_\lambda : M \to M$ be a sequence of maps on a complete metric space (M,d), and $\varepsilon > 0$. We assume that:*

(i) $F_\varepsilon(T_\lambda) \neq \emptyset$;

(ii) $\forall\theta > 0,\ \exists\phi(\theta) > 0$ *such that*;

$$d(x,y) - d(T_\lambda x, T_\lambda y) \leq \theta \Rightarrow d(x,y) \leq \phi(\theta),\ \forall x,y \in F_\varepsilon(T_\lambda) \neq \emptyset.$$

Then:

$$\delta(F_\varepsilon(T_\lambda)) \leq \phi(2\varepsilon).$$

Proof: The proof of Lemma is the same as the proof of Lemma 2.6 for T_λ.

Theorem 3.8. *Let $T_\lambda : M \to M$ be a sequence of maps on a complete metric space (M,d), $x_0 \in M$ and $\varepsilon > 0$. If $||(T_\lambda^m(x_0), T_\lambda^{m+k}(x_0)|| \to 0$ as $m \to \infty$ for some $k > 0$, then T_λ^k has an $\varepsilon-$ common fixed point.*

Proof:. The proof of Lemma is the same as the proof of Theorem 2.7 for $x \in M$. $\square$

Definition 3.9. [9] A mapping $T : X \to X$ is a $\alpha-$contraction if there exists $\alpha \in (0,1)$ such that

$$d(Tx,Ty) \leq \alpha d(x,y), \forall x,y \in X.$$

Definition 3.10. A sequence $T_\lambda : M \to M$ of maps on a complete metric space (M,d) is a $\alpha-$contraction if there exists $\alpha \in (0,1)$ such that

$$d(T_\lambda x, T_\lambda y) \leq \alpha d(x,y), \forall x,y \in M.$$

Theorem 3.11. *Let T_λ be a sequence of maps on a complete metric space (M,d). Suppose that the mapping $T_\lambda : M \to M$ is a $\alpha-$contraction. Then has an $\varepsilon-$common fixed point.*

Proof: Let $\varepsilon > 0$ and $x \in M$.

$$\begin{aligned} d(T_\lambda^k x, T_\lambda^{k+1} x) &= d(T_\lambda(T_\lambda^{k-1}x), T_\lambda(T_\lambda^k x)) \\ &\leq \alpha d(T_\lambda^{k-1}x, T_\lambda^k x) \\ &\vdots \\ &\leq (\alpha)^k d(x, T_\lambda x). \end{aligned}$$

But $\alpha \in (0, \frac{1}{2})$. Hence

$$lim_{n\to\infty} d(T_\lambda^k, T_\lambda^{k+1}) = 0, \forall x \in M.$$

Hence by Lemma 3.5 it follows that $F_\varepsilon(T_\lambda) \neq \emptyset, \forall \varepsilon > 0$. □

In 1972, Chatterjea (see [3]) considered another operator in which continuity is not imposed. Now, the appoximate fixed point theorems by using a family of contractive maps are obtained.

Definition 3.12. A sequence $T_\lambda : M \to M$ of maps on a complete metric space (M,d) is a Chatterjea operator if there exists $\alpha \in (0, \frac{1}{2})$ such that

$$d(T_\lambda x, T_\lambda y) \leq \alpha[d(x, T_\lambda(y)) + d(y, T_\lambda(x))],\ \forall x, y \in M. \tag{3.1}$$

Theorem 3.13. *Let T_λ be a sequence of maps on a complete metric space (M,d). Suppose that the mapping $T_\lambda : M \to M$ is a Chatterjea operator. Then T_λ has an $\varepsilon-$common fixed point.*

Proof: Let $\varepsilon > 0$ and $x \in M$.

$$\begin{aligned} d(T_\lambda^k x, T_\lambda^{k+1} x) &= d(T_\lambda(T_\lambda^{k-1} x), T_\lambda(T_\lambda^k x)) \\ &\leq \alpha[d(T_\lambda^{k-1} x, T_\lambda(T_\lambda^k x)) + d(T_\lambda^k x, T_\lambda(T_\lambda^{k-1} x))] \\ &= \alpha[d(T_\lambda^{k-1} x, T_\lambda^{k+1} x) + d(T_\lambda^k x, T_\lambda^k x)] = \alpha d(T_\lambda^{k-1} x, T_\lambda^{k+1} x). \end{aligned}$$

On the other hand

$$d(T_\lambda^{k-1} x, T_\lambda^{k+1} x) \leq d(T_\lambda^{k-1} x, T_\lambda^k x) + d(T_\lambda^k x, T_\lambda^{k+1} x).$$

Then

$$(1-\alpha) d(T_\lambda^k x, T_\lambda^{k+1} x) \leq \alpha d(T_\lambda^{k-1} x, T_\lambda^k x),$$

hence

$$\begin{aligned} d(T_\lambda^k x, T_\lambda^{k+1} x) &\leq \frac{\alpha}{1-\alpha} d(T_\lambda^{k-1} x, T_\lambda^k x) \\ &\vdots \\ &\leq (\frac{\alpha}{1-\alpha})^n d(x, T_\lambda x). \end{aligned}$$

But $\alpha \in (0, \frac{1}{2})$ hence $\frac{\alpha}{1-\alpha} \in (0,1)$. Therfore

$$Lim_{k\to\infty} d(T_\lambda^k x, T_\lambda^{k+1} x) = 0,\ \forall x \in M.$$

Now by Lemma 3.5 it follows that $F_\varepsilon(T_\lambda) \neq \emptyset, \forall \varepsilon > 0$. □

Definition 3.14. A sequence $T_\lambda : M \to M$ of maps on a complete metric space (M,d) is a Mohseni-saheli operator if there exists $\alpha \in (0, \frac{1}{2})$ such that

$$d(T_\lambda x, T_\lambda y) \leq \alpha[d(x,y) + d(T_\lambda x, T_\lambda y)].$$

Theorem 3.15. *Let T_λ be a sequence of maps on a complete metric space (M,d). Suppose that the mapping $T_\lambda : M \to M$ is a Mohseni-saheli operator. Then T_λ has an $\varepsilon-$common fixed point.*

Proof: Let $\varepsilon > 0$ and $x \in M$.

$$\begin{aligned} d(T_\lambda^k x, T_\lambda^{k+1} x) &= d(T_\lambda(T_\lambda^{k-1} x), T_\lambda(T_\lambda^k x)) \\ &\leq \alpha[d(T_\lambda^{k-1} x, T_\lambda^k x) + d(T_\lambda^k, T_\lambda^{k+1})]. \end{aligned}$$

Therefore,

$$(1-\alpha) d(T_\lambda^k x, T_\lambda^{k+1} x) \leq \alpha d(T_\lambda^{k-1}, T_\lambda^k).$$

So,

$$\begin{aligned} d(T_\lambda^k x, T_\lambda^{k+1} x) &\leq \tfrac{\alpha}{(1-\alpha)} d(T_\lambda^{k-1}, T_\lambda^k) \\ &\vdots \\ &\leq (\tfrac{\alpha}{(1-\alpha)})^k d(x, T_\lambda x). \end{aligned}$$

But $\alpha \in (0, \frac{1}{2})$, therefore $(\dfrac{\alpha}{1-\alpha}) \in (0,1)$. Hence

$$lim_{k\to\infty} d(T_\lambda^k, T_\lambda^{k+1}) = 0, \forall x \in M.$$

Hence by Lemma 3.5 it follows that $F_\varepsilon(T_\lambda) \neq \emptyset, \forall \varepsilon > 0$. □

By combining the three independent contraction conditions: $\alpha-$ contraction, Mohseni-saheli, and Chatterjea operators we obtain another approximate common fixed point result for operators which satisfy the following.

Definition 3.16. A sequence $T_\lambda : M \to M$ of maps on a complete metric space (M,d) is a Mohsenialhosseini-saheli operator if there exists $\alpha, \beta, \gamma \in R$, $\alpha \in [0,1[, \beta \in [0, \frac{1}{2}[, \gamma \in [0, \frac{1}{2}[$ such that for all $x, y \in M$ at least one of the following is true:

(i) $d(T_\lambda x, T_\lambda y) \leq \alpha d(x,y)$;

(ii) $d(T_\lambda x, T_\lambda y) \leq \beta[d(x,y) + d(T_\lambda x, T_\lambda y)]$;

(iii) $d(T_\lambda x, T_\lambda y) \leq \gamma[d(x, T_\lambda(y)) + d(y, T_\lambda(x))]$.

Theorem 3.17. *Let T_λ be a sequence of maps on a complete metric space (M,d). Suppose that the mapping $T_\lambda : M \to M$ is a Mohsenialhosseini-saheli operator. Then T_λ has an $\varepsilon-$common fixed point.*

Proof: Let $x, y \in M$. Supposing *ii*) holds, we have that:

$$\begin{aligned} d(T_\lambda x, T_\lambda y) &\leq \beta[d(x,y) + d(T_\lambda x, T_\lambda y)] \\ &\leq \beta[d(x, T_\lambda x) + d(T_\lambda x, y) + d(T_\lambda x, T_\lambda y)] \\ &\leq \beta[d(x, T_\lambda x) + d(T_\lambda x, x) + d(x,y) + d(T_\lambda x, T_\lambda y)] \\ &= 2\beta d(x, T_\lambda x) + \beta d(x,y) + \beta d(T_\lambda x, T_\lambda y)] \end{aligned}$$

Thus

$$d(T_\lambda x, T_\lambda y) \leq \frac{2\beta}{1-\beta} d(x, T_\lambda x) + \frac{\beta}{1-\beta} d(x,y). \tag{3.2}$$

Supposing *iii*) holds, we have that:

$$\begin{aligned} d(T_\lambda x, T_\lambda y) &\leq \gamma[d(x, T_\lambda y) + d(y, T_\lambda x)] \\ &\leq \gamma[d(x,y) + d(y, T_\lambda y)] + \gamma[d(y, T_\lambda y) + d(T_\lambda y, T_\lambda x)] \\ &= \gamma d(T_\lambda x, T_\lambda y) + 2\gamma d(y, T_\lambda y) + \gamma d(x,y). \end{aligned}$$

Thus

$$d(T_\lambda x, T_\lambda y) \leq \frac{2\gamma}{1-\gamma} d(y, T_\lambda y) + \frac{\gamma}{1-\gamma} d(x,y). \tag{3.3}$$

Similarly:

$$\begin{aligned} d(T_\lambda x, T_\lambda y) &\leq \gamma[d(x, T_\lambda y) + d(y, T_\lambda x)] \\ &\leq \gamma[d(x, T_\lambda x) + d(T_\lambda x, T_\lambda y)] + \gamma[d(y,x) + d(x, T_\lambda x)] \\ &= \gamma d(T_\lambda x, T_\lambda y) + 2\gamma d(x, T_\lambda x) + \gamma d(x,y). \end{aligned}$$

Then

$$d(T_\lambda x, T_\lambda y) \leq \frac{2\gamma}{1-\gamma} d(x, T_\lambda x) + \frac{\gamma}{1-\gamma} d(x,y). \tag{3.4}$$

Therefore for T_λ satisfying at least one of the conditions (i), (ii), (iii) we have that

$$d(T_\lambda x, T_\lambda y) \leq 2\eta d(x, T_\lambda x) + \eta d(x,y), \tag{3.5}$$

and

$$d(T_\lambda x, T_\lambda y) \leq 2\eta d(y, T_\lambda y) + \eta d(x,y), \tag{3.6}$$

where $\eta := max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$, hold. Using these conditions implied by (i) - (iii) and taking $x \in M$, we have:

$$\begin{aligned} d(T_\lambda^k x, T_\lambda^{k+1} x) &= d(T_\lambda(T_\lambda^{k-1} x), T_\lambda(T_\lambda^k x)) \\ &\leq^{(2.5)} 2\eta d(T_\lambda^{k-1} x, T_\lambda(T_\lambda^{k-1} x) + \eta d(T_\lambda^{k-1} x, T_\lambda^k x) \\ &= 3\eta d(T_\lambda^{k-1} x, T_\lambda^k x). \end{aligned}$$

Then

$$d(T_\lambda^k x, T_\lambda^{k+1} x) \leq \cdots \leq (3\eta)^n d(x, T_\lambda x).$$

Therfore

$$Lim_{n\to\infty} d(T_\lambda^k x, T_\lambda^{k+1} x) = 0, \ \forall x \in M.$$

Now by Lemma 3.5 it follows that $F_\varepsilon(T_\lambda) \neq \emptyset, \forall \varepsilon > 0$. □

4. Diameter Approximate Common Fixed Point for a Family of Contractive Maps

In this section a series of qualitative and quantitative results will be obtained regarding the properties of diameter approximate common fixed point. Also, we prove diameter approximate common fixed point theorems for various types of well known operators on a metric space.

Theorem 4.1. *Let T_λ be a sequence of maps on a complete metric space (M,d). Suppose that the mapping $T_\lambda : M \to M$ is a Mohseni-saheli operator. Then for every $\varepsilon > 0$,*

$$\delta(F_\varepsilon(T_\lambda)) \leq \frac{2\varepsilon(1+\alpha)}{1-2\alpha}.$$

Proof: Let $\varepsilon > 0$. and $x \in M$. Condition i) in Lemma 3.7 is satisfied, as one can see in the proof of Theorem 3.15. Now we only verify that condition 2) in Lemma 3.7, holds.

Let $\theta > 0$ and $x, y \in F_\varepsilon(T_\lambda)$. We also assume that $d(x,y) - d(T_\lambda x, T_\lambda y) \leq \theta$. Then:

$$d(x,y) \quad \leq \quad \alpha[d(x,y) + d(T_\lambda x, T_\lambda y)] + \theta.$$

Therfore

$$d(x,y) \quad \leq \quad \alpha[d(x,y) + d(T_\lambda x, x) + d(x,y) + d(y, T_\lambda y)] + \theta.$$

As $x, y \in F_\varepsilon(T_\lambda)$, we know that

$$d(x, T_\lambda x) \leq \varepsilon, d(y, T_\lambda y) \leq \varepsilon.$$

Therfore,

$$d(x,y) \leq \frac{2\alpha\varepsilon + \theta}{1 - 2\alpha}.$$

So for every $\theta > 0$ there exists $\phi(\theta) = \frac{2\alpha\varepsilon+\theta}{1-2\alpha} > 0$ such that

$$d(x,y) - d(T_\lambda x, T_\lambda y) \leq \theta \Rightarrow d(x,y) \leq \phi(\theta).$$

Now by Lemma 3.7, it follows that

$$\delta(F_\varepsilon(T_\lambda)) \leq \phi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(T_\lambda)) \leq \frac{2\varepsilon(1+\alpha)}{1-2\alpha}. \square$$

Theorem 4.2. *Let T_λ be a sequence of maps on a complete metric space (M,d). Suppose that the mapping $T_\lambda : M \to M$ is a Mohsenialhosseini-saheli operator. Then for every $\varepsilon > 0$,*

$$\delta(F_\varepsilon(T_\lambda)) \leq 2\varepsilon \frac{1+\eta}{1-\eta},$$

where $\eta := max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\}$, and α, β, γ as in Definition 3.16.

Proof: Proof: In the proof of Theorem 3.17, we have already shown that if T satisfies at least one of the conditions $(i),(ii),(iii)$ from Definition 3.16, then

$$d(T_\lambda x, T_\lambda y) \leq 2\eta d(x, T_\lambda x) + \eta d(x,y),$$

and

$$d(T_\lambda x, T_\lambda y) \leq 2\eta d(y, T_\lambda y) + \eta d(x,y),$$

hold.

Let $\varepsilon > 0$. We will only verify that condition (ii) in Lemma 3.7 is satisfied, as (i) holds, see the Proof of Theorem 3.17.

Let $\theta > 0$ and $x,y \in F_\varepsilon(T_\lambda)$, and assume that $d(x,y) - d(T_\lambda, T_\lambda y) \leq \theta$. Then

$$\begin{aligned} d(x,y) &\leq d(T_\lambda, T_\lambda y) + \theta \Rightarrow \\ d(x,y) &\leq 2\eta d(x, T_\lambda x) + \eta d(x,y) + \theta \Rightarrow \end{aligned}$$

$$(1-\eta) d(x,y) \leq 2\eta\varepsilon + \theta$$

$$d(x,y) \leq \frac{2\eta\varepsilon + \theta}{1-\eta}.$$

So for every $\theta > 0$ there exists $\phi(\theta) = \frac{2\eta\varepsilon+\theta}{1-\eta} > 0$ such that

$$d(x,y) - d(T_\lambda x, T_\lambda y) \leq \theta \Rightarrow d(x,y) \leq \phi(\theta).$$

Now by Lemma 3.7, it follows that

$$\delta(F_\varepsilon(T_\lambda)) \leq \phi(2\varepsilon), \forall \varepsilon > 0,$$

which means exactly that

$$\delta(F_\varepsilon(T_\lambda)) \leq 2\varepsilon \frac{1+\eta}{1-\eta}, \forall \varepsilon > 0. \square$$

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] M. Berinde, Approximate fixed point theorems. Stud. Univ."Babes-Bolayi", Mathematica, LI (1) (2006), 11-25.

[2] Lj. B. Ćirić, On a family of contractive maps and fixed points, Publ. Inst. Math. 17 (31) (1974), 45-51.

[3] S. K. Chatterjea, Fixed-point Theorems, C.R. Acad. Bulgare Sci. 25 (1972), 727-730.

[4] S. A. M. Mohsenalhosseini, H. Mazaheri, M. A. Dehghan, Approximate best proximity pairs in metric space, Abstr. Appl. Anal. 2011 (2011), Article ID 596971.

[5] S. A. M. Mohsenalhosseini, H. Mazaheri, Fixed Point for Completely Norm Space and Map T_α, Math. Moravica. 16 (2012), 25-35.

[6] S. A. M. Mohsenalhosseini, Approximate best proximity pairs in metric space for Contraction Maps, Adv. Fixed Point Theory. 4 (2014), 310-324.

[7] S. A. M. Mohsenalhosseini, T. Ahmadi, Approximate fixed point in $G-$metric spaces for various types of operators, J. Math. Comput. Sci. 6 (2016), 767-782.

[8] S. A. M. Mohsenialhosseini, Approximate fixed points of operators on G-metric spaces, U.P.B. Sci. Bull. Ser. A. 79 (2017), 85-96.

[9] I. A. Rus, Generalized Contractions and Applications, Cluj University Press, Cluj- Napoca, 2001.

[10] S. H. Tijs, A. Torre, R. Branzei, Approximate Fixed Point Theorems, Libertas Math. 23 (2003), 35-39.