Available online at http://jfpt.scik.org
J. Fixed Point Theory, 2022, 2022:3
ISSN: 2052-5338

APPLICATIONS OF VARIOUS MAXIMUM PRINCIPLES

SEHIE PARK[1,2,*]

[1]The National Academy of Sciences, Republic of Korea; Seoul 06579

[2]Department of Mathematical Sciences, Seoul National University, Seoul 08826, Korea

Abstract. Certain maximum theorems can be reformulated to various types of fixed point theorems and conversely. Based on our Metatheorem, we show that many works of other authors after 2000 can be reformulated. Such authors are Suzuki, Lin-Du, Khamsi, Eshghinezhat-Fakhar, Fierro, Alegre-Marin, Al-Homidan-Ansari-Kassay, Cotrina-Théra-Zúñiga, Alghamd-Alzumi-Shahzad, and possibly some others.

Keywords: Caristi fixed point theorem; Ekeland variational principle; pre-order; quasi-metric space; fixed point; stationary point; w-distance.

2020 AMS Subject Classification: 06A75, 47H10, 54E35, 54H25, 58E30, 65K10.

1. INTRODUCTION

Zorn's lemma, the Banach contraction principle, Caristi's fixed point theorem, Ekeland's variational principle, and Takahashi's nonconvex minimization theorem are forceful tools in nonlinear analysis, control theory, economic theory, and global analysis. These theorems are extended by a large number of authors. Note that those principles are closely related to order theoretic fixed point theorems; see [14], [18], [21], [23].

---

[*] Corresponding author

E-mail addresses: park35@snu.ac.kr; sehiepark@gmail.com; parksehie.com

Received December 22, 2021

In 1985-2000, we had published several articles mainly related to the Ekeland variational principle for approximate solutions of minimization problems and its equivalent formulations with some applications; see [15]–[20]. From the beginning of such study, we obtained a Metatheorem for some equivalent statements on maximality, fixed points, stationary points, common fixed points, and common stationary points. We applied Metatheorem for various occasions. However, for a long period it was not attracted any other one.

Recently in 2022, we obtained an extended version of Metatheorem in [21] and applied it to Zorn's lemma, Banach contraction principle, Nadler's fixed point theorem, Brézis-Browder principle, Caristi's fixed point theorem, Ekeland's variational principle, Takahashi's nonconvex minimization theorem, some others and their variants, generalizations or equivalent formulations. Consequently, we have many new theorems equivalent to known results on fixed point, common fixed point, stationary point, common stationary point, and others.

In the present article, Section 2 devotes Metatheorem with the proof for completeness and its direct consequence for pre-ordered sets (Theorem 2.1). In Sections 3–11, by applying Metatheorem, we obtain logically equivalent formulations of existence of maximal elements or fixed points in other author's works after 2000. Such authors are, in the chronological order, Suzuki [22], Lin-Du [13], Khamsi [12], Eshghinezhat-Fakhar [8], Fierro [9], Alegre-Marin [2], Al-Homidan-Ansari-Kassay [3], Cotrina-Théra-Zúñiga [5], Alghamd-Alzumi-Shahzad [1], and possibly some others.

Finally, Section 12 deals with some conclusion.

2. A METATHEOREM RELATED TO THE EKELAND PRINCIPLE

The well-known central result of I. Ekeland [6], [7] on the variational principle for approximate solutions of minimization problems runs as follows:

Theorem E. (Ekeland [6]) *Let V be a complete metric space, and $F : V \to \mathbb{R} \cup \{+\infty\}$ a l.s.c. function, $\not\equiv +\infty$, bounded from below. Let $\varepsilon > 0$ be given, and a point $u \in V$ such that $F(u) \leqq \inf_V F + \varepsilon$. Then for every $\lambda > 0$, there exists a point $v \in \overline{\mathrm{B}}(u, \lambda)$ such that $F(v) \leqq F(u)$ and $F(w) > F(v) - \varepsilon\lambda^{-1} d(v, w)$ for any $w \in V$, $w \neq v$.*

When $\lambda = 1$, this is called the ε-variational principle. In order to get some equivalents of this principle, we obtained a Metatheorem in [15, 17] and its applications in 1985-2000. Later in 2022 we found a new extended version of the Metatheorem [21]. Now we add its proof for completeness.

Metatheorem. *Let X be a set, A its nonempty subset, and $G(x,y)$ a sentence formula for $x,y \in X$. Then the following are equivalent:*

(i) *There exists an element $v \in A$ such that $G(v,w)$ for any $w \in X \backslash \{v\}$.*

(ii) *If $T : A \multimap X$ is a multimap such that for any $x \in A \backslash T(x)$ there exists a $y \in X \backslash \{x\}$ satisfying $\neg G(x,y)$, then T has a fixed element $v \in A$, that is, $v \in T(v)$.*

(iii) *If $f : A \to X$ is a map such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $\neg G(x,y)$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

(iv) *If $f : A \to X$ is a map such that $\neg G(x, f(x))$ for any $x \in A$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

(v) *If $T : A \multimap X$ is a multimap such that $\neg G(x,y)$ holds for any $x \in A$ and any $y \in T(x) \backslash \{x\}$, then T has a stationary element $v \in A$, that is, $\{v\} = T(v)$.*

(vi) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $\neg G(x, f(x))$ for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(vii) *If $\mathfrak{F}$ is a family of multimaps $T_i : A \multimap X$ for $i \in I$ with an index set I such that $\neg G(x,y)$ holds for any $x \in A$ and any $y \in T_i(x) \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = T_i(v)$ for all $i \in I$.*

(viii) *If Y is a subset of X such that for each $x \in A \backslash Y$ there exists a $z \in X \backslash \{x\}$ satisfying $\neg G(x,z)$, then there exists a $v \in A \cap Y$.*

Here, multimaps have nonempty values and $\neg$ denotes the negation.

Proof. (i)$\Longrightarrow$(ii): Suppose $v \notin T(v)$. Then there exists a $y \in X \backslash \{v\}$ satisfying $\neg G(v,y)$. This is a contradiction.

(ii)$\Longrightarrow$(iii): Clear.

(iii)$\Longrightarrow$(iv): Clear.

(iv)$\Longrightarrow$(v): Suppose T has no stationary element, that is, $T(x)\backslash\{x\} \neq \emptyset$ for any $x \in A$. Choose a choice function f on $\{T(x)\backslash\{x\} : x \in A\}$. Then f has no fixed element by its definition. However, for any $x \in A$, we have $\neg G(x, f(x))$. Therefore, by (iv), f has a fixed element, a contradiction.

(v)$\Longrightarrow$(vi): Define a multimap $T : A \multimap X$ by $T(x) := \{f(x) : f \in \mathscr{F}\} \neq \emptyset$ for all $x \in A$. Since $\neg G(x, f(x))$ for any $x \in A$ and any $f \in \mathfrak{F}$, by (iv), T has a stationary element $v \in A$, which is a common fixed element of $\mathfrak{F}$.

(vi)$\Longrightarrow$(i): Suppose that for any $x \in A$, there exists a $y \in X\backslash\{x\}$ satisfying $\neg G(x, y)$. Choose $f(x)$ to be one of such y. Then $f : A \to X$ has no fixed element by its definition. However, $\neg G(x, f(x))$ for all $x \in A$. Let $\mathfrak{F} = \{f\}$. By (vi), f has a fixed element, a contradiction.

(i)+(vi) $\Longrightarrow$ (vii): By (i), there exists a $v \in A$ such that $G(v, w)$ for all $w \in X\backslash\{v\}$. For each $i \in I$, by (vi), we have a $v_i \in A$ such that $\{v_i\} = T_i(v_i)$. Suppose $v \neq v_i$. Then $G(v, v_1)$ holds by (i) and $\neg G(v, v_1)$ holds by assumption on (vii). This is a contradiction. Therefore $v = v_i$ for all $i \in I$.

(vii) $\Longrightarrow$ (vi): Clear.

(i) $\Longrightarrow$ (viii): By (i), there exists a $v \in A$ such that $G(v, w)$ for all $w \neq v$. Then by the hypothesis, we have $v \in Y$. Therefore, $v \in A \cap Y$.

(viii) $\Longrightarrow$ (i): For all $x \in A$, let

$$A(x) := \{y \in X : x \neq y,\ \neg G(x, y)\}.$$

Choose $Y = \{x \in X : A(x) = \emptyset\}$. If $x \notin Y$, then there exists a $z \in A(x)$. Hence the hypothesis of (vii) is satisfied. Therefore, by (viii), there exists a $v \in A \cap Y$. Hence $A(v) = \emptyset$; that is, $G(v, w)$ for all $w \neq v$, which implies (i).

This completes our proof. □

Note that all v's in (i)–(viii) are the same, and (viii) suggests its location.

As a first application of Metatheorem, we have the following:

Let $(X,\preceq)$ be a *pre-ordered set*; that is, X is a nonempty set and $\preceq$ is reflexive and transitive. For each $x\in X$, we denote $S(x)=\{y\in X : x\preceq y\}$ and $G(x,y)$ means $x\preceq y$.

Theorem 2.1. *Let $x_0\in X$ and $A=S(x_0)$. Then the following seven statements are equivalent:*

(i) *There exists a maximal element $v\in A$ such that $v\not\preceq w$ for any $w\in X\backslash\{v\}$.*

(ii) *If $T: A\multimap X$ is a multimap such that for any $x\in A\backslash T(x)$ there exists a $y\in X\backslash\{x\}$ satisfying $x\preceq y$, then T has a fixed element $v\in A$, that is, $v\in T(v)$.*

(iii) *If $f: A\to X$ is a map such that for any $x\in A$ with $x\neq f(x)$, there exists a $y\in X\backslash\{x\}$ satisfying $x\preceq y$, then f has a fixed element $v\in A$, that is, $v=f(v)$.*

(iv) *If $f: A\to X$ is a map such that $x\preceq f(x)$ for any $x\in A$, then f has a fixed element $v\in A$, that is, $v=f(v)$.*

(v) *If $T: A\multimap X$ is a multimap such that $x\preceq y$ holds for any $x\in A$ and any $y\in T(x)\backslash\{x\}$, then T has a stationary element $v\in A$, that is, $\{v\}=T(v)$.*

(vi) *If $\mathfrak{F}$ is a family of maps $f: A\to X$ satisfying $x\preceq f(x)$ for all $x\in A$ with $x\neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v\in A$, that is, $v=f(v)$ for all $f\in\mathfrak{F}$.*

(vii) *If $\mathfrak{F}$ is a family of multimaps $T_i: A\multimap X$ for $i\in I$ with an index set I such that $x\preceq y$ holds for any $x\in A$ and any $y\in T_i(x)\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v\in A$, that is, $\{v\}=T_i(v)$ for all $i\in I$.*

(viii) *If Y is a subset of X such that for each $x\in A\backslash Y$ there exists a $z\in X\backslash\{x\}$ such that $x\preceq z$, then there exists an element $v\in A\cap Y$.*

Proof. In Metatheorem, put $A:=S(x_0)$ and let $G(v,w)$ be the statement $v\not\preceq w$. Then each of (i)–(viii) follows from the corresponding ones in Metatheorem.

This completes our proof. □

Note that we claimed that (i)–(viii) are equivalent in Theorem 2.1 and did not say that they are true. For a counter-example, the real line does not have any maximal element in the natural order.

From now on, we are going to give several examples or applications or direct consequences of Metatheorem or Theorem 2.1.

3. SUZUKI (2001) [22]

Suzuki [22] introduced the concept of τ-distance on a metric space, which is a generalized concept of both w-distance and Tataru's distance. He also improved the generalizations of the Banach contraction principle, Caristi's fixed point theorem, Ekeland's variational principle, and the nonconvex minimization theorem according to Takahashi.

Let (X,d) be a metric space. Then a function $p : X \times X \to \mathbb{R}_+ = [0,\infty)$ is called τ-*distance* on X if there exists a function $\eta : X \times \mathbb{R}_+ \to \mathbb{R}_+$ and the following are satisfied:

(τ1) $p(x,z) \leq p(x,y) + p(y,z)$ for all $x,y,z \in X$;
(τ2) $\eta(x,0) = 0$ and $\eta(x,t) \geq t$ for all $x \in X$ and $t \in \mathbb{R}_+$, and η is concave and continuous in its second variable;
(τ3) $\lim_n x_n = x$ and $\lim_n \sup\{\eta(z_n, p(z_n,z_m)) : m \geq n\} = 0$ imply $p(w,x) \leq \liminf_n p(w,x_n)$ for all $w \in X$;
(τ4) $\lim_n \sup\{p(x_n,y_m) : m \geq n\} = 0$ and $\lim_n \eta(x_n,t_n) = 0$ imply $\lim_n \eta(y_n,t_n) = 0$;
(τ5) $\lim_n \eta(z_n, p(z_n, p(z_n,x_n)) = 0$ and $\lim_n \eta(z_n,y_n)) = 0$ imply $\lim_n d(x_n,y_n) = 0$.

After giving this definition, Suzuki [22] showed several examples of τ-distances.

In this section, we derive some equivalent formulations of the main theorems of [22] by applying our Metatheorem:

Theorem 3.1. *Let X be a complete metric space, let p be a τ-distance on X, and let $\phi : X \to (-\infty,\infty]$ be proper lower semicontinuous and bounded from below. Let $\varepsilon > 0$, $u \in X$ such that $p(u,u) = 0$, and $A = \{x \in X : \phi(x) \leq \phi(u) - \varepsilon p(u,x)\}$.*

Then the following equivalent conditions (i)–(viii) hold:

(i) *There exists $v \in A$ such that $\phi(w) > \phi(v) - \varepsilon p(v,w)$ for all $w \in X \backslash \{v\}$.*

(ii) *If $T : A \multimap X$ is a multimap such that for any $x \in A \backslash T(x)$ there exists a $y \in X \backslash \{x\}$ satisfying $\phi(y) \leq \phi(x) - \varepsilon p(x,y)$, then T has a fixed element $v \in A$, that is, $v \in T(v)$.*

(iii) *If* $f : A \to X$ *is a map such that for any* $x \in A$ *with* $x \neq fx$, *there exists a* $y \in X \backslash \{x\}$ *satisfying* $\phi(y) \leq \phi(x) - \varepsilon p(x,y)$, *then* f *has a fixed element* $v \in A$, *that is,* $v = f(v)$.

(iv) *If* $f : A \to X$ *is a map such that* $\phi(f(x)) \leq \phi(x) - \varepsilon p(x, f(x))$ *for any* $x \in A$, *then* f *has a fixed element* $v \in A$, *that is,* $v = f(v)$.

(v) *If* $T : A \multimap X$ *is a multimap such that* $\phi(y) \leq \phi(x) - \varepsilon p(x,y)$ *holds for any* $x \in A$ *and any* $y \in T(x) \backslash \{x\}$, *then* T *has a stationary element* $v \in A$, *that is,* $\{v\} = T(v)$.

(vi) *If* $\mathfrak{F}$ *is a family of maps* $f : A \to X$ *satisfying* $\phi(f(x)) \leq \phi(x) - \varepsilon p(x, f(x))$ *for all* $x \in A$ *with* $x \neq f(x)$, *then* $\mathfrak{F}$ *has a common fixed element* $v \in A$, *that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$.

(vii) *If* $\mathfrak{F}$ *is a family of multimaps* $T_i : A \multimap X$ *for* $i \in I$ *with an index set* I *such that* $\phi(y) \leq \phi(x) - \varepsilon p(x,y)$ *holds for any* $x \in A$ *and any* $y \in T_i(x) \backslash \{x\}$, *then* $\mathfrak{F}$ *has a common stationary element* $v \in A$, *that is,* $\{v\} = T_i(v)$ *for all* $i \in I$.

(viii) *If* Y *is a subset of* X *such that for each* $x \in A \backslash Y$ *there exists a* $z \in X \backslash \{x\}$ *such that* $\phi(z) \leq \phi(x) - \varepsilon p(x,z)$, *then there exists an element* $v \in A \cap Y$.

Proof. Define

$$x \preceq y \quad iff \quad \phi(y) \leq \phi(x) - \varepsilon p(x,y)$$

for any $x, y \in X$. Then Statement (i) holds by Suzuki [22, Theorem 4(ii)]. Therefore, by Metatheorem, all (i)–(viii) hold. □

Note that, for $\varepsilon = 1$, (iii) reduces to Theorem 3 of [22] and (iv) is a generalization of Caristi's theorem, and our Theorem 3.1 can be regarded its equivalent formulations.

In this case, for the fixed point $v = f(v)$, we have $p(v,v) = 0$ since $\phi(v) < 0$ and $\phi(v) + p(v,v) = \phi(f(v)) + p(v, f(v)) \leq \phi(v)$.

Suzuki noted that a direct consequence of (iv) is a result of Jachymski (1998).

Now, we obtain equivalent formulations of Suzuki's generalization of Takahashi's nonconvex minimization theorem [23] in 1991.

Theorem 3.2. *Let* X *be a complete metric space and let* $\phi : X \to (-\infty, \infty]$ *be a proper l.s.c. function, bounded from below. Assume that there exists a* τ*-distance* p *on* X *such that for each* $u \in X$ *with* $\inf_{x \in X} \phi(x) < \phi(u)$, *there exists a* $v \in X \backslash \{u\}$ *such that* $\phi(v) + p(u,v) \leq \phi(u)$.

Then the following equivalent statements hold:

(i) *There exists an element* $v \in X$ *such that* $\phi(v) \leq \phi(w)$ *for any* $w \in X \backslash \{v\}$*, that is,* $\phi(v) = \inf_{x \in X} \phi(x)$.

(ii) *If* $T : X \multimap X$ *is a multimap such that for any* $x \in X \backslash T(x)$ *there exists a* $y \in X \backslash \{x\}$ *satisfying* $\phi(x) > \phi(y)$*, then* T *has a fixed element* $v \in X$*, that is,* $v \in T(v)$.

(iii) *If* $f : X \to X$ *is a map such that for any* $x \in X$ *with* $x \neq f(x)$*, there exists a* $y \in X \backslash \{x\}$ *satisfying* $\phi(x) > \phi(y)$*, then* f *has a fixed element* $v \in X$*, that is,* $v = f(v)$.

(iv) *If* $f : X \to X$ *is a map such that* $\phi(x) > \phi(f(x))$ *for any* $x \in X$*, then* f *has a fixed element* $v \in X$*, that is,* $v = f(v)$.

(v) *If* $T : X \multimap X$ *is a multimap such that* $\phi(x) > \phi(y)$ *holds for any* $x \in X$ *and any* $y \in T(x) \backslash \{x\}$*, then* T *has a stationary element* $v \in X$*, that is,* $\{v\} = T(v)$.

(vi) *If* $\mathfrak{F}$ *is a family of maps* $f : X \to X$ *satisfying* $\phi(x) > \phi(fx)$ *for all* $x \in X$ *with* $x \neq f(x)$*, then* $\mathfrak{F}$ *has a common fixed element* $v \in X$*, that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$.

(vii) *If* $\mathfrak{F}$ *is a family of multimaps* $T_i : X \multimap X$ *for* $i \in I$ *with an index set* I *such that* $\phi(x) > \phi(y)$ *holds for any* $x \in A$ *and any* $y \in T_i(x) \backslash \{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in X$*, that is,* $\{v\} = T_i(v)$ *for all* $i \in I$.

(viii) *If* Y *is a subset of* X *such that for each* $x \in X \backslash Y$ *there exists a* $z \in X \backslash \{x\}$ *satisfying* $\phi(x) > \phi(z)$*, then there exists a* $v \in X \cap Y = Y$.

Proof. Notice that (i) is the generalization of Takahashi's theorem given by Suzuki [22, Theorem 5]. In Metatheorem, let $G(v, w)$ be the statement $\phi(v) \leq \phi(w)$ and let $A = X$. Then each of (i)–(viii) follows from the corresponding ones in Metatheorem. This completes our proof. □

Note that Corollary 4 [22] is the particular form of (i) for $\max\{d(f(u), v), d(f(u), f(v))\}$ instead of $p(u, v)$. In this case, Suzuki said (i) slightly generalize results of Takahashi (1993), Ume (1994), and Kim et al. (1997).

Later in 2010, motivated by Suzuki, Wu [24] extended Caristi-Kirk's fixed point theorem, Ekeland's variational principle, and Takahashi's minimization theorem in a complete metric space by replacing the distance with a τ-distance. In addition, these extensions were shown to be equivalent. When the τ-distance is l.s.c. in its second variable, they are applicable to establish more equivalent results about the generalized weak sharp minima and error bounds, which are in turn useful for extending some existing results such as the petal theorem.

In fact, there are several maximum theorems in [24], which can be applicable Metatheorem.

4. LIN AND DU (2006) [13]

Lin and Du [13] introduced the τ-function which generalizes the w-distance. They established a generalized Ekeland's variational principle in the setting of lower semi-continuous from above and τ-functions. As applications of their Ekeland variational principle, they derived generalized Caristi's (common) fixed point theorems, a generalized Takahashi's nonconvex minimization theorem, a nonconvex minimax theorem, a nonconvex equilibrium theorem and a generalized flower petal theorem for l.s.c. from above functions or l.s.c. functions in the complete metric spaces. They also proved that these theorems also imply their Ekeland variational principle.

Throughout this section, unless specified otherwise, (X,d) is a metric space and $\varphi : (-\infty,\infty] \to (0,\infty)$ is a nondecreasing function.

An extended real-valued function $f : X \to (-\infty,\infty]$ is said to be

(i) *lower semicontinuous from above* (in short *lsca*) at $x_0 \in X$ if for any sequence $\{x_n\}$ in X with $x_n \to x_0$ and $f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq \cdots$ implies that $f(x_0) \leq \lim_{n\to\infty} f(x_n)$;

(ii) *upper semicontinuous from below* (in short *uscb*) at $x_0 \in X$ if for any sequence $\{x_n\}$ in X with $x_n \to x_0$ and $f(x_1) \leq f(x_2) \leq \cdots \leq f(x_n) \leq \cdots$ implies that $f(x_0) \geq \lim_{n\to\infty} f(x_n)$.

The function f is said to be *lsca* (resp. *uscb*) on X if f is lsca (resp. *uscb*) at every point of X. The function f is said to be *proper* if $f \not\equiv \infty$. The following definition of τ-*function* is different from the definition of τ-distance, it is a generalization of w-distance in [10].

Definition. A function $p : X \times X \to [0,\infty)$ is called a τ-*function* if the following conditions hold:

(τ1) for all $x,\ y,\ z \in X,\ p(x,z) \leq p(x,y) + p(y,z)$;
(τ2) if $x \in X$ and $\{x_n\}$ in X with $\lim_{n\to\infty} y_n = y$ and $p(x,y_n) \leq M$ for some $M = M(x) > 0$, then $p(x,y) \leq M$;

(τ3) for any sequence $\{x_n\}$ in X with $\lim_{n\to\infty}\sup\{p(x_n,x_m) : m > n\} = 0$, and if there exists a sequence $\{y_n\}$ in X such that $\lim_{n\to\infty} p(x_n,y_n) = 0$, then $\lim_{n\to\infty} d(x_n,y_n) = 0$;
(τ4) for $x,\ y,\ z \in X,\ p(x,y) = 0$ and $p(x,z) = 0$ imply $y = z$.

It is known [10] that if p is a w-distance on $X \times X$, then for all $x,\ y,\ z \in X,\ p(x,y) = 0$ and $p(x,z) = 0$ imply $y = z$.

Remark. Every w-distance, introduced and studied by Kada et al. [10], is a τ-function.

After such preparation, Lin and Du [13] in 2006 obtained the following generalization of Ekeland's variational principle for l.s.c. from above functions:

Theorem 4.1. (Lin-Du) *Let $g : X \to (-\infty,\infty]$ be a proper lsca and bounded below function and p be a τ-function on $X \times X$. Then there exists $v \in X$ such that $p(v,x) > \varphi(g(v))(g(v) - g(x))$ for all $x \in X\backslash\{v\}$.*

Now we apply our Metatheorem to Theorem 4.1:

Theorem 4.2. *Let $g : X \to (-\infty,\infty]$ be a proper lsca and bounded below function and p be a τ-function on $X \times X$.*

Then the following equivalent statements hold:

(i) *There exists an element $v \in X$ such that $v \in X$ such that $p(v,w) > \varphi(g(v))(g(v) - g(w))$ for any $w \in X\backslash\{v\}$.*

(ii) *If $T : X \multimap X$ is a multimap such that for any $x \in X\backslash T(x)$ there exists a $y \in X\backslash\{x\}$ satisfying $p(x,y) \le \varphi(g(x))(g(x) - g(y))$, then T has a fixed element $v \in X$, that is, $v \in T(v)$.*

(iii) *If $f : X \to X$ is a map such that for any $x \in X$ with $x \ne f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $p(x,y) \le \varphi(g(x))(g(x) - g(y))$, then f has a fixed element $v \in X$, that is, $v = f(v)$.*

(iv) *If $f : X \to X$ is a map such that $p(x, f(x)) \le \varphi(g(x))(g(x) - g(f(x)))$ for any $x \in X$, then f has a fixed element $v \in X$, that is, $v = f(v)$.*

(v) *If $T : X \multimap X$ is a multimap such that $p(x,y) \le \varphi(g(x))(g(x) - g(y))$ holds for any $x \in X$ and any $y \in T(x)\backslash\{x\}$, then T has a stationary element $v \in X$, that is, $\{v\} = T(v)$*

(vi) *If* $\mathfrak{F}$ *is a family of maps* $f : X \to X$ *satisfying* $p(x, f(x)) \leq \varphi(g(x))(g(x) - g(f(x)))$ *for all* $x \in X$ *with* $x \neq f(x)$*, then* $\mathfrak{F}$ *has a common fixed element* $v \in X$*, that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$*.*

(vii) *If* $\mathfrak{F}$ *is a family of multimaps* $T_i : X \multimap X$ *for* $i \in I$ *with an index set* I *such that* $p(x,y) \leq \varphi(g(x))(g(x) - g(y))$ *holds for any* $x \in X$ *and any* $y \in T_i(x) \backslash \{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in X$*, that is,* $\{v\} = T_i(v)$ *for all* $i \in I$*.*

(viii) *If* Y *is a subset of* X *such that for each* $x \in X \backslash Y$ *there exists a* $z \in X \backslash \{x\}$ *satisfying* $p(x,z) \leq \varphi(g(x))(g(x) - g(z))$*, then there exists a* $v \in X \cap Y = Y$*.*

Proof. Recall that (i) holds by Theorem 4.1. Since (i)–(viii) are logically equivalent by Metatheorem. This completes our proof. □

Note that (viii) simply tells that Y is nonempty when $X = A$ in Metatheorem and the location of the common point v.

As a first application of their generalized Ekeland variational principle, Lin and Du [13] derived the following generalization of Caristi's theorem for a family of multimaps:

Theorem 4.3. *Let* p *and* g *be the same as in Theorem 4.1. Let* I *be any index set and for each* $i \in I$*, let* $T_i : X \multimap X$ *be a multimap such that for each* $x \in X$*, there exists* $y = y(x,i) \in T_i(x)$ *with*

$$p(x,y) \leq \varphi(g(x))(g(x) - g(y)).$$

Then there exists $v \in X$ *such that* $v \in \bigcap_{i \in I} T_i(v)$*, that is, the family of multimaps* $\{T_i\}_{i \in I}$ *has a common fixed point in* X*, and* $p(v,v) = 0$*.*

In [13], Corollary 2.1 is an equivalent form of Theorem 2.2 for a family of single-valued maps. Therefore, we have Theorem 4.2(vii)$\Longrightarrow$ Theorem 2.2 [13] $\Longrightarrow$ Corollary 2.1 [13] $\Longrightarrow$ Theorem 4.2(vi). Hence they are all equivalent.

5. KHAMSI (2010) [11]

In [11], Khamsi gave a characterization of the existence of minimal elements in partially ordered sets in terms of fixed point of multimaps. This characterization shows that the assumptions in Caristi's fixed point theorem can, a priori, be weakened.

Theorem 5.1. (Khamsi) *Let $(A, \prec)$ be a partially ordered set. Then the following statements are equivalent:*

(1) *A contains a minimal element $a \in A$, that is, $b \prec a$ implies $b = a$.*

(2) *Any multimap $T : A \multimap A$ such that for any $x \in A$, there exists $y \in T(x)$ with $y \prec x$, has a fixed point.*

Note that this follows from Theorem 2.1 (i), (ii) or more earlier versions of Metatheorem in 1985-1986.

Again from Khamsi [12]: Let (M,d) be a metric space and $\phi : M \to [0,\infty)$ be a function. Define the partial order $\prec_\phi$ (due to Brøndsted) on M by

$$x \prec_\phi y \quad iff \quad d(x,y) \leq \phi(y) - \phi(x)$$

for any $x, y \in M$.

Theorem 5.2. *Let (M,d) be a metric space and $\phi : M \to [0,\infty)$ be a function. Consider the poset $(M, \prec)$.*

The following equivalent statements holds:

(i) *$(M, \prec)$ has a minimal element $v \in M$.*

(iii) *Any map $f : M \to M$ such that for all $x \in M$*

$$d(x, f(x)) \leq \phi(x) - \phi(f(x)) \;\; (i.e. f(x) \prec_\phi x)$$

fixes v, i.e. f(v) = v.

Proof. Note that (i) is proved by Khamsi [11]. Therefore, each of the equivalent (i)–(viii) of Theorem 2.1 holds. □

Note that Khamsi [11, Corollary 1] proved only (i) $\Longrightarrow$ (iii). This corollary can be a generalization of Caristi's result.

Further Khamsi [11] assumed that $\eta : [0,\infty) \to [0,\infty)$ is nondecreasing, continuous, such that there exist $c > 0$ and $\delta_0 > 0$ such that for any $t \in [0,\delta_0]$ we have $\eta(t) \geq ct$. Because η is continuous, then there exists $\varepsilon_0 > 0$ such that $\eta^{-1}([0,\varepsilon_0]) \subset [0,\delta_0]$.

Under these assumptions Khamsi obtained the following (i):

Theorem 5.3. *Let M be a complete metric space. Define the relation $\prec$ by*

$$x \prec y \Longleftrightarrow \eta(d(x,y)) \leq \phi(y) - \phi(x)$$

where η and ϕ satisfy all the above assumptions. Then

(i) *$(M,\prec)$ has a minimal element $v \in M$, i.e. if $w \prec v$ then we must have $w = v$.*

(ii) *If $T : M \multimap M$ is a multimap such that for any $x \in M\backslash T(x)$ there exists a $y \in X\backslash\{x\}$ satisfying $\eta(d(x,y)) \leq \phi(x) - \phi(y)$, then T has a fixed element $v \in A$, that is, $v \in T(v)$.*

(iii) *If $f : M \to M$ is a map such that for any $x \in M$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $\eta(d(x,y)) \leq \phi(x) - \phi(y)$, then f has a fixed element $v \in M$, that is, $v = f(v)$.*

(iv) *If $f : M \to M$ is a map such that $\eta(d(x,f(x))) \leq \phi(x) - \phi(f(x))$ for any $x \in M$, then f has a fixed element $v \in M$, that is, $v = f(v)$.*

(v) *If $T : M \multimap M$ is a multimap such that $\eta(d(x,y)) \leq \phi(x) - \phi(y)$ holds for any $x \in M$ and any $y \in T(x)\backslash\{x\}$, then T has a stationary element $v \in M$, that is, $\{v\} = T(v)$.*

(vi) *If $\mathfrak{F}$ is a family of maps $f : M \to M$ satisfying $\eta(d(x,f(x))) \leq \phi(x) - \phi(f(x))$ for all $x \in M$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in M$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(vii) *If $\mathfrak{F}$ is a family of multimaps $T_i : M \multimap M$ for $i \in I$ with an index set I such that $\eta(d(x,y)) \leq \phi(x) - \phi(y)$ holds for any $x \in M$ and any $y \in T_i(x)\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in M$, that is, $\{v\} = T_i(v)$ for all $i \in I$.*

(viii) *If Y is a subset of M such that for each $x \in M\backslash Y$ there exists a $z \in M\backslash\{x\}$ such that $\eta(d(x,z)) \leq \phi(x) - \phi(z)$, then there exists an element $v \in M \cap Y = Y$.*

Proof. In Metatheorem, put $X = A = M$ and let $G(x,y)$ be the statement $\eta(d(x,y)) > \phi(x) - \phi(y)$. Then each of (i)–(viii) follows from the corresponding ones in Metatheorem. Since (i) was proved by Khamsi, this completes our proof. □

6. ESHGHINEZHAT AND FAKHAR (2013) [8]

In [8], some extensions of the Ekeland variational principle in metric spaces are given for a generalized pseudodistance. As an application the authors obtained Caristi's fixed point theorem. Then, by using this result, the authors established some fixed point theorems for set-valued contractive maps.

Definition 6.1. Assume that (X,d) is a metric space. A function $w : X \times X \to \mathbb{R}^+$ is called a *generalized pseudodistance* on X if the following conditions hold.

$(w1)$ (triangle inequality) $w(x,z) \leq w(x,y) + w(y,z)$, for all $x,y,z \in X$.

$(w2)$ For any sequence (x_n) in X such that $\lim_{n\to\infty} \sup_{m>n} w(x_n,x_m) = 0$,
if there exists a sequence (y_n) in X satisfying $\lim_{n\to\infty} w(x_n,y_n) = 0$,
then $\lim_{n\to\infty} d(x_n,y_n) = 0$.

Throughout this section, let (X,d) be a metric space, w be a generalized pseudodistance on X, and $f : X \times X \to \mathbb{R}$ be a function satisfying the condition:

$$\text{(A)}\ \max\{f(x,z), f(z,y)\} \leq 0 \Longrightarrow f(x,y) \leq f(x,z) + f(z,y),\ for\ x,y,z \in X.$$

After giving such preparation, the following is given as [8, Theorem 2.8]:

Theorem 6.2. *Let* $\eta : [0,\infty) \to [0,\infty)$ *and* $\eta(t) > 0$ *for all* $t > 0$.

Then the following statements are equivalent:

(I) *There exists* $\bar{x} \in X$ *such that*

$$f(\bar{x},x) + \eta(w(\bar{x},x)) > 0 \ \ \forall x \neq \bar{x}.$$

(II) *If* $T : X \multimap X$ *is a multimap such that*

$$\forall x \in X \ \ \exists y \in Tx \ \ such\ that\ f(x,y) + \eta(w(x,y)) \leq 0,$$

then T *has a fixed point.*

This can be improved by our Metatheorem as follows:

Theorem 6.3. *Let* A *be a nonempty subset of* (X,d) *and* $\eta : [0,\infty) \to [0,\infty)$ *and* $\eta(t) > 0$ *for all* $t > 0$.

Then the following statements are equivalent:

(i) *There exists* $v \in A$ *such that*

$$f(v,x)+\eta(w(v,x))>0 \ \ \forall x \neq v.$$

(ii) *If* $T : A \multimap X$ *is a multimap such that for any* $x \in X \backslash T(x)$ *there exists a* $y \in X \backslash \{x\}$ *satisfying* $f(x,y)+\eta(w(x,y)) \leq 0$*, then* T *has a fixed element* $v \in A$*, that is,* $v \in T(v)$*.*

(iii) *If* $g : A \to X$ *is a map such that for any* $x \in X$ *with* $x \neq g(x)$*, there exists a* $y \in X \backslash \{x\}$ *satisfying* $f(x,y)+\eta(w(x,y)) \leq 0$*, then* g *has a fixed element* $v \in A$*, that is,* $v = g(v)$*.*

(iv) *If* $g : A \to X$ *is a map such that* $f(x,g(x))+\eta(w(x,g(x))) \leq 0$ *for any* $x \in X$*, then* g *has a fixed element* $v \in A$*, that is,* $v = g(v)$*.*

(v) *If* $T : A \multimap X$ *is a multimap such that* $f(x,y)+\eta(w(x,y)) \leq 0$ *holds for any* $x \in A$ *and any* $y \in T(x) \backslash \{x\}$*, then* T *has a stationary element* $v \in A$*, that is,* $\{v\} = T(v)$*.*

(vi) *If* $\mathfrak{F}$ *is a family of maps* $g : A \to X$ *satisfying* $f(x,g(x))+\eta(w(x,g(x))) \leq 0$ *for all* $x \in X$ *with* $x \neq g(x)$*, then* $\mathfrak{F}$ *has a common fixed element* $v \in A$*, that is,* $v = g(v)$ *for all* $g \in \mathfrak{F}$*.*

(vii) *If* $\mathfrak{F}$ *is a family of multimaps* $T_i : A \multimap X$ *for* $i \in I$ *with an index set* I *such that* $f(x,y)+\eta(w(x,y)) \leq 0$ *holds for any* $x \in A$ *and any* $y \in T_i(x) \backslash \{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in A$*, that is,* $\{v\} = T_i(v)$ *for all* $i \in I$*.*

(viii) *If* Y *is a subset of* X *such that for each* $x \in A \backslash Y$ *there exists a* $y \in X \backslash \{x\}$ *satisfying* $f(x,y)+\eta(w(x,y)) \leq 0$*, then there exists a* $v \in A \cap Y$*.*

Note that this holds by Metatheorem and, for $A = X$, (i) reduces to Theorem 6.2(I) and (II) implies (ii).

7. FIERRO (2015) [9]

In the context of TVS-cone metric spaces, Fierro [9] proved a Bishop-Phelps and a Caristi type theorem. These results allow him to prove a fixed point theorem for (δ, L)-weak contraction according to a pseudo Hausdorff metric defined by means of a cone metric.

Let E be a topological vector space with θ as zero element and usual notations for addition and scalar product. Given a cone P of E, a partial order is defined on E as $x \preceq y$, if and only if $y - x \in P$. We denote by $x \prec y$ whenever $x \preceq y$ and $x \neq y$. In the sequel, (X,d) stands for a cone metric space; see Fierro [9].

A function $\varphi : X \to E$ is lower semicontinuous, if and only if, for any $\alpha \in E$, the set $\{x \in X : \varphi(x) \preceq \alpha\}$ is closed. For this function, a Brønsted type order $\preceq_\varphi$ is defined on X as follows:

$$x \preceq_\varphi y \text{ iff } d(x,y) \preceq \varphi(x) - \varphi(y).$$

It is easy to see that $\preceq_\varphi$ is in effect a partial order relation on X.

In the sequel, $\mathscr{LS}(X)$ stands for the space of all lower semicontinuous and bounded below functions $X \to E$.

Theorem 7.1. *Suppose X is d-complete, $\varphi \in \mathscr{LS}(X)$, and $x_0 \in X$ with $A = \{x \in X : x_0 \preceq x\}$. Then the following equivalent statements hold.*

(i) *There exists a maximal element $v \in A$ such that $v \not\preceq w$ or $d(v,w) \not\preceq \varphi(v) - \varphi(w)$ for any $w \in X \backslash \{v\}$.*

(ii) *If $T : A \multimap X$ is a multimap such that for any $x \in A \backslash T(x)$ there exists a $y \in X \backslash \{x\}$ satisfying $d(x,y) \preceq \varphi(x) - \varphi(y)$, then T has a fixed element $v \in A$, that is, $v \in T(v)$.*

(iii) *If $f : A \to X$ is a map such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $d(x,y) \preceq \varphi(x) - \varphi(y)$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

(iv) *If $f : A \to X$ is a map such that $d(x, f(x)) \preceq \varphi(x) - \varphi(f(x))$ for any $x \in A$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

(v) *If $T : A \multimap X$ is a multimap such that $d(x,y) \preceq \varphi(x) - \varphi(y)$, holds for any $x \in A$ and any $y \in T(x) \backslash \{x\}$, then T has a stationary element $v \in A$, that is, $\{v\} = T(v)$.*

(vi) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $d(x, f(x)) \preceq \varphi(x) - \varphi(f(x))$, for all $x \in A$ with $x \neq fx$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(vii) *If $\mathfrak{F}$ is a family of multimaps $T_i : A \multimap X$ for $i \in I$ with an index set I such that $d(x,y) \preceq \varphi(x) - \varphi(y)$, holds for any $x \in A$ and any $y \in T_i(x) \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = T_i(v)$ for all $i \in I$.*

(viii) *If Y is a subset of X such that for each $x \in A \backslash Y$ there exists a $z \in X \backslash \{x\}$ such that $d(x,z) \preceq \varphi(x) - \varphi(z)$, then there exists an element $v \in A \cap Y$.*

Proof. Note that (i), (ii), (v) are Theorems 5. 6(6.1), (6.2) of [9], resp. Hence, by Metatheorem or Theorem 2.1, we have the conclusion. □

Note that (ii), (iv) and (v) are generalized versions of Caristi's theorem. Moreover, Fierro [9, Theorem 8] deduced a cone metric version of the nonconvex minimization theorem of Takahashi from (i).

8. ALEGRE AND MARIN (2016) [1]

Alegre and Marin [1] introduced the notion of modified *w*-distance (*mw*-distance) on a quasi-metric space which generalizes the concept of quasi-metric. They obtained a fixed point theorem for generalized contractions with respect to *mw*-distances on complete quasi-metric spaces.

Theorem 8.1. *Let f be a self-map of a complete quasi-metric space (X,d). If there exists a strong-mw-distance q on (X,d) and a Jachymski function $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\phi(t) < t$ for all $t > 0$, and*

$$q(f(x), f(y)) \leq \phi(q(x,y)), \; for\ all\ x,y \in X,$$

then f has a unique fixed point $z \in X$. Moreover $q(z,z) = 0$.

From this we obtain our routine equivalent formulation by our Metatheorem. We only show the case (i):

Theorem 8.2. *Under the hypothesis of Theorem 8.1, we have equivalency of Metatheorem (i)–(viii). For example,*

(i) *There exists a maximal element $v \in A$ such that*

$$v \not\preceq w \;\; or \;\; q(f(v), f(w)) > \phi(q(v,w))$$

for any $w \in X \backslash \{v\}$.

9. AL-HOMIDAN, ANSARI, AND KASSAY (2019) [3]

In [3], the authors established Takahashi's minimization theorem in the setting of quasi-metric spaces and provide its equivalence with Ekeland's variational principle given in Cobzaş [4] in 2011. After sufficient preparation, the authors stated the following:

Cobzaş [11] established the following Caristi-Kirk fixed point theorem for multimaps in the setting of quasi-metric spaces.

Theorem 9.1. *Let* (X,ρ) *be a quasi-metric space,* $T : X \multimap X$ *be a multimap with nonempty values and* $\varphi : X \to \mathbb{R}$ *be a function. If* (X,ρ) *is right* $\overline{\rho}$*-K-complete and* φ *is bounded below and* $\overline{\rho}$*-lower semicontinuous such that the condition*

$$\forall x \in X,\ \exists y \in T(x) : \rho(x,y) \leq \varphi(x) - \varphi(y)$$

is satisfied, then T *has a fixed point* $v \in X$.

If the condition

$$\forall x \in X, \forall y \in T(x) : \rho(x,y) \leq \varphi(x) - \varphi(y)$$

is satisfied, then T *has an invariant point in* X*, that is, there exists* $v \in X$ *such that* $\{v\} = T(v)$.

This gives an excellent example our Metatheorem as follows:

Theorem 9.2. *Under the hypothesis of Theorem 9.1, the following equivalent statements hold:*

(i) *There exists an element* $v \in X$ *such that* $\rho(v,w) > \varphi(v) - \varphi(w)$ *for any* $w \in X\backslash\{v\}$.

(ii) *If* $T : X \multimap X$ *is a multimap such that for any* $x \in X\backslash T(x)$ *there exists a* $y \in X\backslash\{x\}$ *satisfying* $\rho(x,y) \leq \varphi(x) - \varphi(y)$*, then* T *has a fixed element* $v \in X$*, that is,* $v \in T(v)$.

(iii) *If* $f : X \to X$ *is a map such that for any* $x \in X$ *with* $x \neq f(x)$*, there exists a* $y \in X\backslash\{x\}$ *satisfying* $\rho(x,y) \leq \varphi(x) - \varphi(y)$*, then* f *has a fixed element* $v \in X$*, that is,* $v = f(v)$.

(iv) *If* $f : X \to X$ *is a map such that* $\rho(x, f(x)) \leq \varphi(x) - \varphi(f(x))$ *for any* $x \in X$*, then* f *has a fixed element* $v \in X$*, that is,* $v = f(v)$.

(v) *If* $T : X \multimap X$ *is a multimap such that* $\rho(x,y) \leq \varphi(x) - \varphi(y)$ *holds for any* $x \in X$ *and any* $y \in T(x)\backslash\{x\}$*, then* T *has a stationary element* $v \in X$*, that is,* $\{v\} = T(v)$

(vi) *If* $\mathfrak{F}$ *is a family of maps* $f : X \to X$ *satisfying* $\rho(x, f(x)) \leq \varphi(x) - \varphi(f(x))$ *for all* $x \in X$ *with* $x \neq f(x)$, *then* $\mathfrak{F}$ *has a common fixed element* $v \in X$, *that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$.

(vii) *If* $\mathfrak{F}$ *is a family of multimaps* $T_i : X \multimap X$ *for* $i \in I$ *with an index set* I *such that* $\rho(x,y) \leq \varphi(x) - \varphi(y)$ *holds for any* $x \in X$ *and any* $y \in T_i(x)\backslash\{x\}$, *then* $\mathfrak{F}$ *has a common stationary element* $v \in X$, *that is,* $\{v\} = T_i(v)$ *for all* $i \in I$.

(viii) *If* Y *is a subset of* X *such that for each* $x \in X\backslash Y$ *there exists a* $z \in X\backslash\{x\}$ *satisfying* $\rho(x,z) \leq \varphi(x) - \varphi(z)$, *then there exists a* $v \in X \cap Y = Y$.

Proof. Note that (ii) and (v) hold by Theorem 9.1. Since (i)–(viii) are logically equivalent by Metatheorem, this completes our proof. □

10. COTRINA, THÉRA, AND ZÚÑIGA (2020) [5]

The paper [5] deals with the existence of solutions to equilibrium and quasi-equilibrium problems without any convexity assumption. Coverage includes some equivalences to the Ekeland variational principle for bifunctions and basic facts about transfer lower continuity. An application is given to systems of quasi-equilibrium problems.

Theorem 10.1. ([5]) *Let* C *be a nonempty closed subset of the complete metric space* (X,d), *and* $h : C \to \mathbb{R}$ *be a function bounded from below. For every* $\varepsilon > 0$, *and for any* $x_0 \in C$, *there exists* $\hat{x} \in C$ *such that*

$h(\hat{x}) + \varepsilon d(x_0, \hat{x}) \leq h(x_0)$, *and*

$h(x) + \varepsilon d(x, \hat{x}) > h(\hat{x})$, *for all* $x \in C\backslash\{\hat{x}\}$.

This can be reformulated to the following:

Theorem 10.2. *Let* (M,d) *be a complete metric space, and* $h : M \to \mathbb{R}$ *be a function bounded from below. For every* $\varepsilon > 0$, *let*

$$x \preceq y \textit{ in } M \text{ iff } h(x) + \varepsilon d(x,y) \leq h(y)$$

and for any $x_0 \in C$, *let* $A = \{x \in M : x \preceq x_0\}$.

Then the following equivalent statements hold:

(i) *There exists $v \in A$ such that $w \npreceq v$ or*

$$h(w) + \varepsilon d(w,v) > h(v) \ for\ all\ w \in M \backslash \{v\}.$$

(ii)–(viii) *Same as in Metatheorem.*

Proof. Note that (i) holds by Theorem 10.1. Hence Theorem 10.2 follows from Metatheorem. □

Note that some equivalent formulations and applications of Theorem 10.1 are given in [5].

11. ALGHAMDI, ALZUMI, AND SHAHZAD (2020) [1]

The aim of [1] is to obtain variants of the nonconvex minimization theorem and the Caristi fixed point theorem in quasi-metric spaces. The authors also proved a generalized Ekeland variational principle. Their results generalize the results of Kada, Suzuki and Takahashi [10], Park [20] and some others by using w-distance.

The following is [1, Theorem 5.1]:

Theorem 11.1. ([1]) *Let (X,ρ) be a $\overline{\rho}$-sequentially complete quasi-metric space, $\varphi : X \to \mathbb{R}$ be a $\overline{\rho}$-lower semicontinuous and bounded below function, and p be a w-distance on X. Then, for any $x \in X$, there exists $y \in X$ such that*

$$\varphi(y) \leq \varphi(x) \ and\ \gamma(p(y,s)) > F(\varphi(y) - \varphi(s)),$$

for all $s \in X \backslash \{y\}$.

Here $F : \mathbb{R} \to \mathbb{R}$ and $\gamma : \mathbb{R}^+ \to \mathbb{R}^+$ are functions of special type defined in [1].

Theorem 11.2. *Let (X,ρ), $\varphi : X \to \mathbb{R}$, p be the same as in Theorem 11.1. Let $x_0 \in X$ and $A = \{x \in X : \varphi(x) \leq \varphi(x_0)\}$. Define an order in X as*

$$x \preceq y \ \ iff \ \ \gamma(p(x,y)) \leq F(\varphi(x) - \varphi(y)),$$

Then (i)–(viii) in Theorem 2.1 hold. For example, we have

(i) *There exists an element $v \in A$ such that $v \npreceq w$ for any $w \in X \backslash \{v\}$.*

(iii) *If $f : A \to X$ is a map such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $x \preceq y$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

(iv) *If* $f : A \to X$ *is a map such that* $x \preceq f(x)$ *for any* $x \in A$*, then* f *has a fixed element* $v \in A$*, that is,* $v = f(v)$.

Proof. Note that (i) holds by Theorem 11.1. Therefore Theorem 11.2 follows from Theorem 2.1 or Metatheorem. □

In [1], the authors generalized the Caristi fixed point theorem as follows [1, Theorem 4.1] with independent proof. Now this is a simple consequence of Theorem 11.2(iii),(iv) by replacing γ by $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ in another class of functions.

Corollary 11.3. ([1]) *Let* (X, ρ) *be a* $\overline{\rho}$*-sequentially complete quasi-metric space and* $\varphi : X \to \mathbb{R}$ *a* $\overline{\rho}$*-lower semicontinuous and bounded below function. Let* $T : X \to X$ *be a map. Assume that there exists a w-distance* p *on* X *such that*

$$\eta(p(x, Tx)) \leq F(\varphi(x) - \varphi(Tx)) \text{ for all } x \in X.$$

Then there is $z_0 \in X$ *such that* $z_0 = Tz_0$ *and* $p(z_0, z_0) = 0$.

This is an example of the productivity of our Metatheorem.

12. CONCLUSION

Our original Metatheorem first appeared in [15] in 1985. and the present one is given in [21] in 2022. The original one was to find some useful equivalents of the Ekeland principle. The present Metatheorem is an extended new form in our previous article entitled “Extensions of various maximum principles”[21] with several applications.

In this article, our Metatheorem is applied to equivalent formulations of a number of known theorems appeared after 2000. These theorems are mainly extensions of the Caristi theorem or the Ekeland principle. Those extensions concern with new spaces, new topologies on them, and new types of order relation. Some of them seem to be too artificial for practical application. Also note that Theorems 6.3 and 8.2 only stated the equivalency of (i)–(vii), so it is open to show whether any of them is true or false.

However, as we did in our previous work [21], in such equivalent formulations, certain maximal points are actually same to fixed points, stationary points, collectively fixed points, collectively stationary points, and we have some information on the location of such points. No one recognized this fact yet. Therefore, if we have a theorem on any of such points, then we can deduce at least six equivalent theorems on other types of points.

In many fields of mathematical sciences, there are plentiful number of theorems concerning maximal points or fixed points that can be applicable our Metatheorem. Some of such theorems can be seen in our previous works in [15]–[21]. Therefore, our Metatheorem is a machine to find new equivalent theorems from some known facts with trivial proofs. This is like an industrial revolution of making new equivalent statements.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

REFERENCES

[1] A. Alghamdi, H.Z. Alzumi, and N. Shahzad, Variants of the nonconvex minimization theorem and the Caristi's fixed point theorem, Appl. Set-Valued Anal. Optim. 2 (2020), 305–315.

[2] C. Alegre and J. Marin, Modified *w*-distances on quasi-metric spaces and a fixed point theorem on complete quasi-metric spaces, Top. Appl. 203 (2016), 32–41.

[3] S. Al-Homidan, Q. H. Ansari, and G. Kassay, Takahashi's minimization theorem and some related results in quasi-metric spaces, J. Fixed Point Theory Appl. 21 (2019), 38.

[4] S. Cobzaş, Completeness in quasi-metric spaces and Ekeland variational principle, Topol. Appl. 158 (2011), 1073–1084.

[5] J. Cotrina, M. Théra, and J. Zúñiga. An existence result for quasi-equilibrium problems via Ekeland's variational principle. 2020. hal-02650953.

[6] I. Ekeland, Sur les problèmes variationnels, C.R. Acad. Sci. Paris 275 (1972), 1057–1059; 276 (1973), 1347–1348.

[7] I. Ekeland, On the variational principle, J. Math. Anal. Appl. 47 (1974), 324–353.

[8] S. Eshghinezhada and M. Fakhar, Some generalizations of Ekeland's variational principle with applications to fixed point theory, Math. Comp. Modell. 57 (2013), 1250–1258.

[9] R. Fierro, Fixed point theorems for set-valued mappings on TVS-cone metric spaces, Fixed Point Theory Appl. 2015 (2015), 221.

[10] O. Kada, T. Suzuki, and W. Takahashi, Nonconvex minimization theorems and fixed point theorems in complete metric spaces, Math. Japonica. 44 (1996), 381–391.

[11] M.A. Khamsi, Remarks on Caristi?s fixed point theorem, Nonlinear Anal., Theory Meth. Appl. 71 (2009), 227–231.

[12] Z. Li, Remarks on Caristi's fixed point theorem and Kirk's problem, Nonlinear Anal., Theory Meth. Appl. 73 (2010) 3751–3755

[13] L.-J. Lin and W.-S. Du, Ekeland's variational principle, minimax theorems and existence of nonconvex equilibria in complete metric spaces, J. Math. Anal. Appl. 323 (2006), 360–370.

[14] N. Mizoguchi and W. Takahashi, Fixed point theorems for multivalued mappings on complete metric spaces, J. Math. Anal. Appl. 141 (1989), 177–188.

[15] S. Park, Some applications of Ekeland's variational principle to fixed point theory, Approximation Theory and Applications (S.P. Singh, ed.), Pitman Res. Notes Math. 133 (1985), 159–172.

[16] S. Park, Countable compactness, l.s.c. functions, and fixed points, J. Korean Math. Soc. 23 (1986), 61–66.

[17] S. Park, Equivalent formulations of Ekeland's variational principle for approximate solutions of minimization problems and their applications, Operator Equations and Fixed Point Theorems (S.P. Singh *et al.*, eds.), MSRI-Korea Publ. 1 (1986), 55–68.

[18] S. Park, Equivalent formulations of Zorn's lemma and other maximmumm principles, J. Korean Soc. Math. Edu. 25 (1987), 19–24.

[19] S. Park, Partial orders and metric completeness, Proc. Coll. Natur. Sci. SNU 12 (1987), 11–17.

[20] S. Park, On generalizations of the Ekeland-type variational principles, Nonlinear Anal., Theory Meth. Appl. 39 (2000), 881–889.

[21] S. Park, Equivalents of various maximum principles, Results Nonlinear Anal. 5 (2022), 169–174.

[22] T. Suzuki, Generalized distance and existence theorems in complete metric spaces, J. Math. Anal. Appl. 253 (2001), 440–458.

[23] W. Takahashi, Existence theorems generalizing fixed point theorems for multivalued mappings, in: Fixed Point Theory and Applications, Marseille, 1989, in: Pitman Res. Notes Math. Ser., vol. 252, Longman Sci. Tech., Harlow, 1991, 397–406

[24] Z. Wu, Equivalent extensions to Caristi-Kirk's fixed point theorem, Ekeland's variational principle, and Takahashi's minimization theorem, Fixed Point Theory Appl. 2010 (2010), 970579.