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# COUPLED COMMON FIXED POINTS IN S-METRIC SPACE

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**Abstract.** The aim of this paper is to prove coupled common fixed point theorem in partially ordered $S$-metric spaces with used the notion of a mixed weakly monotone pair of mappings of Gordji *et al.* (Fixed Point Theory Appl. 2012:95, 2012) under $(\phi, \psi)$-weakly contractive condition.Our results extend and generalize several well-known results in the literature. Also, an illustrative example and an application for existence of solutions of system for nonlinear Volterra integral equations are presented to support the obtained results.

**Keywords:** S-metric space; common fixed point; altering distance function; mixed weakly monotone mappings; coupled common fixed point; integral equation.

**2010 AMS Subject Classification:** 54H25, 47H10.

## 1. Introduction and Preliminaries

The Banach contraction principle is the most celebrated fixed point theorem. This result has been generalised in various directions, see [13] -[19]. As a generalisation of metric space Sedghi et al. [11] generalised metric space to $S$-metric space. On the other hand, Guo and Lakshmikantham [5] in 1987 introduced the concept of coupled fixed point. In a recent paper, Gnana-Bhaskar and Lakshmikantham [1] introduced the concept of mixed monotone property

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for contractive operators of the form $F : X \times X \to X$, where $X$ is a partially ordered metric space, and then established some coupled fixed point theorems. They also illustrated these results by proving the existence and uniqueness of the solution for a periodic boundary value problem. Later, Lakshmikantham and Ćirić [9] investigated some more coupled fixed point theorems in partially ordered complete metric spaces. In [4], Gordji et al. have introduced the concept of a mixed weakly monotone pair of maps and proved some coupled common fixed point theorems for a contractive-type maps with the mixed weakly monotone property in partially ordered metric spaces. Recently, Dung [2] generalized the results of Gordji et al. [4] in the framework of an $S$-metric space.

The aim of this paper is to prove coupled common fixed point theorem for pair of mixed weakly monotone maps under $(\phi, \psi)$-weakly contractive condition in partially ordered $S$-metric space. Our result generalize the results of [1], [4], [6]. We support our results by examples. As an application, the existence of solutions of system for nonlinear Volterra integral equations is presented.

First, we recall the necessary definitions and results which will be useful for the rest of the paper.

**Definition 1.1.** [11] Let $X$ be a nonempty set. A function $S : X^3 \to [0,\infty)$ is said to be an S-metric on $X$, if for each $x,y,z,a \in X$,

(1) $S(x,y,z) = 0$ if and only if $x = y = z$,
(2) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair $(X,S)$ is called an S-metric space.

**Example 1.2.** [11] *We can easily check that the following examples are S-metric spaces.*

*(1) Let $X = \mathbb{R}^n$ and $\|\cdot\|$ be a norm on $X$. Then $S(x,y,z) = \|y+z-2x\| + \|y-z\|$ is an S-metric on $X$. In general, if $X$ is a vector space over $\mathbb{R}$ and $\|\cdot\|$ is a norm on $X$. Then it is easy to see that*

$$S(x,y,z) = ||\alpha y + \beta z - \lambda x|| + ||y-z||,$$

*where $\alpha + \beta = \lambda$ for every $\alpha, \beta \geq 1$, is an S-metric on $X$.*

(2) *Let $X$ be a nonempty set and $d_1, d_2$ be two ordinary metric on $X$. Then*

$$S(x,y,z) = d_1(x,z) + d_2(y,z),$$

*is an S-metric on $X$.*

**Lemma 1.3.** [13] *Let $(X,S)$ be an S-metric space. Then, we have $S(x,x,y) = S(y,y,x)$, $x,y \in X$.*

The following lemma is a direct consequence of Definition 1.2 and Lemma 1.3.

**Lemma 1.4.** [2] *Let $(X,S)$ be an S-metric space. Then*

$$S(x,x,z) \leq 2S(x,x,y) + S(y,y,z)$$

*and*

$$S(x,x,z) \leq 2S(x,x,y) + S(z,z,y)$$

*for all $x,y,z \in X$.*

**Definition 1.5.** [12] Let $(X,S)$ be an S-metric space

(1) A sequence $\{x_n\}$ in $X$ converges to $x$ if $S(x_n,x_n,x) \to 0$ as $n \to \infty$, that is for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$, $S(x_n,x_n,x) < \varepsilon$. This case, we denote by $\lim_{n\to\infty} x_n = x$ and we say that $x$ is the limit of $\{x_n\}$ in $X$.

(2) A sequence $\{x_n\}$ in $X$ is said to be Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \varepsilon$ for each $n,m \geq n_0$.

(3) The S-metric space $(X,S)$ is said to be complete if every Cauchy sequence is convergent.

**Definition 1.6.** [8] Let $(X,S)$ and $(\acute{X},\acute{S})$ be two S-metric spaces, and let $f : (X,S) \to (\acute{X},\acute{S})$ be a function. Then $f$ is said to be continuous at a point $a \in X$ if and only if for every sequence $x_n$ in $X$, $S(x_n,x_n,a) \to 0$ implies $\acute{S}(f(x_n),f(x_n),f(a)) \to 0$. A function $f$ is continuous at $X$ if and only if it is continuous at all $a \in X$.

**Lemma 1.7.** [12] *Let $(X,S)$ be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n,x_n,y_n) = S(x,x,y)$.*

**Lemma 1.8.** [14] *Let $(X,S)$ be an S-metric space. If there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n\to\infty} S(x_n,x_n,y_n)=0$, whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} x_n = t$ for some $t\in X$, then $\lim_{n\to\infty} y_n = t$.*

**Definition 1.9.** [1] Let $(X,\preceq)$ be a partially ordered set and $f : X\times X \longrightarrow X$ be mapping. We say that $f$ has the mixed monotone property on $X$ if, for all $x,y\in X$,

$$x_1,x_2\in X, x_1\preceq x_2 \Rightarrow f(x_1,y)\preceq f(x_2,y)$$

and

$$y_1,y_2\in X, y_1\preceq y_2 \Rightarrow f(x,y_1)\succeq f(x,y_2)$$

**Definition 1.10.** [4] Let $(X,\preceq)$ be a partially ordered set and $f,g : X\times X \longrightarrow X$ be two maps. We say that a pair $(f,g)$ has the mixed weakly monotone property on $X$ if, for all $x,y\in X$, we have

$$x\preceq f(x,y), f(y,x)\preceq y \Rightarrow f(x,y)\preceq g(f(x,y),f(y,x)), g(f(y,x),f(x,y))\preceq f(y,x)$$

and

$$x\preceq g(x,y), g(y,x)\preceq y \Rightarrow g(x,y)\preceq f(g(x,y),g(y,x)), f(g(y,x),g(x,y))\preceq g(y,x).$$

**Definition 1.11.** [7] The function $\phi : [0,\infty)\to[0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) $\phi$ is continuous and nondecreasing ;

(ii) $\phi(t)=0 \Leftrightarrow t=0$;

Let $\Psi$ be the set of all the functions $\psi : [0,\infty)\to[0,\infty)$ such that $\lim_{t\to r}\psi(t)>0$ for all $r>0$ and $\lim_{t\to 0^+}\psi(t)=0$.

## 2. Main Results

**Theorem 2.1.** *Let $(X,\preceq,S)$ be a partially ordered S-metric space; $f,g : X\times X\to X$ be two maps such that*

*(1) $X$ is complete;*

(2) *The pair* $(f,g)$ *has the mixed weakly monotone property on* $X$*;*

$$x_0 \preceq f(x_0,y_0), y_0 \succeq f(y_0,x_0) \text{ or } x_0 \preceq g(x_0,y_0), y_0 \succeq g(y_0,x_0) \text{ for some } x_0,y_0 \in X$$

(3) *Suppose there exist* $\phi$ *is an altering distance function and* $\psi \in \Psi$ *such that*

$$\begin{aligned} \phi(S(f(x,y),f(x,y),g(u,v))) &\leq \phi(\max\{S(x,x,u),S(y,y,v)\}) \\ &\quad -\psi(\max\{S(x,x,u),S(y,y,v)\}) \end{aligned} \tag{2.1}$$

*for all* $x,y,u,v \in X$ *with* $x \preceq u$ *and* $y \succeq v$.

(4) $f$ *or* $g$ *is continuous or* $X$ *has the following property:*

***i*:** *If* $x_n$ *is an increasing sequence with* $x_n \to x$*, then* $x_n \preceq x$ *for all* $n \in \mathbb{N}$*;*

***ii*:** *If* $x_n$ *is a decreasing sequence with* $x_n \to x$*, then* $x \preceq x_n$ *for all* $n \in \mathbb{N}$.

*Then* $f$ *and* $g$ *have a coupled common fixed point in* $X$.

*Proof.* We prove the theorem in several steps.

Step 1. Let $x_0,y_0 \in X$ be arbitrary points such that $x_0 \preceq f(x_0,y_0)$ and $y_0 \succeq f(y_0,x_0)$ (the case $x_0 \preceq g(x_0,y_0)$ and $y_0 \succeq g(y_0,x_0)$ is proved similarly by interchanging the roles of $f$ and $g$).

Taking $x_1 = f(x_0,y_0), y_1 = f(y_0,x_0)$. Since $(f,g)$ has the mixed weakly monotone property, we have

$$x_1 = f(x_0,y_0) \preceq g(f(x_0,y_0),f(y_0,x_0)) = g(x_1,y_1)$$

and

$$y_1 = f(y_0,x_0) \succeq g(f(y_0,x_0),f(x_0,y_0)) = g(y_1,x_1).$$

Taking $x_2 = g(x_1,y_1), y_2 = g(y_1,x_1)$. Then we have

$$x_2 = g(x_1,y_1) \preceq f(g(x_1,y_1),g(y_1,x_1)) = f(x_2,y_2)$$

and

$$y_2 = g(y_1,x_1) \succeq f(g(y_1,x_1),g(x_1,y_1)) = f(y_2,x_2).$$

Continuously, for all $n = 0,1,2,\ldots$, we put

$$\begin{aligned} x_{2n+1} &= f(x_{2n},y_{2n}), \qquad y_{2n+1} = f(y_{2n},x_{2n}) \\ x_{2n+2} &= g(x_{2n+1},y_{2n+1}), \qquad y_{2n+2} = g(y_{2n+1},x_{2n+1}). \end{aligned} \tag{2.2}$$

Therefore, we obtain that

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq \cdots \text{ and } y_0 \succeq y_1 \succeq \cdots \succeq y_n \succeq \cdots \tag{2.3}$$

Now, we shall show that $\{x_n\}$ and $\{y_n\}$ are two Cauchy sequences. Since $x_{2n} \preceq x_{2n+1}$ and $y_{2n} \succeq y_{2n+1}$, it follows from (2.1) that

$$\begin{aligned} \phi(S(x_{2n+1}, x_{2n+1}, x_{2n+2})) &= \phi(S(f(x_{2n}, y_{2n}), f(x_{2n}, y_{2n}), g(x_{2n+1}, y_{2n+1}))) \\ &\leq \phi(\max\{S(x_{2n}, x_{2n}, x_{2n+1}), S(y_{2n}, y_{2n}, y_{2n+1})\}) - \\ &\quad \psi(\max\{S(x_{2n}, x_{2n}, x_{2n+1}), S(y_{2n}, y_{2n}, y_{2n+1})\}). \end{aligned} \tag{2.4}$$

Analogously to (2.4), we have

$$\begin{aligned} \phi(S(y_{2n+1}, y_{2n+1}, y_{2n+2})) &= \phi(S(f(y_{2n}, x_{2n}), f(y_{2n}, x_{2n}), g(y_{2n+1}, x_{2n+1}))) \\ &\leq \phi(\max\{S(y_{2n}, y_{2n}, y_{2n+1}), S(x_{2n}, x_{2n}, x_{2n+1})\}) - \\ &\quad \psi(\max\{S(y_{2n}, y_{2n}, y_{2n+1}), S(x_{2n}, x_{2n}, x_{2n+1})\}). \end{aligned} \tag{2.5}$$

For all $n \geq 0$, (2.4) and (2.5) combine to give

$$\begin{aligned} &\max\{\phi(S(x_{2n+1}, x_{2n+1}, x_{2n+2})), \phi(S(y_{2n+1}, y_{2n+1}, y_{2n+2}))\} \\ &\qquad \leq \phi(\max\{S(x_{2n}, x_{2n}, x_{2n+1}), S(y_{2n}, y_{2n}, y_{2n+1})\}) \\ &\qquad - \psi(\max\{S(y_{2n}, y_{2n}, y_{2n+1}), S(x_{2n}, x_{2n}, x_{2n+1})\}) \end{aligned}$$

but we have

$$\begin{aligned} &\phi(\max\{S(x_{2n+1}, x_{2n+1}, x_{2n+2}), S(y_{2n+1}, y_{2n+1}, y_{2n+2})\}) \\ &\qquad = \max\{\phi(S(x_{2n+1}, x_{2n+1}, x_{2n+2})), \phi(S(y_{2n+1}, y_{2n+1}, y_{2n+2}))\}. \end{aligned}$$

That is

$$\begin{aligned}(2.6)\quad \phi(\max\{S(x_{2n+1},x_{2n+1},x_{2n+2}),S(y_{2n+1},y_{2n+1},y_{2n+2})\}) \\ \leq \phi(\max\{S(x_{2n},x_{2n},x_{2n+1}),S(y_{2n},y_{2n},y_{2n+1})\}) \\ -\psi(\max\{S(x_{2n},x_{2n},x_{2n+1}),S(y_{2n},y_{2n},y_{2n+1})\}).\end{aligned}$$

Similarly, we have

$$\begin{aligned}(2.7)\quad \phi(\max\{S(x_{2n},x_{2n},x_{2n+1}),S(y_{2n},y_{2n},y_{2n+1})\}) \\ \leq \phi(\max\{S(x_{2n-1},x_{2n-1},x_{2n}),S(y_{2n-1},y_{2n-1},y_{2n})\}) \\ -\psi(\max\{S(x_{2n-1},x_{2n-1},x_{2n}),S(y_{2n-1},y_{2n-1},y_{2n})\}).\end{aligned}$$

Hence, it follows from (2.6) and (2.7), for all $n \in \mathbb{N}$, that

$$\begin{aligned}\phi(\max\{S(x_{n+1},x_{n+1},x_{n+2}),S(y_{n+1},y_{n+1},y_{n+2})\}) \\ \leq \phi(\max\{S(x_n,x_n,x_{n+1}),S(y_n,y_n,y_{n+1})\}) \\ -\psi(\max\{S(x_n,x_n,x_{n+1}),S(y_n,y_n,y_{n+1})\})\end{aligned}$$

Now, denote $\delta_n = \max\{S(x_n,x_n,x_{n+1}),S(y_n,y_n,y_{n+1})\}$ for all $n \in \mathbb{N}$. So we have

$$\phi(\delta_{n+1}) \leq \phi(\delta_n) - \psi(\delta_n). \tag{2.8}$$

Since $\phi$ is non-decreasing, it follows that the sequence $\delta_n$ is monotone decreasing. Therefore, there is some $\delta \geq 0$ such that $\lim_{n\to\infty}\delta_n = \delta$. We want to show that $\delta = 0$. Assume on contrary that $\delta > 0$. Then taking the limit as $n \to \infty$ (equivalently, $\delta_n \to \delta$) in (2.8), using the fact that $\lim_{t\to r}\psi(t) > 0$ for all $r > 0$ and $\phi$ is continuous, we obtain

$$\phi(\delta) = \lim_{n\to\infty}\phi(\delta_{n+1}) \leq \lim_{n\to\infty}[\phi(\delta_n) - \psi(\delta_n)] = \phi(\delta) - \lim_{\delta_n\to\delta}\psi(\delta_n) < \phi(\delta)$$

a contradiction. Thus $\delta = 0$, that is

$$\lim_{n\to\infty} \max\{S(x_n,x_n,x_{n+1}),S(y_n,y_n,y_{n+1})\} = \lim_{n\to\infty} \delta_n = 0. \tag{2.9}$$

Since $\lim_{n\to\infty} \max\{S(x_n,x_n,x_{n+1}),S(y_n,y_n,y_{n+1})\} = 0$. That is

$$\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = \lim_{n\to\infty} S(y_n,y_n,y_{n+1}) = 0.$$

Step 2. We prove that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in $X$. Suppose that $x_n$ and $y_n$ are not Cauchy, then there exists $\varepsilon > 0$ for which we can find two sequences of positive integers $n(k)$ and $m(k)$ such that for all positive integer $k$ with $n(k) > m(k) \geq k$, we have

$$\max\{S(x_{n(k)},x_{n(k)},x_{m(k)}),S(y_{n(k)},y_{n(k)},y_{m(k)})\} > \frac{\varepsilon}{k} \tag{2.10}$$

Furthermore, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k)$ and satisfying (2.10). Then

$$\max\{S(x_{n(k)-1},x_{n(k)-1},x_{m(k)}),S(y_{n(k)-1},y_{n(k)-1},y_{m(k)})\} \leq \frac{\varepsilon}{k}.$$

It follows from (2.10) and Lemma 1.4 that

$$\begin{aligned}
\frac{\varepsilon}{k} &< \max\{S(x_{n(k)},x_{n(k)},x_{m(k)}),S(y_{n(k)},y_{n(k)},y_{m(k)})\} \\
&\leq \max\{2S(x_{n(k)},x_{n(k)},x_{n(k)-1}) + S(x_{m(k)},x_{m(k)},x_{n(k)-1}), \\
&\quad 2S(y_{n(k)},y_{n(k)},y_{n(k)-1}) + S(y_{m(k)},y_{m(k)},y_{n(k)-1})\} \\
&\leq 2\max\{S(x_{n(k)},x_{n(k)},x_{n(k)-1}),S(y_{n(k)},y_{n(k)},y_{n(k)-1})\} \\
&\quad + \max\{S(x_{m(k)},x_{m(k)},x_{n(k)-1}),S(y_{m(k)},y_{m(k)},y_{n(k)-1}) \\
&\leq 2\delta_{n(k)-1} + \frac{\varepsilon}{k}.
\end{aligned}$$

Taking the limit as $k \to \infty$ in the above inequality, we have, by (2.9)

$$\lim_{k\to\infty} \max\{S(x_{n(k)},x_{n(k)},x_{m(k)}),S(y_{n(k)},y_{n(k)},y_{m(k)})\} = 0$$

which is a contradiction. So $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in a complete metric space $X$. Hence there exist $x,y \in X$ such that $x_n \to x$ and $y_n \to y$ in $X$ as $n \to \infty$.

Step 3. We prove that $(x,y)$ is a coupled common fixed point of $f$ and $g$. We consider the following three cases.

Case 3.1. $f$ is continuous. Thus

$$x = \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} f(x_{2n}, y_{2n}) = f(\lim_{n\to\infty} x_{2n}, \lim_{n\to\infty} y_{2n}) = f(x,y)$$

and

$$y = \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} f(y_{2n}, x_{2n}) = f(\lim_{n\to\infty} y_{2n}, \lim_{n\to\infty} x_{2n}) = f(y,x)$$

Now using (2.1)

$$\begin{aligned}
\phi(S(f(x,y), f(x,y), g(x,y))) &\leq \phi(\max\{S(x,x,x), S(y,y,y)\}) \\
&\quad -\psi(\max\{S(x,x,x), S(y,y,y)\}) \\
&= \phi(0) - \psi(0) = 0.
\end{aligned}$$

Therefore,

$$\phi(S(f(x,y), f(x,y), g(x,y))) = 0.$$

Using the property of $\phi$, we have

$$S(f(x,y), f(x,y), g(x,y)) = 0.$$

That is,

$$S(x,x,g(x,y)) = 0.$$

Thus

$$g(x,y) = x.$$

By similar way we can prove that $g(y,x) = y$. Therefore, $(x,y)$ is a coupled common fixed point of $f$ and $g$.

Case 3.2. $g$ is continuous. We can also prove that $(x,y)$ is a coupled common fixed point of $f$ and $g$ similarly as in Case 3.1.

Case 3.3. $X$ satisfies two assumptions $(i)$ and $(ii)$. Then by (2.3) we have $x_{2n} \preceq x$ and $y_{2n} \succeq y$ for all $n \in \mathbb{N}$. By using (2.1) and (2.2), we have

$$\begin{aligned}
\phi(S(x_{2n+1}, x_{2n+1}, g(x,y))) &= \phi(S(f(x_{2n}, y_{2n}), f(x_{2n}, y_{2n}), g(x,y))) \\
&\leq \phi(\max\{S(x_{2n}, x_{2n}, x), S(, y_{2n}, , y_{2n}, y)\}) \\
&\quad -\psi(\max\{S(x_{2n}, x_{2n}, x), S(, y_{2n}, , y_{2n}, y)\}).
\end{aligned}$$

By taking the limit as $n \to \infty$, and using the properties of $\phi$ and $\psi$, we obtain

$$\phi(S(x,x,g(x,y))) \leq \phi(\max\{S(x,x,x),S(,y,,y,y)\}) = 0.$$

That is, $\phi(S(x,x,g(x,y))) = 0$. So $S(x,x,g(x,y)) = 0$ and then $g(x,y) = x$. Now, we have

$$\begin{aligned}\phi(S(f(x,y),f(x,y),g(x,y))) &\leq \phi(\max\{S(x,x,x),S(,y,,y,y)\}) \\ &\quad -\psi(\max\{S(x,x,x),S(,y,,y,y)\}).\end{aligned}$$

Thus, $\phi(S(f(x,y),f(x,y),g(x,y))) = 0$, and then $S(f(x,y),f(x,y),g(x,y)) = 0$. Therefore, $f(x,y) = g(x,y) = x$. In similar way we can get that $f(y,x) = g(y,x) = y$. This proves that $(x,y)$ is a coupled common fixed point of $f$ and $g$. □

Now, we will show that many results can be deduced from our previous obtained result.

**Corollary 2.2.** *Let $(X,\preceq,S)$ be a partially ordered S-metric space; $f,g : X \times X \to X$ be two maps such that*

*(1) $X$ is complete;*

*(2) The pair $(f,g)$ has the mixed weakly monotone property on $X$;*

$$x_0 \preceq f(x_0,y_0), y_0 \succeq f(y_0,x_0) \text{ or } x_0 \preceq g(x_0,y_0), y_0 \succeq g(y_0,x_0) \text{ for some } x_0,y_0 \in X$$

*(3) Suppose that for all $x,y,u,v \in X$ with $x \preceq u$ and $y \succeq v$, we have*

$$S(f(x,y),f(x,y),g(u,v)) \leq \frac{k}{2}[S(x,x,u)+S(y,y,v)] \tag{2.11}$$

*where $k \in [0,1)$.*

*(4) $f$ or $g$ is continuous or $X$ has the following property:*

**i:** *If $x_n$ is an increasing sequence with $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;*

**ii:** *If $x_n$ is a decreasing sequence with $x_n \to x$, then $x \preceq x_n$ for all $n \in \mathbb{N}$.*

*Then $f$ and $g$ have a coupled common fixed point in $X$.*

*Proof.* If $S$ satisfies (2.11), then $S$ satisfies (2.1) with $\phi(t) = t$ and $\psi(t) = (1-k)t$. Then, the result follows from Theorem 2.1. □

*Remark* 2.3. Under the assumptions of Theorem 2.1 and Corollary 2.2 , if we set $g(u,v) = f(u,v)$ for all $(u,v)$, then we obtain an extension of main results of Harjani *et al.* [6] and Bhaskar and Lakshmikantham [1] into the structure of S-metric spaces.

Other consequences of our theorem is the following result for mappings with the mixed weakly monotone property satisfying a contraction of integral type.

Denote by $\Lambda$ the set of functions $\mu : [0,\infty) \to [0,\infty)$ satisfying the following hypotheses:

(1) $\mu$ is a Lebesgue-integrable mapping on each compact of $[0,\infty)$.
(2) For every $\varepsilon > 0$, we have $\int_0^{\varepsilon} \mu(t)dt > 0$.

**Corollary 2.4.** *Let $(X,\preceq,S)$ be a partially ordered S-metric space; $f,g : X\times X \to X$ be two maps such that*

*(1) $X$ is complete;*
*(2) The pair $(f,g)$ has the mixed weakly monotone property on $X$;*

$$x_0 \preceq f(x_0,y_0), y_0 \succeq f(y_0,x_0) \text{ or } x_0 \preceq g(x_0,y_0), y_0 \succeq g(y_0,x_0) \text{ for some } x_0,y_0 \in X$$

*(3) Suppose that for all $x,y,u,v \in X$ with $x \preceq u$ and $y \succeq v$, we have*

$$\int_0^{S(f(x,y),f(x,y),g(u,v))} \mu_1(t)dt \leq \int_0^{\max\{S(x,x,u),S(y,y,v)\}} \mu_1(t)dt - \int_0^{\max\{S(x,x,u),S(y,y,v)\}} \mu_2(t)dt$$

*(4) $f$ or $g$ is continuous or $X$ has the following property:*

***i*: If $x_n$ is an increasing sequence with $x_n \to x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$;**
***ii*: If $x_n$ is a decreasing sequence with $x_n \to x$, then $x \preceq x_n$ for all $n \in \mathbb{N}$.**

*Then $f$ and $g$ have a coupled common fixed point in $X$.*

*Proof.* Define $\phi(r) = \int_0^r \mu_1(t)dt$ and $\psi(r) = \int_0^r \mu_2(t)dt$ for all $r \in [0,\infty)$. Then the result follows from Theorem 2.1. □

**Example 2.5.** *Let $(\mathbb{R},\preceq,S)$ be a partially ordered complete S-metric space where $S(x,y,z) = |y+z-2x|+|y-z|$ for all $x,y,z \in \mathbb{R}$. Define the altering distance function $\phi : [0,\infty)$ by $\phi(t) = \frac{t}{3}$ and $\psi \in \Psi$ where $\psi(t) = \frac{t}{6}$. Let $f,g : \mathbb{R}^2 \to \mathbb{R}$ be two maps defined by $f(x,y) = \frac{6x-3y+69}{72}$*

*and* $g(x,y) = \frac{8x-4u+92}{96}$. $(f,g)$ *has the mixed weakly monotone property on* $\mathbb{R}$. $f$ *and* $g$ *are continuous. Also for all* $x,y,u,v \in \mathbb{R}$ *with* $x \preceq u$ *and* $y \succeq v$ *we have*

$$\begin{aligned}
\phi(S(f(x,y),f(x,y),g(u,v))) &= \phi(2 \mid f(x,y)-g(u,v) \mid) \\
&= \phi(\mid \frac{24(x-u)-12(y-v)}{144} \mid) \\
&= \mid \frac{24(x-u)-12(y-v)}{432} \mid \\
&\leq \mid \frac{x-u}{18} \mid + \mid \frac{y-v}{36} \mid \\
&\leq 2\max\{\frac{\mid x-u \mid}{18}, \frac{\mid y-v \mid}{18}\} \\
&\leq \frac{1}{18}\max\{2 \mid x-u \mid, 2 \mid y-v \mid\} \\
&\leq \frac{1}{6}\max\{2 \mid x-u \mid, 2 \mid y-v \mid\} \\
&= \frac{1}{3}\max\{2 \mid x-u \mid, 2 \mid y-v \mid\} \\
&\quad -\frac{1}{6}\max\{2 \mid x-u \mid, 2 \mid y-v \mid\} \\
&= \phi(\max\{2 \mid x-u \mid, 2 \mid y-v \mid\}) \\
&\quad -\psi(\max\{2 \mid x-u \mid, 2 \mid y-v \mid\})
\end{aligned}$$

*Then the hypothesis of Theorem 2.1 are holds, so (1,1) is a coupled common fixed point of* $f$ *and* $g$.

## 3. Application

In this part, we deal with Volterra-type integral equations. We will apply the result of Theorem 2.1 to prove the existence of solutions of a system of nonlinear integral equations in complete $S$-metric space.

Consider the Volterra integral equations in the following system:

$$\begin{aligned}
x(t) &= q(t) + \int_a^t [k(t,s,x(s)) + l(t,s,y(s))]ds \\
y(t) &= q(t) + \int_a^t [k(t,s,y(s)) + l(t,s,x(s))]ds
\end{aligned} \tag{3.1}$$

Denote by $X = C([a,b],\mathbb{R})$ the space of all continuous functions defined on $[a,b]$. Let $S : X^3 \to [0,\infty)$ be a $S$-metric space define by $S(x,y,z) = \|y+z-2x\| + \|y-z\|$ where $\|x-y\| = \max_{t\in[a,b]} |x(t)-y(t)|$, for all $x,y,z \in X$. We endow $X$ with the partial ordered $\leq$ given by: $x,y \in X$, $x \leq y \Leftrightarrow x(t) \leq y(t)$ for all $t \in [a,b]$. Obviously, $(X,\leq,S)$ is a complete $S$-metric space.

On the other hand, $(X,\leq,S)$ is regular see [10].

**Theorem 3.1.** *Let $X = C([a,b],\mathbb{R})$ for all $t \in [a,b]$. Assume that*

(i) $k,l : [a,b]^2 \times \mathbb{R} \to \mathbb{R}$ *are continuous,*

(ii) $q : [a,b] \to \mathbb{R}$ *is continuous,*

(iii) *there exists $\alpha > 0$, $\beta < \frac{1}{2\alpha}$ for all $x,y \in X$, $x \leq y$, $t \in [a,b]$*

$$0 \leq k(t,s,y(s)) - k(t,s,x(s)) \leq \frac{\alpha\beta}{(b-a)}(y(s)-x(s)),$$
$$0 \leq l(t,s,x(s)) - l(t,s,y(s)) \leq \frac{\alpha\beta}{(b-a)}(y(s)-x(s)),$$

(iv) *there exist continuous functions $x_0, y_0 : [a,b] \to \mathbb{R}$ such that*

$$x_0(t) \leq q(t) + \int_a^t [k(t,s,x_0(s)) + l(t,s,y_0(s))]ds,$$
$$y_0(t) \geq q(t) + \int_a^t [k(t,s,y_0(s)) + l(t,s,x_0(s))]ds.$$

*Then the system of Volterra integral equations 3.1 has a solution in $X^2 = (C[a,b]^2,\mathbb{R})$.*

*Proof.* Define the mappings $f,g : X \times X \to X$ by

$$f(x,y)(t) = g(x,y)(t) = q(t) + \int_a^t [k(t,s,x(s)) + l(t,s,y(s))]ds$$

for all $x,y \in X$ and $t \in [a,b]$. First, we will prove that $f$ has the mixed monotone property. If $x_1 \leq x_2$ and $t \in [a,b]$, we have

$$f(x_2,y)(t) - f(x_1,y)(t) = \int_a^t [k(t,s,x_2(s)) - k(t,s,x_1(s))]ds,$$

since $x_1(t) \leq x_2(t)$ for all $t \in [a,b]$, and by (iii) we have $k(t,s,x_2(s)) \geq k(t,s,x_1(s))$. Hence $f(x_2,y)(t) \geq f(x_1,y)(t)$, $\forall t \in [a,b]$. So

$$f(x_1,y) \leq f(x_2,y).$$

Similarly, for $y_1 \leq y_2$ and $t \in [a,b]$,

$$f(x,y_1)(t) - f(x,y_2)(t) = \int_a^t [l(t,s,y_1(s)) - l(t,s,y_2(s))]ds,$$

since $y_1(t) \leq y_2(t)$ for all $t \in [a,b]$, and by (iii) we have $l(t,s,y_1(s)) \geq l(t,s,y_2(s))$. Hence $f(x,y_1)(t) \geq f(x,y_2)(t)$, $\forall t \in [a,b]$. Then

$$f(x,y_1) \geq f(x,y_2).$$

Hence, $f$ has the mixed monotone property. Now, for $x,y,u,v \in X$ with $x \leq u, y \geq v$ we have

$$S(f(x,y), f(x,y), f(u,v)) = 2\|f(x,y) - f(u,v)\| = 2\max_{t\in[a,b]} |f(x,y)(t) - f(u,v)(t)|.$$

$$\begin{aligned}
|f(x,y)(t) - f(u,v)(t)| &= |\int_a^t [k(t,s,x(s)) - k(t,s,u(s))]ds + \int_a^t [l(t,s,y(s)) - l(t,s,v(s))]ds| \\
&\leq \int_a^t |k(t,s,x(s)) - k(t,s,u(s))|ds + \int_a^t |l(t,s,y(s)) - l(t,s,v(s))|ds \\
&\leq \int_a^t \frac{\alpha\beta}{(b-a)}|x(s) - u(s)|ds + \int_a^t \frac{\alpha\beta}{(b-a)}|y(s) - v(s)|ds \\
&= \frac{\alpha\beta}{(b-a)}[\int_a^t |x(s) - u(s)|ds + \int_a^t |y(s) - v(s)|ds] \\
&\leq \frac{\alpha\beta}{(b-a)}(\max_{t\in[a,b]} |x(t) - u(t)| + \max_{t\in[a,b]} |y(t) - v(t)|) \int_a^t ds.
\end{aligned}$$

Thus

$$\begin{aligned}
\phi(S(f(x,y), f(x,y), f(u,v))) &= \phi(2\max_{t\in[a,b]} |f(x,y)(t) - f(u,v)(t)|) \\
&= 2\max_{t\in[a,b]} |f(x,y)(t) - f(u,v)(t)| \\
&\leq \alpha\beta(S(x,x,u) + S(y,y,v)) \\
&\leq 2\alpha\beta \max\{S(x,x,u), S(y,y,v)\} \\
&= \max\{S(x,x,u), S(y,y,v)\} - (1-2\alpha\beta)\max\{S(x,x,u), S(y,y,v)\} \\
&= \phi(\max\{S(x,x,u), S(y,y,v)\}) - \psi(\max\{S(x,x,u), S(y,y,v)\}).
\end{aligned}$$

This show that the mapping $f$ satisfies the contractive condition 2.1 with $f=g$. Let $x_0, y_0$ be the functions in assumption (iv), so we have

$$x_0 \leq f(x_0, y_0), \qquad y_0 \geq f(y_0, x_0)$$

Hence, all the hypotheses of Theorem 2.1 are satisfied with $\phi(t)=t$, $\psi=(1-2\alpha\beta)t$ where $\alpha>0$, $\beta<\frac{1}{2\alpha}$. Therefore $f$ has a coupled fixed point $(x,y)$ in $X$, that is the system of integral equations has a solution. □

## Conflict of Interests

The author(s) declare that there is no conflict of interests.

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