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REMARKS ON COLLECTIVELY MAXIMAL ELEMENTS

SEHIE PARK[1,2,*]

[1]The National Academy of Sciences, Republic of Korea; Seoul 06579

[2]Department of Mathematical Sciences, Seoul National University, Seoul 08826, Korea

Abstract. In 1983, Yannelis and Prahbakar [19] presented some mathematical theorems which were used to generalize previous results on the existence of maximal elements and of equilibrium. Since then a large number of authors have worked on generalizations or applications of some basic theorems in [19]. In the present article, we show that theorems on collectively maximal elements can be deduced from one of our versions of generalized KKM type theorems for abstract convex spaces. As an example, an existence result of Kim and Yuan [6] can be extended and some of its modifications by several authors are introduced.

Keywords: abstract convex space; KKM theorem; KKM space; mapping classes $\mathfrak{KC}$; $\mathfrak{KO}$; collectively maximal element.

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1. Introduction

In 1983, Yannelis and Prahbakar [19] presented some mathematical theorems which were used to generalize previous results on the existence of maximal elements and of equilibrium. Since then a large number of authors have worked on generalizations or applications of some basic theorems in [19].

*Corresponding author

E-mail addresses: park35@snu.ac.kr; sehiepark@gmail.com

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For a multimap $F : X \multimap Y$, where X and Y are topological spaces, a point $x \in X$ is termed a *maximal element* if $F(x) = \emptyset$. For a nonempty index set I and each $i \in I$, let X_i be a topological space and $F_i : X = \prod_{i \in I} X_i \multimap X_i$ a multimap. Then a point $x = (x_i)_{i \in I} \in X$ is called a *collectively maximal element* for the family $\{F_i\}_{i \in I}$ if $F_i(x) = \emptyset$ for all $i \in I$. The existence of maximal elements and its important applications to mathematical economics have been studied by many authors in both mathematics and economics; for earlier works, see [2, 5, 19] and the references at the end of this article.

Our main aim in this article is to show that some of known existence theorems on collectively maximal elements are consequences of one of our generalized forms of the Knaster-Kuratowski-Mazurkiewicz (simply, KKM) type theorem for abstract convex spaces. Consequently, the works related to collectively maximal elements can be deduced from the KKM theory on abstract convex spaces due to ourselves since 2006.

Especially in 2001, Kim and Yuan [6] gave a such theorem on collectively maximal elements for topological vector spaces. Several versions of the theorem were obtained and applied by Lin and Ansari [8, Corollary 4.4] in 2004, Al-Homidan and Ansari [1, Theorem 2.1], Hai and Khan [4, Theorem 2.1], Lin, Chen, and Ansari [9, Theorem 2.1.1], Lin and Tu [11, Theorem 2.2] in 2007, and others. Note that all of those theorems are stated for Hausdorff topological vector spaces.

In fact, we show that all of such theorems can be extended to the corresponding ones for abstract convex spaces due to ourselves. More precisely, from an abstract version of the Kim-Yuan theorem, we can deduce all of such above-mentioned theorems.

This article is organized as follows: Section 2 is a preliminary for abstract convex spaces. In Section 3, we introduce our most general KKM type theorem (Theorem C) and derive an extended form (Theorem D) of various existence theorems for collectively maximal elements in abstract convex spaces. Section 4 devotes to find existing examples of our main Theorem D in the literature. Here we give an extended version of a theorem of Kim and Yuan [6] on collectively maximal elements. In Section 5, examples of known existence theorems which follow from our new theorem. Finally, our conclusion is given in Section 6.

2. PRELIMINARIES FOR ABSTRACT CONVEX SPACES

In order to upgrade the KKM theory, in 2006-10, we proposed new concepts of abstract convex spaces and the KKM spaces, which are proper generalizations of various known types of particular spaces and adequate to establish the KKM theory.

In this paper, multimaps are also called simply *maps*. Let $\langle D \rangle$ denote the set of all nonempty finite subsets of a set D.

Recall the following in [13, 14, 3]:

Definition. An *abstract convex space* $(E,D;\Gamma)$ consists of a topological space E, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap E$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$, such that the Γ-*convex hull* of any $D' \subset D$ is denoted and defined by

$$\mathrm{co}_\Gamma D' := \bigcup \{\Gamma_A \mid A \in \langle D' \rangle\} \subset E.$$

A subset X of E is called a Γ-*convex subset* of $(E,D;\Gamma)$ relative to $D' \subset D$ if for any $N \in \langle D' \rangle$, we have $\Gamma_N \subset X$, i.e., $\mathrm{co}_\Gamma D' \subset X$.

For the case $E = D$, we denote $(E;\Gamma) = (E,E;\Gamma)$.

Definition. Let $(E,D;\Gamma)$ be an abstract convex space and Z a topological space. For a multimap $F : E \multimap Z$ with nonempty values, if a multimap $G : D \multimap Z$ satisfies

$$F(\Gamma_A) \subset G(A) := \bigcup_{y \in A} G(y) \qquad \text{for all } A \in \langle D \rangle,$$

then G is called a *KKM map with respect to F*. A *KKM map* $G : D \multimap E$ is a KKM map with respect to the identity map 1_E of E.

A multimap $F : E \multimap Z$ is called a $\mathfrak{KC}$-*map* [resp. a $\mathfrak{KO}$-*map*] if, for any closed-valued [resp. open-valued] KKM map $G : D \multimap Z$ with respect to F, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. In this case, we denote $F \in \mathfrak{KC}(E,Z)$ [resp. $F \in \mathfrak{KO}(E,Z)$].

Definition. The *partial KKM principle* for an abstract convex space $(E,D;\Gamma)$ is the statement $1_E \in \mathfrak{KC}(E,E)$, that is, for any closed-valued KKM map $G : D \multimap E$, the family $\{G(y)\}_{y \in D}$ has the finite intersection property. The *KKM principle* is the statement $1_E \in \mathfrak{KC}(E,E) \cap \mathfrak{KO}(E,E)$, that is, the same property also holds for any open-valued KKM map.

3. The Basic Theorem for Collectively Maximal Elements

In this section, from one of the most general KKM type theorems due to ourselves, we derive an extended form of a basic existence theorem for collectively maximal elements in abstract convex spaces.

In recent works of the author, we obtained three general KKM type theorems (Theorems A, B, C) for abstract convex spaces. In fact, two of them were stated for intersectionally closed-valued KKM maps in the sense of Luc, Sarabi, and Soubeyran [12].

In order to introduce our most general KKM type theorem, consider the following related four conditions for a map $G : D \multimap Z$ with a topological space Z:

(a) $\bigcap_{y \in D} \overline{G(y)} \neq \emptyset$ implies $\bigcap_{y \in D} G(y) \neq \emptyset$.

(b) $\bigcap_{y \in D} \overline{G(y)} = \overline{\bigcap_{y \in D} G(y)}$ (G is *intersectionally closed-valued*).

(c) $\bigcap_{y \in D} \overline{G(y)} = \bigcap_{y \in D} G(y)$ (G is *transfer closed-valued*).

(d) G is closed-valued.

Recall that Luc, Sarabi, and Soubeyran [12] introduced condition (b) and showed that (a) $\Longleftarrow$ (b) $\Longleftarrow$ (c) $\Longleftarrow$ (d). Moreover, they called that complements of intersectionally closed sets are unionly open sets.

From the partial KKM principle we have the following one of the most general KKM type theorems in [15, 16]: ,

Theorem C. *Let $(E, D; \Gamma)$ be an abstract convex space, Z a topological space, $F \in \mathfrak{KC}(E, Z)$, and $G : D \multimap Z$ a map such that*

(1) $\overline{G}$ is a KKM map w.r.t. F; and

(2) there exists a nonempty compact subset K of Z such that either

(i) $K = Z$;

(ii) $\bigcap \{\overline{G(y)} \mid y \in M\} \subset K$ for some $M \in \langle D \rangle$; or

(iii) for each $N \in \langle D \rangle$, there exists a Γ-convex subset L_N of E relative to some $D' \subset D$ such that $N \subset D'$, $\overline{F(L_N)}$ is compact, and

$$\overline{F(L_N)} \cap \bigcap_{y \in D'} \overline{G(y)} \subset K.$$

Then we have

$$\overline{F(E)} \cap K \cap \bigcap_{y \in D} \overline{G(y)} \neq \emptyset.$$

Furthermore,

(α) *if G is transfer closed-valued, then $\overline{F(E)} \cap K \cap \bigcap \{G(y) \mid y \in D\} \neq \emptyset$; and*

(β) *if G is intersectionally closed-valued, then $\bigcap \{G(y) \mid y \in D\} \neq \emptyset$.*

Note that if $D = E$, then we can choose $D' = L_N$ in (iii).

Let I be a finite or infinite index set. For each $i \in I$, let $(E_i, D_i; \Gamma_i)$ be an abstract convex space, $E = \prod_{i \in I} E_i$ be equipped the product topology and $D = \prod_{i \in I} D_i$. For each $i \in I$, let π_i be the projections of E onto E_i and D onto D_i. Define $\Gamma : \langle D \rangle \multimap E$ by $\Gamma_N = \Gamma(N) = \prod_{i \in I} \Gamma_i(\pi_i(N))$ for each $N \in \langle D \rangle$. Then $(E, D; \Gamma)$ is the product abstract convex space.

Let Z be a topological space, $F \in \mathfrak{KC}(E, Z)$, and $A_i : Z \multimap D_i$ for each $i \in I$ be a map satisfying

(1) $F(\Gamma_N) \subset \bigcup_{y \in N} [A_i^{-1}(\pi_i(y))]^c$ for each $N \in \langle D \rangle$ and

(2) $A_i^{-1}(y_i)$ is open in Z for each $y_i \in D_i$.

Let $G : D \multimap Z$ be defined by $G(y) = \bigcap_{i \in I} [A_i^{-1}(\pi_i(y))]^c$ for $y \in D$. Then G is closed valued.

Now we need the following in this article:

Theorem D. *Let (E, D, Γ) be a product abstract convex space as above and Z be a topological space. Let $F \in \mathfrak{KC}(E, Z)$ and $A_i : Z \multimap D_i$, $G : D \multimap Z$ be given as above. If G satisfies one of the conditions (i)-(iii) of Theorem C, then there exists a point $\hat{z} \in Z$ such that $A_i(\hat{z}) = \emptyset$ for all $i \in I$.*

PROOF. By (1), we have

$$F(\Gamma_N) \subset G(N) = \bigcup_{y \in N} G(y) = \bigcup_{y \in N} \bigcap_{i \in I} [A_i^{-1}(\pi_i(y))]^c$$

for each $N \in \langle D \rangle$. Since $F \in \mathfrak{KC}(E,Z)$, $\{G(y)\}_{y \in D}$ has the finite intersection property. Moreover, since G satisfies one of the conditions (i)-(iii) of Theorem C, we have

$$\begin{aligned}
\bigcap_{y \in D} G(y) \neq \emptyset \ &\Longrightarrow\ \exists \hat{z} \in G(y)\ \ \forall y \in D \\
&\Longrightarrow \exists \hat{z} \in [A_i^{-1}(\pi_i(y))]^c\ \ \forall i \in I\ \ \forall y \in D \\
&\Longrightarrow \exists \hat{z} \notin A_i^{-1}(\pi_i(y))\ \ \forall i \in I\ \ \forall y \in D \\
&\Longrightarrow \pi_i(y) \notin A_i(\hat{z})\ \ \forall i \in I\ \ \forall y \in D \\
&\Longrightarrow A_i(\hat{z}) = \emptyset\ \ \forall i \in I.
\end{aligned}$$

This completes our proof. □

Theorem D has a large number of particular forms as like as Theorem C has. Instead of listing all of such possible forms, we try to find existing examples in the literature in the following sections.

4. Main Existence Theorem on Collectively Maximal Elements

The following result of Kim and Yuan [6] in 2001 was one of the main tools to study the existence of equilibria of generalized abstract economy:

Theorem 4.1. ([6]) *For each $i \in I$, let X_i be a nonempty convex subset of a Hausdorff topological vector space E_i. For each $i \in I$, let $S_i : X = \prod_{i \in I} X_i \multimap X_i$ be a multimap such that*

(i) for all $x = (x_i)_{i \in I} \in X$, $x_i \notin \operatorname{co} S_i(x)$;

(ii) for all $y_i \in X_i$, $S_i^{-1}(y_i)$ is open in X; and

(iii) there exist a nonempty compact subset K of X and a nonempty compact convex subset Y_i of X_i for each $i \in I$ with the property that for each $x \in X \setminus K$ there exists $j \in I$ such that $S_j(x) \cap Y_j \neq \emptyset$.

Then there exists $\hat{x} \in K$ such that $S_i(\hat{x}) = \emptyset$ for each $i \in I$.

Several versions of Theorem 4.1 were obtained or applied by Lin and Ansari [8, Corollary 4.4] in 2004, Al-Homidan and Ansari [1, Theorem 2.1], Hai and Khan [4, Theorem 2.1], Lin, Chen, and Ansari [9, Theorem 2.1.1], Lin and Tu [11, Theorem 2.2] in 2007, and others.

In order to generalize Theorem 4.1 for abstract convex spaces, we need the following:

Definition. Let X be a convex space in the sense of Lassonde [7]. A nonempty set $L \subset X$ is called a *c-compact* set if for each finite subset $N \subset X$ there is a compact convex set $L_N \subset X$ such that $K \cup N \subset L_N$.

Definition. Let $(X;\Gamma)$ be an abstract convex space. A nonempty set $L \subset X$ is called a Γ-*compact* set if for each finite subset $N \subset X$ there is a compact Γ-convex set $L_N \subset X$ such that $L \cup N \subset L_N$.

Note that a c-compact set is a Γ-compact set, and hence a nonempty compact convex subset of a Hausdorff topological vector space is Γ-compact.

Theorem 4.1 can be extended as follows:

Theorem 4.2. *For each $i \in I$, let $(X_i;\Gamma_i)$ be an abstract convex space such that $(X;\Gamma) = (\prod_{i\in I} X_i; \prod_{i\in I} \Gamma_i)$ is a partial KKM space, and $S_i : X \multimap X_i$ be a multimap such that*

(1) for all $x = (x_i)_{i\in I} \in X,\ x_i \notin \mathrm{co}_{\Gamma_i} S_i(x)$;

(2) for all $y_i \in X_i,\ (\mathrm{co}_{\Gamma_i} S_i)^{-1}(y_i)$ is open in X;

(3) there exist a nonempty compact subset K of X and a nonempty Γ_i-compact and Γ_i-convex subset $L_i \subset X_i$ for each $i \in I$ with the property that for each $x \in X \setminus K$, there exists $i \in I$ such that $S_i(x) \cap L_i \neq \emptyset$.

Then there exists $\hat{x} \in X$ such that $S_i(\hat{x}) = \emptyset$ for each $i \in I$.

PROOF. We apply our Theorem D-(iii) for $X := E = D = Z$, $X_i = D_i$, $S_i : X \to X_i$, and $G(y) := \bigcap_{i\in I}[S_i^{-1}(\pi_i(y))]^c$. For each $i \in I$ and $N \in \langle X \rangle$, let L_N^i be a compact Γ_i-convex subset of X_i containing $\pi_i(N)$ and $L_i \subset X_i$. Then $L_N := \prod_{i\in I} L_N^i$ is a compact Γ-convex subset of X containing $\prod_{i\in I} \pi_i(N)$ and $\prod_{i\in I} L_i$. In order to show condition (iii), it suffices to show

$$L_N \cap \bigcap_{x\in L_N} G(x) \subset K.$$

For each $x \in X \setminus K$, there exists $y_j \in S_j(x) \cap L_N^j$ for some $j \in I$. Choose an element $y := (y_i)_{i\in I} \in L_N$ such that $x \in S_j^{-1}(\pi_j(y))$ for some $j \in I$. Therefore we have

$$L_N \setminus K \subset X \setminus K \subset \bigcup_{y\in L_N} \bigcup_{i\in I} S_i^{-1}(\pi_i(y)) \subset \bigcup_{y\in L_N} [G(y)]^c.$$

Then we have

$$L_N \cap \bigcap_{y \in L_N} G(y) \subset K.$$

Then we have the conclusion by Theorem D. □

Note that Theorem 4.2 has a large number of examples. We show some of them in the next section.

5. Several Particular Forms

In this section, we introduce several known particular forms of Theorem 4.2 in the chronological order:

Lin and Ansari 2004 [8]

In [8], after giving several existence theorems for collectively maximal elements, the authors gave the following [8, Corollary 4.4]:

Theorem 5.1. ([8]) *For each $i \in I$, let X_i be a nonempty convex subset of a Hausdorff topological vector space E_i and $Q_i,\ T_i : X = \prod_{i \in I} X_i \multimap X_i$ be multimaps satisfying the following conditions:*

(a) For each $i \in I$ and for all $x \in X$, $\operatorname{co} Q_i(x) \subset T_i(x)$.

(b) For each $i \in I$ and for all $x = (x_i)_{i \in I} \in X$, $x_i \notin T_i(x)$, where x_i is the i-th projection of x.

(c) For each $i \in I$ and for all $y_i \in X_i$, $Q_i^{-1}(y_i)$ is compactly open in X.

(d) There exist a nonempty compact subset K of X and a nonempty compact convex subset $C_i \subset X_i$ for each $i \in I$ such that for all $x \in X \setminus K$, there exists $i \in I$ such that $Q_i(x) \cap C_i \neq \emptyset$.

Then there exists $\bar{x} \in X$ such that $Q_i(\bar{x}) = \emptyset$ for all $i \in I$.

Note that, by putting $\operatorname{co} Q_i(x) = T_i(x)$ for all $x \in X$, Theorem 5.1 reduces to Theorem 4.1 and can be extended to Theorem 4.2. Moreover, many of other results in [8] can be extended to abstract convex spaces.

In [8], some artificial terminology like compactly open, compactly closed, compact interior, transfer compactly open valued are introduced. However, these can be eliminated by adopting compactly generated topology instead of original topology.

Al-Homidan and Ansari 2007 [1]

In [1], the authors consider systems of quasi-equilibrium problems with lower and upper bounds and establish the existence of their solutions by using some known maximal element theorems for a family of multimaps.

This paper [1] is based on Theorem 5.1 of Lin and Ansari [8] in 2004. Hence certain results in [1] can be improved by applying our Theorem 4.2.

Hai and Khanh 2007 [4]

The authors of [4] propose four kinds of systems of set-valued quasivariational inclusion problems in product spaces, which include many known systems of equilibrium problems and systems of variational inequalities as well as inclusion problems.

In [4], the following is stated as [4, Theorem 2.1], which is an existence theorem of maximal elements for a family of multifunctions and established in [2] in a slightly stronger form:

Theorem 5.2. ([4]) *For each $i \in I$, let X_i be a Hausdorff topological vector space, let $A_i \subset X_i$ be a nonempty convex subset, and let $S_i : A = \prod_{i \in I} A_i \multimap A_i$ have convex values. Assume that the following conditions hold:*

(i) $S_i^{-1}(x_i)$ is open in A for all $x_i \in A_i$ and $i \in I$.

(ii) $x_i \notin S_i(x)$ for each $x \in A$ and $i \in I$.

(iii) If A is not compact, there exist a nonempty compact subset N of A and, $\forall i \in I$, a nonempty compact convex subset B_i of A_i such that, for each $x \in A \setminus N$, there exists $i \in I$ such that $B_i \cap S_i(x) \neq \emptyset$.

Then, there exists $\bar{x} \in A$ such that $S_i(\bar{x}) = \emptyset$ for all $i \in I$.

This is same to Theorem 4.1 and can be extended to Theorem 4.2. Similarly, certain results applying Theorem 5.2 in [4] can be extended.

Lin, Chen, and Ansari 2007 [9]

In [9], Theorem 4.1 of Kim and Yuan [6] is one of the main tools to study the existence of equilibria of generalized abstract economy. Therefore its extended form (our Theorem 4.2) can be used to improve certain results in [9].

Lin and Chuang 2007 [10]

In [10], the following lemma [10, Lemma 2.3] is stated due to Deguire, Tan, and Yuan [2] and applied as an important tool to study systems of nonempty intersection theorems. This is a slightly incorrect form of Theorem 4.1. Therefore, some results in this paper can be improved by adopting our Theorem 4.2.

Lemma 5.3. ([10]) *Let I be any index set. For each $i \in I$, let X_i be a nonempty closed convex subset of a Hausdorff t.v.s. E_i. For each $i \in I$, let $P_i : X = \prod_{i \in I} X_i \multimap X_i$ and suppose that:*

(i) for each $x = (x_i)_{i\in I} \in X$, $x_i \notin P_i(x)$;

(ii) P_i has convex values;

(iii) for each $y_i \in X_i$, $P_i^{-1}(y_i)$ is open in X;

(iv) there exist a nonempty compact subset K of X and a nonempty compact convex subset D_i of X_i for all $i \in I$ such that for each $x \in X \setminus K$, there exists $j \in I$ such that $P_j(x) \cap D_j \neq \emptyset$.

Then there exists $\bar{x} \in X$ such that for each $i \in I$, $P_i(\bar{x}) = \emptyset$.

Note that each X_i is not necessarily closed and that each D_i is c-compact.

Lin and Tu 2008 [11]

Lin and Tu [11] study existence theorems of solutions for systems of variational inclusions problems and systems of variational disclusions problems. From these existence results, they establish existence theorems of solutions for systems of generalized vector quasiequilibrium problems and systems of quasi-optimization problems.

This paper [11] is based on Theorem 4.1 (quoted as Theorem 2.2 by Deguire-Tan-Yuan [2] incorrectly) and can be improved by applying our Theorem 4.2.

6. Conclusion

Since 2006, we have been establishing the Grand KKM Theory on abstract convex spaces $(E,D;\Gamma)$. The present article aims to obtain new extended versions of collectively maximal element theorems for abstract convex spaces. In fact, we obtained Theorem D which unifies extended versions of many of known results.

Since Theorem D in Section 3 has many examples for various types of spaces, it has notable applications in corresponding papers and our improved theorems would be applicable to them. But we restricted to extend only basic theorems not for their applications. Moreover, most of our examples are stated for simple $(E;\Gamma)$, but they can be further extended to spaces of the form $(E,D;\Gamma)$.

Each new result in this article is stated for (partial) KKM spaces instead of their subclasses like convex subsets of Hausdorff topological vector spaces, G-convex spaces, and some other extensions or modifications. Recently we found scores of KKM spaces for which our new theorems on the KKM theory can be applicable. See [17, 18].

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