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SOME APPROXIMATION PROBLEMS IN G-METRIC SPACES

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Abstract. In this paper, we will first introduce the concept of best approximation in G- metric spaces. We prove a few theorems which occur frequently, and we use new best approximation to prove some properties and show that some properties are valid for this new definition Although some are not. Its application on vertical operation in spaces $C(Q)$ is shown.

Keywords: M-M best approximation; M-M-proximinal; M-M-Chebyshev; approximatively compact.

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1. Introduction

The theory of best approximation is an important topic in functional analysis. It is a very extensive field which has various applications [6], [2]. The place to begin is with the theory of near-best approximation being developed by J. C. Mason [8, 10, 11, 12, 13, 14] and others [5, 15] for analytic functions on a disk. Their basic idea is to replace the nonlinear problem of determining the optimal approximate of an analytic function relative to a given subspace and fix norm with that of solving an appropriately defined linear problem. The method is constructive so that the results are useful when the errors in making this replacement are acceptably small.

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The remark that normed linear spaces constitute the natural framework for the study of the problem of best approximation was made in 1938 by M. Nicolescu [16] and, independently, by M. G. Krein [9], who have also obtained the first results in this direction. After a relatively stagnant period of about twenty years, new methods and results have been given in the papers [26, 19], which have made possible the development of a general theory of best approximation in normed linear spaces. Thus, important results on best approximation by elements of linear subspaces, especially in concrete spaces, have been obtained by [20, 21] and others. A substantial part of the present monograph is based on the papers [26, 19]. Although in the first ones of these papers the results have been given only for real Banach spaces, in the papers [22, 23] it has been remarked that they remain valid, with the same proof, for general (real or complex) normed linear spaces; in the monograph these results are given in their general form, with reference to the first papers in which they appear. we deals with some properties of the sets $P_G(x,y)$ and $P_G^{-1}(x,y)$ in convex metric spaces. Also, we prove a few theorems that occur frequently in the paper. Furthermore, we give some illustrative examples of our main results.

2. Preliminaries

This section recalls the following notations and the ones that will be used in what follows.

Definition 2.1. [18] Let (X,d) be a metic space and G a closed subset of X. For $x \in X$, let $d(x,G) = inf_{z\in G} d(x,z)$. A point $z_0 \in G$ satisfying: $d(x,z_0) = d(x,G)$ is called a point of best approximation or a nearest point to $x \in G$.

Remark 2.2. [18] *In this paper we will denote the set of all best approximation of x in G, by:*

$$P_G(x) = \{z \in G \mid d(x,z) = d(x,G)\}. \tag{2.1}$$

Theorem 2.3. [18] *Let G be a subspace of a normed linear space X, $x \in X \setminus \overline{G}$ and $z_0 \in G$. We have: $z_0 \in P_G(x)$ if and only if there exists an $f \in X^*$ with the following properties:*

$$\|f\| = 1 \tag{2.2}$$

$$f|_G = 0 \tag{2.3}$$

$$f(x - z_0) = \|x - z_0\|. \tag{2.4}$$

Lemma 2.4. [18] *Let G be a subspace of a normed linear space X, $x \in X \setminus \overline{G}$, $z_0 \in G$, and $f \in X^*$. Then:*

i) *f satisfies (2.2) and (2.4) if and only if it satisfies (2.1) and $Ref(x-z_0) = \|x-z_0\|$.*

ii) *f satisfies (2.3) if and only if $Ref(z) = 0$ $(z \in G)$.*

iii) *f satisfies (2.2) and (2.3) and $|f(x-z_0)| = \|x-z_0\|$ if and only if $f_1 = [sing f(x-z_0)]$ satisfies (2.2), (2.3) and (2.4).*

iiii) *f satisfies (2.2), (2.3) and $Ref(x-z_0) = \|x-z_0\|$, if and only if either $f_1 = f$, $f_2 = -f$ satisfies (2.2), (2.3) and (2.4).*

Corollary 2.5. [18] *Let G be a subspace of a normed linear space X, $x \in X \setminus \overline{G}$ such that x and $z_0 \in G$. The following statements are equivalent :*

a) $z_0 \in P_G(x)$.

b) *There exists an $f \in X^*$ satisfying (2.2), (2.3) and (2.5).*

c *There exists an $f \in X^*$ satisfying (2.2), (2.3) and (2.7).*

d) *There exists an $f \in X^*$ satisfying (2.2), (2.3) and (2.8).*

e) *There exists an $f \in X^*$ satisfying (2.2), (2.6) and (2.5).*

f) *There exists an $f \in X^*$ satisfying (2.2), (2.6) and (2.8).*

Corollary 2.6 ([26], p. 509)**.** *Let X be a normed linear space, G a linear subspace of X, and $x \in X \setminus \overline{G}$ such that x be linearly independent and $M \subseteq G$. We have $M \subset P_G(x)$ if and only if there exists an $f \in X^*$ satisfying (2.2), (2.3) and $f(x-z_0) = \|x-z_0\| \qquad (z_0 \in M)$.*

$$f(x-z_0) = \|x-z_0\| \qquad (z_0 \in M). \tag{2.5}$$

Definition 2.7. [25] Let (X,d) be a metric space and $x,y,z \in X$. We say that z is between x and y if $d(x,z)+d(z,y) = d(x,y)$. A point z is called a mid point of x and y if $d(x,z) = d(z,y) = \frac{1}{2}d(x,y)$. For any two points $x,y \in X$, the set $\{z \in X : d(x,z)+d(z,y) = d(x,y)\}$ is called a metric segment and is denoted by $[x,y]$. The set $[x,y,-[= \{z \in X : d(x,y)+d(y,z) = d(x,z)\}$ denotes a half ray starting from x and passing through y i.e. it is the union of line segments $[x,z]$ where $[x,y] \subseteq [x,z]$.

Correspondingly, $]-,x,y]$ is a half ray starting from y and passing through x, $]-x,y,-[$ is a line passing through x and y.

Definition 2.8. [25] A subset K of metric space (X,d) is called a convex cone if $[x,y,-[\subset K$ whenever $x,y\in K$.

Definition 2.9. [25] (Takahashi(1970)) A mapping $W : X\times X\times[0,1]\rightarrow X$ is said to be a convex structure on X if for all $x,y\in X$ and $\lambda\in[0,1]$:

$$d(u,W(x,y,\lambda))\leq\lambda d(u,x)+(1-\lambda)d(u,y). \tag{2.6}$$

holds for all $u\in X$. A metric space (X,d) together with a convex structure is called a convex metric space.

Definition 2.10. [25] A nonempty subset G of metric space (X,d) is is said to be
(i) starshaped (Guay et al.(1982)) if there exists some $u\in G$ such that $W(x,u,\lambda)\in G$ for every $x\in G$ and for every $\lambda\in I$,
(ii): convex (Takahashi(1970)) if $W(x,y,\lambda)\in G$ for every $x,y\in G$ and $\lambda\in I$.

Definition 2.11. [25] A subset G of a metric space (X,d) is said to be Chebyshev or uniquely proximinal (respectively, semi-Chebyshev) if $P_G(x)$ is exactly singleton (atmost singleton) for each $x\in X$ i.e for each $x\in X$ there exists exactly one (respectively, atmost one) point g_0 in G such that $d(x,g_0)=d(x,G)$.

Lemma 2.12. [25] *Let E be a normed linear space and G a Chebyshev set in E. We have:*

$$P_G[\lambda x+(1-\lambda)P_G(x)]=\pi_G(x)\quad for\quad 0\leq\lambda\leq 1. \tag{2.7}$$

3. Main Results

In this section we deals with some properties of the sets $P_G(x,y)$ and $P_G^{-1}(x,y)$ in convex metric spaces. Also, we prove a few theorems which occur frequently in the paper. we will be used of new best approximation prove some properties and show that some properties is valid for this new definition but some are not true.

Definition 3.1. [24] Let X be a nonempty set and let $G : X\times X\times X\longrightarrow R^+$ be a function satisfying the following properties:
(G1) $G(x,y,z)=0\ if\ and\ only\ if\ x=y=z$;

(G2) $0 < G(x,x,y) for\ all\ x,y \in X\ with\ x \neq y$;

(G3) $G(x,x,y) \leq G(x,y,z) for\ all\ x,y,z \in X with\ z \neq y$;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots (symmetry\ in\ all\ three\ variables)$;

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z) for\ all\ x,y,z,a \in X(\ rectangle\ inequality)$.

Then, the function G is called generalized metric or, more specifically $G-metric$ on X, and the pair (X,G) is called a $G-metric$ space.

Proposition 3.2. [24] *Every G-metric (X,G) defines a metric space (X,d_G) by*
1) $d_G(x,y) = G(x,y,y) + G(y,x,x)$.
if (X,G) is a symmetric G- metric space. Then
2) $d_G(x,y) = 2G(x,y,y)$.

Definition 3.3. Let (X,G) be a G-metic space and A a closed subset of X. For $x,y \in X$, let $G(x,y,A) = inf_{z\in A}\{G(x,y,z)\}$. A point $z_0 \in A$ satisfying $G(x,y,z_0) = G(x,y,A)$ is called a point of M-M best approximation for x,y in A.

Example 3.4. Let (X,d) be a metic space. Then $G : X \times X \times X \longrightarrow R^+$ defiened by $G(x,y,z) = d(x,z) + d(z,y)$ for all $x,y,z \in X$ is an $G-$metric on X.

Remark 3.5. In this paper we will denote the set of all M-M best approximation for x,y in A, by:

$$P_A(x,y) = \{z \in A \mid G(x,y,z) = G(x,y,A)\}. \tag{3.1}$$

The set A is said to be M-M-proximinal or an M-M-existence set if $P_A(x,y) \neq \emptyset$ for each $x,y \in X$. Also, for $a_0 \in A$, we have $P_A^{-1}(a_0) = \{(x,y) : a_0 \in P_A(x,y)\}$. The set $P_A^{-1}(a_0)$ is called the M-M-a_0-nearest points set of A. Since we can choose $x = y$ so every M-M-proximinal set is closed.

Example 3.6. Let A be the closed unit disc in $\mathbb{C}^2$ and let a_0 be the point $(1,0)$ which lie on the boundary of A. Then $P_A^{-1}(a_0)$ contains all focuses of the horizontal and vertical ellipses which center on $[1,\infty)$ and tanAent to closed unit disc at a_0. Because y' in $\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$ on the point a_0 is infinity when $y_0 \to 0$. Hence $P_A^{-1}(a_0)$ contains all the ellipses centered on the x-axis. Thus $P_A^{-1}(a_0)$ is the $\{(z,w) : z = \bar{w} \vee Im\, z = Im\, w = 0\}$.

Definition 3.7. A subset A of an $G-$metric space (X,G) is said to be approximatively compact if for every $x,y \in X$ and every sequence $\{w_n\}$ in W with

$$lim_{n\to\infty} G(x,y,w_n) = G(x,y,W) \tag{3.2}$$

there exists a subsequence $\{w_{n_i}\}$ converging to an element of W. Any sequence satisfyinA 3.2 is called a minimizing sequence for x,y in W.

In 2001 Deutsch et al. [6] show that a closed set need not be proximinal. this is ture for $P_G(x,y)$.

Example 3.8. [6] Let

$$M = \{z \in C_2[-1,1] : \int_0^1 z(t)dt = 0\}.$$

Then M is a closed subspace of $C_2[-1,1]$ that is not proximinal, Also is not M-M-proximinal.

Example 3.9. (Yates(1969)) Let K be the closed disc in R^2 and let k_0 be the point $(1,0)$ which lie on the boundary of K. Then $P_G^{-1}(g_0)$ is the ray $\{(x,0) : x \geq 1\}$. It may be noted that K is convex and $P_G^{-1}(g_0)$ is a closed convex cone.

Example 3.10. Let K be the closed disc in R^2 and let k_0 be the point $(1,0)$ which lie on the boundary of K. Then $P_G^{-1}(g_0)$ is the ray $\{(x,0) : x \geq 1\}$. It may be noted that K is convex and $P_G^{-1}(g_0)$ is a closed convex cone.

Theorem 3.11. *Let G be a subset of a metric space (X,d) and let $g_0 \in G$ Then the set $P_G^{-1}(g_0)$ is closed and*

$$x \in P_G^{-1}(g_0) \Longrightarrow \alpha x + (1-\alpha x)g_0 \in P_G^{-1}(g_0) : (\alpha = scalar). \tag{3.3}$$

Theorem 3.12. *In a convex metric space (X,d), if G is starshaped w.r.t. g_0 then $P_G(x,y)$ is starshaped w.r.t g_0 if $g_0 \in P_G(x,y)$.*

Proof. Let $z \in P_G(x,y)$. Then $d(x,y;z) = d(x,y;G)$. Since G is starshaped w.r.t g_0, $W(z,g_0,\lambda) \in G$ for each $\lambda \in I$. We claim that $W(z,g_0,\lambda) \in P_G(x,y)$ for all $\lambda \in I$. Consider

$$\begin{aligned} d(x,y,W(z,g_0,\lambda)) &\leq \lambda d(x,y,z) + (1-\lambda) d(x,y,g_0) \\ &= \lambda d(x,y,G) + (1-\lambda) d(x,y,G) \\ &= d(x,y,G) \\ &\leq d(x,y,W(z,g_0,\lambda)). \end{aligned}$$

Therefore $d(x,y,W(z,g_0,\lambda)) = d(x,y,G)$ for all $\lambda \in I$ and so $W(z,g_0,\lambda) \in P_G(x,y)$ for all $z \in P_G(x,y)$ and $\lambda \in I$. Hence $P_G(x,y)$ is starshaped w.r.t g_0. □

Theorem 3.13. *If G is convex subset of a convex metric space (X,d) then the set $P_G(x,y)$ is convex for each $x \in X$.*

Proof. Suppose $z_1, z_2 \in P_G(x,y)$. and $\lambda \in I$. Then $z_1, z_2 \in G$ and so $W(z_1,z_2,\lambda) \in G$. Consider

$$\begin{aligned} d(x,y,W(z_1,z_2,\lambda)) &\leq \lambda d(x,y,z_1) + (1-\lambda) d(x,y,z_2) \\ &= \lambda d(x,y,G) + (1-\lambda) d(x,y,G) \\ &= d(x,y,G) \\ &\leq d(x,y,W(z_1,z_2,\lambda)). \end{aligned}$$

Therefore $d(x,y,W(z_1,z_2,\lambda)) = d(x,y,G)$ for all $\lambda \in I$ and so $W(z_1,z_2,\lambda) \in P_G(x,y)$ for all $\lambda \in I$. Hence $P_G(x,y)$ is aconvex set. □

Example 3.14. Let M be the set $\{x = (\alpha_1, \alpha_2) : |\alpha_1| \geq 1\}$ in the Euclidean plane, endowed with the metric induced by the Euclidean metric, let $N = \{x = (\alpha_1, \alpha_2) \in M : |\alpha_1 + 2| \leq 1, |\alpha_2| \leq 1\}$ and let $x = (2,0), y = (3,0) \in M$ then $P_G(x,y)$ contains single point $N_0 = (-1,0) \in N^{\circ}$.

Definition 3.15. A subset G of a metric space (X,d) is said to be M-M-Chebyshev or uniquely M-M-proximinal (respectively, semi-M-M-Chebyshev) if $P_G(x,y)$ is exactly singleton (atmost singleton) for each $x,y \in X$ i.e for each $x,y \in X$ there exists exactly one (respectively, atmost one) point g_0 in G such that $d(x,y,g_0) = d(x,y,G)$.

Let X be a normed linear space and G a linear subspace of X. Then one defines, in a natural way, a mapping $\pi_G : \mathtt{D}(\pi_G) \to G$ by the condition

$$\pi_G(x,y) \in P_G(x,y) \quad (x,y \in \mathtt{D}(\pi_{\mathtt{G}})). \tag{3.4}$$

In general $\mathtt{D}(\pi_G) \neq X$ and the mapping π_G is multivalued on $\mathtt{D}(\pi_G) \setminus G$, but we have always $G \subset \mathtt{D}(\pi_G)$ and the restriction of the mapping π_G to G is one-valued. We have $\mathtt{D}(\pi_G) = X$ if and only if G is proximinal; on the other hand, π_G is one-valued on $\mathtt{D}(\pi_G)$ if and only if G is a semi-Chebyshev subspace. Even in the case when these conditions are simultaneously satisfied (i.e. G is a Chebyshev subspace),

Lemma 3.16. *Let X be a normed linear space and G a Chebyshev set in X. We have:*

$$\pi_G[(\lambda(x,y) + (1-\lambda)(\pi_G(x,y), \pi_G(x,y))] = \pi_G(x,y) \quad for \quad 0 \le \lambda \le 1. \tag{3.5}$$

Proof. Putting $z_1 = \lambda(x) + (1-\lambda)(\pi_G(x,y)), z_2 = \lambda(y + (1-\lambda)(\pi_G(x,y))$ (where $x,y \in X$ and $0 \le \lambda \le 1,$) we have

$$\begin{aligned} \|x - z_1\| + \|z_1 - \pi_G(x,y)\| &+ \|y - z_2\| + \|z_2 - \pi_G(x,y)\| \\ &= (1-\lambda)\|x - \pi_G(x,y)\| + (1-\lambda)\|y - \pi_G(x,y)\| \\ &+ \lambda\|x - \pi_G(x,y)\| + \|y - \pi_G(x,y)\| \\ &= \|x - \pi_G(x,y)\| + \|y - \pi_G(x,y))\|, \end{aligned}$$

whence

$$\begin{aligned} \|z_1 - g\| + \|z_2 - g\| &\ge \|x - g\| - \|x - z_1\| + \|y - g\| - \|y - z_2\| \\ &\ge \|x - \pi_G(x,y)\| - \|x - z_1\| + \|x - \pi_G(x,y)\| - \|x - z_1\| \\ &= \|z_1 - \pi_G(x,y)\| + \|z_2 - \pi_G(x,y)\| \quad (g \in G), \end{aligned}$$

whence $\pi_G(x,y) \in P_G(x,y)$. Since G is a Chebyshev set, it follows that we have (3.4), which completes the proof of lemma. □

Lemma 3.17. [25] *If (X,d) is a convex metric space then for $x,y \in X$ and $\lambda \in I$, we have $d(x,y) = d(x, W(x,y,\lambda)) + d(W(x,y,\lambda),y)$.*

Lemma 3.18. *If (X,d) is a convex metric space then for $x,y,z\in X$ and $\lambda\in I$, we have*

$$d(x,y;z)=d(x,W(x,z,\lambda))+d(W(x,z,\lambda),z)+d(z,W(z,y,\lambda))+d(W(z,y,\lambda),y).$$

Proof. Consider

$$\begin{aligned} d(x,y;z) &\le d(x,W(x,z,\lambda))+d(W(x,z,\lambda),z)\\ &+ d(z,W(z,y,\lambda))+d(W(z,y,\lambda),y)\\ &\le \lambda d(x,x)+(1-\lambda)d(x,z)+\lambda d(z,x)\\ &+ (1-\lambda)d(z,z)+\lambda d(z,z)\\ &+ (1-\lambda)d(z,y)+\lambda d(z,y)\\ &+ (1-\lambda)d(y,y)\\ &\le d(x,y;z). \end{aligned}$$

The result now follows. □

Theorem 3.19. *If G is a Chebyshev set in a convex metric space (X,d) then*

$$P_G(W(x,P_G(x,y),\lambda),W(y,P_G(x,y),\lambda))=P_G(x,y) \tag{3.6}$$

for every $\lambda\in I$.

Proof. By the above lemma

$$\begin{aligned} d(x,W(x,P_G(x,y),\lambda)) &+ d(W(x,P_G(x,y),\lambda),P_G(x,y))\\ &+ d(y,W(P_G(x,y),y,\lambda))+d(W(P_G(x,y),y,\lambda),P_G(x,y))\\ &= d(x,P_G(x,y))+d(P_G(x,y),y)\\ &= d(x,y;P_G(x,y)). \end{aligned}$$

Now for any $z\in G$,

$$\begin{aligned} d(x,y;z) &\le d(x,W(x,P_G(x,y),\lambda))+d(W(x,P_G(x,y),\lambda),z)\\ &+ d(z,W(y,P_G(x,y),\lambda))+d(W(z,P_G(x,y),\lambda),y) \end{aligned}$$

implies

$$\begin{aligned} d(W(x,P_G(x,y),\lambda),z) &+ d(W(y,P_G(x,y),\lambda),z) \\ &\geq d(x,y;z) - d(x,W(x,P_G(x,y),\lambda)) + d(z,W(y,P_G(x,y),\lambda)) \\ &\geq d(x,y;G) - d(x,W(x,P_G(x,y),\lambda)) - d(z,W(y,P_G(x,y),\lambda)) \\ &= d(x,y;P_G(x,y)) - d(x,W(x,P_G(x,y),\lambda)) - d(z,W(y,P_G(x,y),\lambda)) \\ &= d(W(x,P_G(x,y),\lambda),P_G(x,y)) + d(W(y,P_G(x,y),\lambda),P_G(x,y)). \end{aligned}$$

i.e.

$$\begin{aligned} d(W(x,P_G(x,y),\lambda),P_G(x,y)) &+ d(W(y,P_G(x,y),\lambda),P_G(x,y)) \\ &\leq d(W(x,P_G(x,y),\lambda),z) + d(W(y,P_G(x,y),\lambda),z) \end{aligned}$$

for all $z \in G$. Therefore

$$\begin{aligned} d(W(x,P_G(x,y),\lambda),P_G(x,y)) &+ d(W(y,P_G(x,y),\lambda),P_G(x,y)) \\ &\leq d(W(x,P_G(x,y),\lambda),G) + d(W(y,P_G(x,y),\lambda),G) \\ &\leq d(W(x,P_G(x,y),\lambda),P_G(x,y)) + d(W(y,P_G(x,y),\lambda),P_G(x,y)). \end{aligned}$$

i.e. $d(W(x,P_G(x,y),\lambda),P_G(x,y)) + d(W(y,P_G(x,y),\lambda),P_G(x,y)) = d(W(x,P_G(x,y),\lambda),G) + d(W(y,P_G(x,y),\lambda),G)$.
and since G is Chebyshev, we have $P_G(W(x,P_G(x,y),\lambda),W(y,P_G(x,y),\lambda)) = P_G(x,y)$ □

Theorem 3.20. *If G is a subset of a convex metric spaces (X,d) and $g_0 \in G$ then*

$$(x,y) \in P_G^{-1}(g_0) \Rightarrow ((W(x,g_0,\lambda),(W(y,g_0,\lambda)) \in P_G^{-1}(g_0)$$

for every $\lambda \in I$ i.e. $P_G^{-1}(g_0)$ is starshaped.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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