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TRIPLED COMMON FIXED POINT RESULTS IN ORDERED S -METRIC SPACES

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Abstract. The notion of tripled fixed point theorem introduced by V. Berinde and M. Borcut [5] and Sedghi et al [22] introduced the concept of S - metric space. Our aim of this article is to extend the concept of tripled fixed point in S - metric space and prove a tripled coincidence and common fixed point theorems for commuting mappings with mixed g -monotone property in partially ordered S -metric spaces. We also give some examples in support of our theorem.

Keywords: tripled coincidence point; tripled fixed point; mixed g - monotone; mixed monotone.

2010 AMS Subject Classification: 45H10, 54H25.

1. INTRODUCTION AND PRELIMINARIES

Fixed point theory is one of the most active field for the researchers. The first known result for fixed point theory in metric space was given by Banach [4] namely as Banach contraction principle. In last few decade this contraction principle was generalized and extend in many ways. Beside this, some authors are interested and have tried to give generalizations of metric spaces in different ways. In 1963 Gahler [10] gave the concepts of 2- metric space further in 1992 Dhage [9] modified the concept of 2- metric space and introduced the concepts of D -

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metric space but in 2005 Mustafa and Sims [12] pointed out that these attempts are not valid and introduced the concepts of G - metric space and proved fixed point theorems in G - metric space. Many authors proved different fixed point theorems in G - metric space in different ways see in [21] and references therein. Sedghi et al. [20] modified the concepts of D - metric space and introduced the concepts of D^* - metric space also proved a common fixed point theorems in D^* - metric space.

Recently, Sedghi et al [22] introduced the concept of S - metric space which is different from other space and proved fixed point theorems in S -metric space. They also gives some examples of S - metric spaces which shows that S - metric space is different form other spaces. In fact they gives following concepts of S - metric space.

Definition 1. Let X be a nonempty set. An S - metric space on X is a function $S : X^3 \rightarrow [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$,

- (1) $S(x, y, z) \geq 0$,
- (2) $S(x, y, z) = 0$ if and only if $x = y = z$,
- (3) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

The pair (X, S) is called an S - metric space.

Examples of such S - metric space are as follows,

Example 2. Let $X = \mathbb{R}^n$ and $\| \cdot \|$ a norm on X , then $S(x, y, z) = \| y + z - 2x \| + \| y - z \|$ is an S - metric on X .

Example 3. Let $X = \mathbb{R}^n$ and $\| \cdot \|$ a norm on X , then $S(x, y, z) = \| x - z \| + \| y - z \|$ is an S - metric on X .

Example 4. Let X be a nonempty set, d is ordinary metric on X , then $S(x, y, z) = d(x, z) + d(y, z)$ is an S - metric on X .

Lemma 5. Let (X, S) be an S - metric space, then we have,

$$S(x, x, y) = S(y, y, x).$$

Proof. By the third condition of S - metric, we have

$$S(x, x, y) \leq S(x, x, x) + S(x, x, x) + S(y, y, x)$$

and similarly

$$S(y, y, x) \leq S(y, y, y) + S(y, y, y) + S(x, x, y)$$

which implies that

$$S(x, x, y) = S(y, y, x).$$

□

Definition 6. Let (X, S) be an S - metric space.

(1) A sequence $\{x_n\}$ in X is said to be converges to x if and only if $S(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$.

That is for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \varepsilon$ and we denote this by $\lim_{n \rightarrow \infty} x_n = x$.

(2) A sequence $\{x_n\}$ in X is said to be Cauchy sequence if and only if $S(x_n, x_m, x) \rightarrow 0$

as $n, m \rightarrow \infty$. That is for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m \geq n_0$,

$$S(x_n, x_m, x) < \varepsilon.$$

Definition 7. The S - metric space (X, S) is said to be complete if every Cauchy sequence is convergent.

Every S - metric on X defines a metric d_S on X by

$$(1.1) \quad d_S(x, y) = S(x, x, y) + S(y, y, x) \quad \forall x, y \in X.$$

Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that $B_S(x, r) \subset A$. Then τ is a topology on X . Also, nonempty subset A in the S - metric space (X, S) is S - closed if $\bar{A} = A$.

Lemma 8. Let (X, S) be a S - metric space and A is a nonempty subset of X . A is said S - closed if for any sequence $\{x_n\}$ in A such that $x_n \rightarrow x$ as $n \rightarrow \infty$, then $x \in A$.

Very recently, In some very recent papers, Berinde and Borcut [5], Borcut and Berinde [6], and Borcut [7] have introduced the concepts of tripled fixed point and tripled coincidence point, respectively, for nonlinear contractive mappings $F : X \times X \times X \rightarrow X$ in partially ordered complete metric spaces and obtained existence and uniqueness theorems of tripled fixed points and tripled coincidence points, respectively, for some general classes of contractive type mappings.

The presented theorems in [5, 6, 7] extend several existing results in the literature [11]. We recall the main concepts needed to present them.

Definition 9. Let (X, \leq) is a partially ordered set and $F : X \times X \times X \rightarrow X$. The mapping F is said to have the mixed monotone property if F is nondecreasing monotone in first argument and is a nonincreasing monotone in its second argument, that is, for any $x, y \in X$

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \implies F(x_1, y, z) \leq F(x_2, y, z)$$

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \implies F(x, y_1, z) \geq F(x, y_2, z)$$

$$z_1, z_2 \in X, \quad z_1 \leq z_2 \implies F(x, y, z_1) \leq F(x, y, z_2)$$

Definition 10. Let (X, \leq) is a partially ordered set and $F : X \times X \times X \rightarrow X$ and $g : X \rightarrow X$. The mapping F is said to have the mixed g - monotone property if F is g - nondecreasing monotone in first and third argument and is a g - nonincreasing monotone in its second argument, that is, for any $x, y, z \in X$

$$(1.2) \quad x_1, x_2 \in X, \quad g(x_1) \leq g(x_2) \implies F(x_1, y, z) \leq F(x_2, y, z)$$

$$(1.3) \quad y_1, y_2 \in X, \quad g(y_1) \leq g(y_2) \implies F(x, y_1, z) \geq F(x, y_2, z)$$

$$(1.4) \quad z_1, z_2 \in X, \quad g(z_1) \leq g(z_2) \implies F(x, y, z_1) \leq F(x, y, z_2)$$

It is clear that Definition 10 reduced to 9 when g is the identity mapping.

Definition 11. An element $(x, y, z) \in X \times X \times X$ is called a tripled fixed point of a mapping $F : X \times X \times X \rightarrow X$ if

$$F(x, y, z) = x \quad F(y, x, y) = y \quad \text{and} \quad F(z, y, x) = z.$$

Definition 12. An element $(x, y, z) \in X \times X \times X$ is called a tripled coincidence point of the mappings $F : X \times X \times X \rightarrow X$ and $g : X \rightarrow X$ if

$$F(x, y, z) = g(x) \quad F(y, x, y) = g(y) \quad \text{and} \quad F(z, y, x) = g(z).$$

It is easy to see that tripled coincidence point can be reduced to tripled fixed point on taking g be an identity mapping.

The aim of this paper is to prove a tripled coincidence and common fixed point theorems for commuting mappings with mixed g -monotone property in partially ordered S -metric spaces.

2. MAIN RESULTS

Our first result is the following

Theorem 13. Let (X, S, \leq) be a partially ordered S -metric space. Let $F : X \times X \times X \rightarrow X$ and $g : X \rightarrow X$ be mappings such that F has the mixed g -monotone property on X and there exists a $k \in [0, \frac{1}{3})$

$$(2.1) \quad S(F(x, y, z), F(u, v, w), F(a, b, c)) \leq k[S(gx, gu, ga) + S(gy, gv, gb) + S(gz, gw, gc)]$$

for all $x, y, z, u, v, w, a, b, c \in X$ for which $gx \geq gu \geq ga$, $gy \leq gv \leq gb$ and $gz \geq gw \geq gc$ where either $gx \neq gu \neq ga$ or $gy \neq gv \neq gb$ or $gz \neq gw \neq gc$. If there exists $x_0, y_0, z_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0, z_0), \quad gy_0 \geq F(y_0, x_0, y_0) \quad \text{and} \quad gz_0 \leq F(z_0, y_0, x_0).$$

We assume the following hypotheses,

- (i). $F(X \times X \times X) \subseteq g(X)$,
- (ii). $g(X)$ is S -complete,
- (iii). g is S -continuous and commutes with F .

Then F and g have a tripled coincidence point. If $gx = gu = ga$, $gy = gv = gb$ and $gz = gw = gc$, then F and g have common fixed point, that is, there exist $x \in X$ such that

$$g(x) = F(x, x, x) = x.$$

Proof. Let $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, z_0)$ and $gz_0 \leq F(z_0, y_0, x_0)$. Since $F : X \times X \times X \subseteq g(X)$, we can choose $gx_1, gy_1, gz_1 \in X$ such that $gx_1 = F(x_0, y_0, z_0)$,

$gy_1 = F(y_0, x_0, y_0)$ and $gz_1 = F(z_0, y_0, x_0)$. Again since $F : X \times X \times X \subseteq g(X)$, we can choose $x_2, y_2, z_2 \in X$ such that $g(x_2) = F(x_1, y_1, z_1)$, $gy_2 = F(y_1, x_1, y_1)$ and $g(z_2) = F(z_1, y_1, x_1)$. Continuing this process, we can construct sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$(2.2) \quad g(x_{n+1}) = F(x_n, y_n, z_n), \quad g(y_{n+1}) = F(y_n, x_n, y_n) \quad \text{and} \quad g(z_{n+1}) = F(z_n, y_n, x_n)$$

for all $n > 0$.

Next, we show that

$$(2.3) \quad g(x_n) \leq g(x_{n+1}), \quad g(y_n) \geq g(y_{n+1}) \quad \text{and} \quad g(z_n) \leq g(z_{n+1})$$

for all $n > 0$.

Since $g(x_0) \leq F(x_0, y_0, z_0) = g(x_1)$, $g(y_0) \geq F(y_0, x_0, y_0) = g(y_1)$ and $g(z_0) \leq F(z_0, y_0, x_0) = g(z_1)$, therefore, (2.3) holds for $n = 0$. Next, suppose that (2.3) holds for some fixed $n \geq 0$, that is,

$$(2.4) \quad g(x_n) \leq g(x_{n+1}), \quad g(y_n) \geq g(y_{n+1}) \quad \text{and} \quad g(z_n) \leq g(z_{n+1})$$

Since F is the mixed g -monotone property, from 2.4 and 1.2, we have

$$(2.5) \quad F(x_n, y, z) \leq F(x_{n+1}, y, z), \quad F(y_{n+1}, x, y) \leq F(y_n, x, y) \quad \text{and} \quad F(z_n, y, x) \leq F(z_{n+1}, y, x)$$

for all $x, y, z \in X$ and from 2.4 and 1.3 we have

$$(2.6) \quad F(y, x_n, y) \geq F(y, x_{n+1}, y), \quad F(x, y_{n+1}, z) \geq F(x, y_n, z) \quad \text{and} \quad F(z, y, x_n) \geq F(z, y, x_{n+1})$$

for all $x, y \in X$. If we take $y = y_n$, $x = x_n$ and $z = z_n$ in 2.5, then we obtain and

$$(2.7) \quad \begin{aligned} g(x_{n+1}) &= F(x_n, y_n, z_n) \leq F(x_{n+1}, y_n, z_n), \\ F(y_{n+1}, x_n, y_{n+1}) &\leq F(y_n, x_n, y_n) = g(y_{n+1}) \\ g(z_{n+1}) &= F(z_n, y_n, x_n) \leq F(z_{n+1}, y_n, x_n) \end{aligned}$$

If we take $y = y_{n+1}$, $x = x_{n+1}$ and $z = z_{n+1}$ in 2.6 then

$$\begin{aligned}
(2.8) \quad & F(y_{n+1}, x_n, y_{n+1}) \geq F(y_{n+1}, x_{n+1}, y_{n+1}) = g(y_{n+2}) \\
& g(x_{n+2}) = F(x_{n+1}, y_{n+1}, z_{n+1}) \geq F(x_{n+1}, y_n, z_{n+1}) \\
& g(z_{n+2}) = F(z_{n+1}, y_{n+1}, x_{n+1}) \geq F(z_{n+1}, y_n, x_{n+1}).
\end{aligned}$$

Now, from 2.7 and 2.8, we have

$$(2.9) \quad g(x_{n+1}) \leq g(x_{n+2}), \quad g(y_{n+1}) \geq g(y_{n+2}) \quad \text{and} \quad g(z_{n+1}) \leq g(z_{n+2}).$$

Therefore, by the mathematical induction, we conclude that 2.3 holds for all $n \geq 0$.

Continuing this process, one can easily verify that

$$g(x_0) \leq g(x_1) \leq g(x_2) \leq \dots \leq g(x_{n+1}) \leq \dots$$

$$g(y_0) \geq g(y_1) \geq g(y_2) \geq \dots \geq g(y_{n+1}) \geq \dots$$

and

$$g(z_0) \leq g(z_1) \leq g(z_2) \leq \dots \leq g(z_{n+1}) \leq \dots$$

If $(x_{n+1}, y_{n+1}, z_{n+1}) = (x_n, y_n, z_n)$, then F and g have a tripled coincidence point. So we assume $(x_{n+1}, y_{n+1}, z_{n+1}) \neq (x_n, y_n, z_n)$ for all $n \geq 0$, that is, we assume that either $g(x_{n+1}) = F(x_n, y_n, z_n) \neq g(x_n)$ or $g(y_{n+1}) = F(y_n, x_n, y_n) \neq g(y_n)$ or $g(z_{n+1}) = F(z_n, y_n, x_n) \neq g(z_n)$.

Next, we claim that, for all $n \geq 0$,

$$(2.10) \quad S(gx_n, gx_n, gx_{n+1}) \leq \frac{1}{3}(3k)^n [S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1) + S(gz_0, gz_0, gz_1)].$$

For $n = 1$, we have

$$\begin{aligned}
(2.11) \quad & S(gx_1, gx_1, gx_2) = S(F(x_0, y_0, z_0), F(x_0, y_0, z_0), F(x_1, y_1, z_1)) \\
& \leq k[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1) + S(gz_0, gz_0, gz_1)] \\
& \leq \frac{1}{3}(3k)[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1) + S(gz_0, gz_0, gz_1)].
\end{aligned}$$

Thus 2.10 holds for $n = 1$. Therefore, we presume that 2.10 holds for $n > 0$. Since $g(x_{n+1}) \geq g(x_n)$, $g(y_{n+1}) \leq g(y_n)$ and $g(z_{n+1}) \geq g(z_n)$, from 2.1 and 2.2 we have

$$\begin{aligned} S(gx_n, gx_n, gx_{n+1}) &= S(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n)) \\ (2.12) \quad &\leq k[S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n) + S(gz_{n-1}, gz_{n-1}, gz_n)] \end{aligned}$$

similarly, we have

$$\begin{aligned} S(gy_n, gy_n, gy_{n+1}) &= S(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_n, x_n, y_n)) \\ (2.13) \quad &\leq k[S(gy_{n-1}, gy_{n-1}, gy_n) + S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n)]. \end{aligned}$$

and

$$\begin{aligned} S(gz_n, gz_n, gz_{n+1}) &= S(F(z_{n-1}, y_{n-1}, x_{n-1}), F(x_{n-1}, y_{n-1}, z_{n-1}), F(z_n, y_n, x_n)) \\ (2.14) \quad &\leq k[S(gz_{n-1}, gz_{n-1}, gz_n) + S(gy_{n-1}, gy_{n-1}, gy_n) + S(gx_{n-1}, gx_{n-1}, gx_n)] \end{aligned}$$

By adding 2.12, 2.13 and 2.14, then we get

$$\begin{aligned} &S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1}) + S(gz_n, gz_n, gz_{n+1}) \\ &\leq 3k[S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n) + S(gz_{n-1}, gz_{n-1}, gz_n)] \end{aligned}$$

Continuing the process, we have for each $n \in N$,

$$\begin{aligned} &S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1}) + S(gz_n, gz_n, gz_{n+1}) \\ &\leq \frac{1}{3}(3k)^n[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1) + S(gz_0, gz_0, gz_1)] \\ &\leq \frac{(3k)^n}{3(1-3k)}[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1) + S(gz_0, gz_0, gz_1)] \end{aligned}$$

Letting $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} [S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1}) + S(gz_n, gz_n, gz_{n+1})] = 0.$$

That is

$$\lim_{n \rightarrow \infty} S(gx_n, gx_n, gx_{n+1}) = 0,$$

$$\lim_{n \rightarrow \infty} S(gy_n, gy_n, gy_{n+1}) = 0$$

and

$$\lim_{n \rightarrow \infty} S(gz_n, gz_n, gz_{n+1}) = 0.$$

Thus $\{gx_n\}, \{gy_n\}$ and $\{gz_n\}$ are S -Cauchy sequences in $g(X)$.

Since $g(X)$ is S -complete, we get $\{gx_n\}, \{gy_n\}$ and $\{gz_n\}$ are converges to some $x \in X, y \in X$ and $z \in X$ respectively. Since g is S -continuous, we have $\{g(gx_n)\}, \{g(gy_n)\}$ and $\{g(gz_n)\}$ are converges to gx, gy and gz respectively. that is

$$(2.15) \quad \lim_{n \rightarrow \infty} g(gx_n) = gx,$$

$$(2.16) \quad \lim_{n \rightarrow \infty} g(gy_n) = gy$$

and

$$(2.17) \quad \lim_{n \rightarrow \infty} g(gz_n) = gz.$$

Also from commutativity of F and g , we have

$$(2.18) \quad F((gx_n), g(y_n), g(z_n)) = gF(x_n, y_n, z_n) = g(gx_{n+1}),$$

$$(2.19) \quad F((gy_n), g(x_n), g(y_n)) = gF(y_n, x_n, y_n) = g(gy_{n+1})$$

and

$$(2.20) \quad F((gz_n), g(y_n), g(x_n)) = gF(z_n, y_n, x_n) = g(gz_{n+1}),$$

Next we claim that (x, y, z) is a tripled coincidence point of F and g .

Now from the condition 2.1, we have

$$\begin{aligned} S(g(gx_{n+1}), g(gx_{n+1}), F(x, y, z)) &= S(F(gx_n, gy_n, gz_n), F(gx_n, gy_n, gz_n), F(x, y, z)) \\ &\leq k[S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy) + S(gz_n, gz_n, gz)] \end{aligned}$$

Letting $n \rightarrow \infty$ and using the fact that S is continuous on its variables, we get that

$$S(gx, gx, F(x, y, z)) \leq k[S(gx, gx, gx) + S(gy, gy, gy) + S(gz, gz, gz)] = 0$$

Hence $gx = F(x, y, z)$. Similarly, we may show that $gy = F(y, x, y)$ and $gz = F(z, y, x)$.

Finally, we claim that x is common fixed point of F and g .

Since (x, y, z) is a tripled coincidence point of the mappings F and g , we have $gx = F(x, y, z)$, $gy = F(y, x)$ and $gz = F(z, y, x)$. Assume that $gx \neq gy \neq gz$. Then by 2.1, we get

$$(2.21) \quad \begin{aligned} S(gx, gx, gy) &= S(F(x, y, z), F(x, y, z), F(y, x, y)) \\ &\leq k[S(gx, gx, gy) + S(gy, gy, gx) + S(gz, gz, gy)] \end{aligned}$$

also

$$(2.22) \quad \begin{aligned} S(gy, gy, gx) &= S(F(y, x, y), F(y, x, y), F(x, y, z)) \\ &\leq k[S(gx, gx, gy) + S(gy, gy, gx) + S(gy, gy, gz)] \end{aligned}$$

and

$$(2.23) \quad \begin{aligned} S(gz, gz, gy) &= S(F(z, y, x), F(z, y, x), F(y, x, y)) \\ &\leq k[S(gz, gz, gy) + S(gy, gy, gx) + S(gx, gx, gy)] \end{aligned}$$

by adding 2.21, 2.22 and 2.23 we have

$$S(gx, gx, gy) + S(gy, gy, gx) + S(gz, gz, gy) \leq 3k[S(gx, gx, gy) + S(gy, gy, gz) + S(gz, gz, gy)]$$

since $k < \frac{1}{3}$, we get

$$S(gx, gx, gy) + S(gy, gy, gx) + S(gz, gz, gy) \leq [S(gx, gx, gy) + S(gy, gy, gx) + S(gz, gz, gy)]$$

Which contradiction. So $gx = gy = gz$, and hence

$$F(x, y, z) = gx = gy = gz = F(y, x, y) = F(z, y, x).$$

Since $\{gx_{n+1}\}$ is subsequence of $\{gy_n\}$ we have $\{gy_{n+1}\}$ is S -convergent to x . Thus

$$\begin{aligned} S(gx_{n+1}, gx_{n+1}, gx) &= S(gx_{n+1}, gx_{n+1}, F(x, y, z)) \\ &= S(F(x_n, y_n, z_n), F(x_n, y_n, z_n), F(x, y, z)) \\ &\leq k[S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy) + S(gz_n, gz_n, gz)] \end{aligned}$$

Letting $n \rightarrow \infty$ and use the fact that S is continuous on its variables, we get

$$S(x, x, gx) \leq k[S(x, x, gx) + S(y, y, gy) + S(z, z, gz)]$$

Similarly, we may show that

$$S(y, y, gy) \leq k[S(x, x, gx) + S(y, y, gy) + S(z, z, gz)]$$

and

$$S(z, z, gz) \leq k[S(x, x, gx) + S(y, y, gy) + S(z, z, gz)]$$

Thus

$$S(x, x, gx) + S(y, y, gy) + S(z, z, gz) \leq 3k[S(x, x, gx) + S(y, y, gy) + S(z, z, gz)].$$

Since $3k < 1$, the last inequality happens only if $S(x, x, gx) = 0$, $S(y, y, gy) = 0$ and $S(z, z, gz) = 0$. Hence $x = gx$, $y = gy$ and $z = gz$. Thus we get $gx = F(x, y, z) = x$, $gy = F(y, x, y) = y$ and $gz = F(z, y, x) = z$. Thus F and g have a common fixed point. This completes the proof of the theorem. \square

Our second result of this paper is following.

Theorem 14. *In the above theorem, in the place of condition (ii), if we assume the following conditions in the complete S - metric space X , namely*

(2.24) *if $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \rightarrow x \in X$, then $x_n \leq x$, $\forall n$,*

(2.25) *if $\{y_n\} \subset X$ is a nonincreasing sequence with $y_n \rightarrow y \in X$, then $y_n \geq y$, $\forall n$*

and

(2.26) *if $\{z_n\} \subset X$ is a nondecreasing sequence with $z_n \rightarrow z \in X$, then $z_n \leq z$, $\forall n$.*

Then, we have the conclusion of Theorem 13, provided g is non decreasing.

Proof. Proceeding exactly as in Theorem 13, we have $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are S - Cauchy sequences in X . Since (X, S) is a complete, there exists $(x, y, z) \in X \times X \times X$ such that

$$(2.27) \quad \lim_{n \rightarrow \infty} F(x_n, y_n, z_n) = \lim_{n \rightarrow \infty} g(x_n) = x,$$

$$(2.28) \quad \lim_{n \rightarrow \infty} F(y_n, x_n, y_n) = \lim_{n \rightarrow \infty} g(y_n) = y$$

and

$$(2.29) \quad \lim_{n \rightarrow \infty} F(z_n, y_n, x_n) = \lim_{n \rightarrow \infty} g(z_n) = z.$$

Therefore, from (iii) we arrive at 2.15, 2.16 2.20 and 2.19. Since $\{gx_n\}$ is a non decreasing sequence and $gx_n \rightarrow x$ and as $\{gy_n\}$ is non increasing sequence and $gy_n \rightarrow y$ also $\{gz_n\}$ is a non decreasing sequence and $gz_n \rightarrow z$, by assumption 2.24 and 2.25 we have, $g(gx_n) \leq gx$, $g(gy_n) \geq gy$ and $g(gz_n) \leq gz$ for all $n \geq 0$. If $g(gx_n) = gx$, $g(gy_n) = gy$ and $g(gz_n) = gz$ for some n , then by construction $g(gx_{n+1}) = gx$, $g(gy_{n+1}) = gy$ and $g(gz_{n+1}) = gz$ and (x, y, z) is tripled fixed point. So we assume either $g(gx_n) \neq gx$ or $g(gy_n) \neq gy$ or $g(gz_n) \neq gz$. Applying the contractive condition 2.1, we have

$$\begin{aligned} & S(F(x, y, z), F(x, y, z), gx) \\ & \leq 3[S(F(x, y, z), F(x, y, z), F(gx_n, gy_n, gz_n)) \\ & \quad + S(F(x, y, z), F(x, y, z), F(gx_n, gy_n, gz_n)) + S(gx, gx, F(gx_n, gy_n, gz_n))] \\ & = 3[S(F(x, y, z), F(x, y, z), F(gx_n, gy_n, gz_n)) \\ & \quad + S(F(x, y, z), F(x, y, z), F(gx_n, gy_n, gz_n)) + S(gx, gx, gF(x_n, y_n, z_n))] \\ & \leq 3\{2k[S(gx, gx, gx_n) + S(gy, gy, gy_n) + S(gz, gz, gz_n)] + S(g(gx, gx, g(gx_{n+1})))\} \end{aligned}$$

Taking $n \rightarrow \infty$ in the above inequality we obtain $S(F(x, y, z), F(x, y, z), gx) = 0$, that is, $F(x, y, z) = g(x)$. Similarly we have that $F(y, x, y) = g(y)$ and $F(z, y, x) = g(z)$. Remaining part of the proof follows from Theorem 13. Hence we have $g(x) = F(x, x, x) = x$. This completes the proof of the theorem. □

Corollary 15. *Let (X, S, \leq) be a partially ordered S - metric space. Let $F : X \times X \times X \rightarrow X$ and $g : X \rightarrow X$ be mappings such that F has the mixed g - monotone property on X and there exists a $k \in [0, \frac{1}{3})$*

$$(2.30) \quad S(F(x, y, z), F(x, y, z), F(u, v, w)) \leq k[S(gx, gx, gu) + S(gy, gy, gv) + S(gz, gz, gw)]$$

for all $x, y, z, u, v, w \in X$ for which $gx \geq gu$, $gy \leq gv$ and $gz \neq gw$. If there exists $x_0, y_0, z_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0, z_0), \quad gy_0 \geq F(y_0, x_0, y_0) \quad \text{and} \quad gz_0 \leq F(z_0, y_0, x_0).$$

We assume the following hypotheses,

- (i). $F : (X \times X \times X) \subseteq g(X)$,
- (ii). $g(X)$ is S -complete,
- (iii). g is S -continuous and commutes with F .

Then F and g have a tripled coincidence point. If $gx = gu$, $gy = gv$ and $gz = gw$, then F and g have common fixed point, that is, there exist $x \in X$ such that

$$g(x) = F(x, x, x) = x.$$

Proof. It follows from Theorem 13 if we take $x = u$, $y = v$ and $z = w$. □

Corollary 16. Let (X, S, \leq) be a partially ordered S -metric space. Let $F : X \times X \times X \rightarrow X$ be mapping such that F has the mixed monotone property on X and there exists a $k \in [0, \frac{1}{3})$

$$(2.31) \quad S(F(x, y, z), F(u, v, w), F(a, b, c)) \leq k[S(x, u, a) + S(y, v, b) + S(z, w, c)]$$

for all $x, y, z, u, v, w, a, b, c \in X$ for which $x \geq u \geq a$, $y \leq v \leq b$ and $z \geq w \geq c$ where either $u \neq a$ or $v \neq b$ or $w \neq c$. If there exists $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0) \quad \text{and} \quad z_0 \leq F(z_0, y_0, x_0).$$

Then F has tripled fixed point in X . If $u = a$ or $v = b$ or $w = c$, then F has fixed point, that is, there exist $x \in X$ such that

$$x = F(x, x, x).$$

Proof. If we define $g : X \rightarrow X$ be an identity mapping in Theorem 13 then result is follows. □

Corollary 17. Let (X, S, \leq) be a partially ordered S -metric space. Let $F : X \times X \times X \rightarrow X$ be mapping such that F has the mixed monotone property on X and there exists a $k \in [0, \frac{1}{3})$

$$(2.32) \quad S(F(x, y, z), F(x, y, z), F(a, b, c)) \leq k[S(gx, gx, ga) + S(gy, gy, gb) + S(gz, gz, gc)]$$

for all $x, y, z, a, b, c \in X$ for which $x \geq a$, $y \leq b$ and $z \geq c$. If there exists $x_0, y_0, z_0 \in X$ such that

$$x_0 \leq F(x_0, y_0, z_0), \quad y_0 \geq F(y_0, x_0, y_0) \quad \text{and} \quad z_0 \leq F(z_0, y_0, x_0).$$

Then F has a tripled fixed point. If $x = a$, $y = b$ and $z = c$, then F has fixed point, that is, there exist $x \in X$ such that

$$x = F(x, x, x).$$

Proof. If we define $g : X \rightarrow X$ be an identity mapping in Corollary 16 then result is follows. \square

Now we present some examples to illustrate our results given by Theorem 13 and Theorem 14.

Example 18. Let $X = \mathbb{R}$ be ordered by the following relation

$$x \leq y \iff x = y \quad \text{or} \quad (x, y \in [0, 1] \quad \text{and} \quad x \leq y).$$

Let a S -metric on X be defined by

$$S(x, y, z) = |x - z| + |y - z|.$$

Then (X, S) is a complete regular ordered S -metric space.

Let $g : X \rightarrow X$ and $F : X \times X \times X \rightarrow X$ be defined by

$$g(x) = \begin{cases} \frac{1}{30}x & \text{if } x < 0 \\ \frac{1}{3}(x) & \text{if } x \in [0, 1] \\ \frac{1}{30}x + \frac{3}{10} & \text{if } x > 1, \end{cases} \quad \text{and} \quad F(x, y, z) = \frac{x+y+z}{30}$$

Take $k = \frac{1}{10}$. Then we found that all the conditions of Theorem 13 and Theorem 14 are satisfied.

Obviously, the mappings g and F have a unique common tripled fixed point $(0, 0, 0)$.

Example 19. Let $X = [0, 1]$, with the usual partial ordered \leq . Let a S -metric on X be defined by

$$S(x, y, z) = |x - z| + |y - z|.$$

Then (X, S) is a complete regular ordered S - metric space.

Let $g : X \rightarrow X$ and $F : X \times X \times X \rightarrow X$ be defined by

$$(2.33) \quad F(x, y, z) = \frac{x+y+z}{24} \text{ and } g(x) = \frac{x}{3}$$

Take $k = \frac{1}{8} \in [0, \frac{1}{3})$. Then we found that all the conditions of Theorem 13 and Theorem 14 are satisfied. Obviously, the mappings g and F have a unique common tripled fixed point $(0, 0, 0)$.

Example 20. Let $X = [0, 1]$, with the usual partial ordered \leq . Let a S - metric on X be defined by

$$S(x, y, z) = |x - z| + |y - z|.$$

Then (X, S) is a complete regular ordered S - metric space.

Let $g : X \rightarrow X$ and $F : X \times X \times X \rightarrow X$ be defined by

$$(2.34) \quad F(x, y) = \frac{1}{20}[\sin x + \sin y + \sin z] \text{ and } g(x) = \frac{x}{5}$$

for all $x, y, z \in X$. Since $|\sin x - \sin y| \leq |x - y|$, $|\sin y - \sin z| \leq |y - z|$ and $|\sin z - \sin x| \leq |z - x|$ holds for all $x, y, z \in X$. Then we have $k = \frac{1}{4} \in [0, \frac{1}{3})$. So all the conditions of Theorem 13 and Theorem 14 are satisfied. Then there exists a tripled fixed point of F . In this case $(0, 0, 0)$ is tripled fixed point of F .

Furthermore we show that Theorem 13 is not true for following example,

Example 21. Let $X = [0, 1]$, with the usual partial ordered \leq . Let a S - metric on X be defined by

$$S(x, y, z) = |x - z| + |y - z|.$$

Then (X, S) is a complete regular ordered S - metric space.

Let $g : X \rightarrow X$ and $F : X \times X \times X \rightarrow X$ be defined by

$$(2.35) \quad F(x, y, z) = \frac{x+y+z}{6} \text{ and } g(x) = \frac{x}{3}$$

Then there is no $k = \frac{1}{8} \in [0, \frac{1}{2})$ for which Theorem 13 is true Take $k = \frac{1}{8} \in [0, \frac{1}{2})$. Then we found that all the conditions of Theorem 13 and Theorem 14 are satisfied. Obviously, the mappings g and F have a unique common tripled fixed point $(0, 0, 0)$.

Now in next section we give an another result for tripled fixed point which is generalization of our Theorem 13.

3. GENERALIZATION OF TRIPLED FIXED POINT THEOREM

We begin this section with the following example,

Example 22. Let X be the set $[0, \infty)$ and

$$S(x, y, z) = |x - z| + |y - z|.$$

We set $g : X \rightarrow X$ and $F : X \times X \times X \rightarrow X$ be defined as $g(x) = x^2$ and

$$F(x, y) = \begin{cases} \frac{x^2 - y^2 - z^2}{4} & \text{if } x \geq y \geq z \\ 0 & \text{if otherwise} \end{cases}$$

Then for $k = \frac{1}{3} \notin [0, \frac{1}{3})$ we have tripled fixed point $(0, 0, 0)$ in X .

Now we give first result of this section which as follows,

Theorem 23. Let (X, S, \leq) be a partially ordered S -metric space. Let $F : X \times X \times X \rightarrow X$ and $g : X \rightarrow X$ be mappings such that F has the mixed g -monotone property on X and there exists a $k \in [0, 1)$

$$(3.1) \quad S(F(x, y, z), F(u, v, w), F(a, b, c)) \leq \frac{k}{3} [S(gx, gu, ga) + S(gy, gv, gb) + S(gz, gw, gc)]$$

for all $x, y, z, u, v, w, a, b, c \in X$ for which $gx \geq gu \geq ga$, $gy \leq gv \leq gb$ and $gz \geq gw \geq gc$ where either $gx \neq gu \neq ga$ or $gy \neq gv \neq gb$ or $gz \neq gw \neq gc$. If there exists $x_0, y_0, z_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0, z_0), \quad gy_0 \geq F(y_0, x_0, z_0) \quad \text{and} \quad gz_0 \leq F(z_0, y_0, x_0).$$

We assume the following hypotheses,

- (i). $F : (X \times X \times X) \subseteq g(X)$,
- (ii). $g(X)$ is S -complete,
- (iii). g is S -continuous and commutes with F .

Then F and g have a tripled coincidence point. If $gx = gu = ga$, $gy = gv = gb$ and $gz = gw = gc$, then F and g have common fixed point, that is, there exist $x \in X$ such that

$$g(x) = F(x, x, x) = x.$$

Proof. Let $x_0, y_0, z_0 \in X$ such that $gx_0 \leq F(x_0, y_0, z_0)$, $gy_0 \geq F(y_0, x_0, z_0)$ and $gz_0 \leq F(z_0, y_0, x_0)$. Since $F : X \times X \times X \subseteq g(X)$, we can choose $gx_1, gy_1, gz_1 \in X$ such that $gx_1 = F(x_0, y_0, z_0)$, $gy_1 = F(y_0, x_0, z_0)$ and $gz_1 = F(z_0, y_0, x_0)$. Again since $F : X \times X \times X \subseteq g(X)$, we can choose $x_2, y_2, z_2 \in X$ such that $gx_2 = F(x_1, y_1, z_1)$, $gy_2 = F(y_1, x_1, z_1)$ and $gz_2 = F(z_1, y_1, x_1)$. Continuing this process, we can construct sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$(3.2) \quad g(x_{n+1}) = F(x_n, y_n, z_n), \quad g(y_{n+1}) = F(y_n, x_n, z_n) \quad \text{and} \quad g(z_{n+1}) = F(z_n, y_n, x_n)$$

for all $n > 0$.

Next, we show that

$$(3.3) \quad g(x_n) \leq g(x_{n+1}), \quad g(y_n) \geq g(y_{n+1}) \quad \text{and} \quad g(z_n) \leq g(z_{n+1}).$$

for all $n > 0$.

Since $g(x_0) \leq F(x_0, y_0, z_0) = g(x_1)$, $g(y_0) \geq F(y_0, x_0, z_0) = g(y_1)$ and $g(z_0) \leq F(z_0, y_0, x_0) = g(z_1)$, therefore, (3.3) holds for $n = 0$. Next, suppose that (3.3) holds for some fixed $n \geq 0$, that is,

$$(3.4) \quad g(x_n) \leq g(x_{n+1}), \quad g(y_n) \geq g(y_{n+1}) \quad \text{and} \quad g(z_n) \leq g(z_{n+1})$$

Since F is the mixed g -monotone property, from 3.4 and 1.2, we have

$$(3.5) \quad F(x_n, y, z) \leq F(x_{n+1}, y, z), \quad F(y_{n+1}, x, y) \leq F(y_n, x, y) \quad \text{and} \quad F(z_n, y, x) \leq F(z_{n+1}, y, x)$$

for all $x, y, z \in X$ and from 3.4 and 1.3 we have

$$(3.6) \quad F(y, x_n, y) \geq F(y, x_{n+1}, y), \quad F(x, y_{n+1}, z) \geq F(x, y_n, z) \quad \text{and} \quad F(z_n, y, x) \geq F(z_{n+1}, y, x)$$

for all $x, y, z \in X$. If we take $y = y_n$, $x = x_n$ and $z = z_n$ in 3.5, then we obtain and

$$(3.7) \quad \begin{aligned} g(x_{n+1}) &= F(x_n, y_n, z_n) \leq F(x_{n+1}, y_n, z_n), \\ F(y_{n+1}, x_n, y_{n+1}) &\leq F(y_n, x_n, y_n) = g(y_{n+1}) \\ g(z_{n+1}) &= F(z_n, y_n, x_n) \leq F(z_{n+1}, y_n, x_n) \end{aligned}$$

If we take $y = y_{n+1}$, $x = x_{n+1}$ and $z = z_{n+1}$ in 3.6 then

$$\begin{aligned}
 (3.8) \quad & F(y_{n+1}, x_n, y_{n+1}) \geq F(y_{n+1}, x_{n+1}, y_{n+1}) = g(y_{n+2}) \\
 & g(x_{n+2}) = F(x_{n+1}, y_{n+1}, z_{n+1}) \geq F(x_{n+1}, y_n, z_{n+1}) \\
 & g(z_{n+2}) = F(z_{n+1}, y_{n+1}, x_{n+1}) \geq F(z_{n+1}, y_n, x_{n+1}).
 \end{aligned}$$

Now, from 3.7 and 3.8, we have

$$(3.9) \quad g(x_{n+1}) \leq g(x_{n+2}), \quad g(y_{n+1}) \geq g(y_{n+2}) \quad \text{and} \quad g(z_{n+1}) \leq g(z_{n+2}).$$

Therefore, by the mathematical induction, we conclude that 2.3 holds for all $n \geq 0$.

Continuing this process, one can easily verify that

$$g(x_0) \leq g(x_1) \leq g(x_2) \leq \dots \leq g(x_{n+1}) \leq \dots$$

$$g(y_0) \geq g(y_1) \geq g(y_2) \geq \dots \geq g(y_{n+1}) \geq \dots$$

and

$$g(z_0) \leq g(z_1) \leq g(z_2) \leq \dots \leq g(z_{n+1}) \leq \dots$$

If $(x_{n+1}, y_{n+1}, z_{n+1}) = (x_n, y_n, z_n)$, then F and g have a tripled coincidence point. So we assume

$$(x_{n+1}, y_{n+1}, z_{n+1}) \neq (x_n, y_n, z_n)$$

for all $n \geq 0$, that is, we assume that either

$$g(x_{n+1}) = F(x_n, y_n, z_n) \neq g(x_n)$$

or

$$g(y_{n+1}) = F(y_n, x_n, y_n) \neq g(y_n)$$

or

$$g(z_{n+1}) = F(z_n, y_n, x_n) \neq g(z_n).$$

Next, we claim that, for all $n \geq 0$,

$$(3.10) \quad S(gx_n, gx_n, gx_{n+1}) \leq \frac{1}{3}(3k)^n [S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1) + S(gz_0, gz_0, gz_1)].$$

For $n = 1$, we have

$$\begin{aligned}
 S(gx_1, gx_1, gx_2) &= S(F(x_0, y_0, z_0), F(x_0, y_0, z_0), F(x_1, y_1, z_1)) \\
 &\leq k[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1) + S(gz_0, gz_0, gz_1)] \\
 &\leq \frac{1}{3}(3k)[S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1) + S(gz_0, gz_0, gz_1)].
 \end{aligned}$$

Thus 2.10 holds for $n = 1$. Therefore, we presume that 2.10 holds for $n > 0$. Since $g(x_{n+1}) \geq g(x_n)$, $g(y_{n+1}) \leq g(y_n)$ and $g(z_{n+1}) \geq g(z_n)$, from 2.1 and 2.2 we have

$$\begin{aligned}
 S(gx_n, gx_n, gx_{n+1}) &= S(F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_{n-1}, y_{n-1}, z_{n-1}), F(x_n, y_n, z_n)) \\
 (3.11) \quad &\leq k[S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n) + S(gz_{n-1}, gz_{n-1}, gz_n)]
 \end{aligned}$$

similarly, we have

$$\begin{aligned}
 S(gy_n, gy_n, gy_{n+1}) &= S(F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1}, y_{n-1}), F(y_n, x_n, y_n)) \\
 (3.12) \quad &\leq k[S(gy_{n-1}, gy_{n-1}, gy_n) + S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n)].
 \end{aligned}$$

and

$$\begin{aligned}
 S(gz_n, gz_n, gz_{n+1}) &= S(F(z_{n-1}, y_{n-1}, x_{n-1}), F(x_{n-1}, y_{n-1}, z_{n-1}), F(z_n, y_n, x_n)) \\
 (3.13) \quad &\leq k[S(gz_{n-1}, gz_{n-1}, gz_n) + S(gy_{n-1}, gy_{n-1}, gy_n) + S(gx_{n-1}, gx_{n-1}, gx_n)]
 \end{aligned}$$

By adding 3.11, 3.12 and 3.13, then we get

$$\begin{aligned}
 &S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1}) + S(gz_n, gz_n, gz_{n+1}) \\
 (3.14) \quad &\leq 3k[S(gx_{n-1}, gx_{n-1}, gx_n) + S(gy_{n-1}, gy_{n-1}, gy_n) + S(gz_{n-1}, gz_{n-1}, gz_n)]
 \end{aligned}$$

Continuing the process, we have for each $n \in N$,

$$\begin{aligned}
 &S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1}) + S(gz_n, gz_n, gz_{n+1}) \\
 &\leq \frac{1}{3}(3k)^n [S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1) + S(gz_0, gz_0, gz_1)] \\
 &\leq \frac{(3k)^n}{3(1-3k)} [S(gx_0, gx_0, gx_1) + S(gy_0, gy_0, gy_1) + S(gz_0, gz_0, gz_1)]
 \end{aligned}$$

Letting $n \rightarrow \infty$, we have

$$\lim_{n \rightarrow \infty} [S(gx_n, gx_n, gx_{n+1}) + S(gy_n, gy_n, gy_{n+1}) + S(gz_n, gz_n, gz_{n+1})] = 0.$$

That is

$$\lim_{n \rightarrow \infty} S(gx_n, gx_n, gx_{n+1}) = 0,$$

$$\lim_{n \rightarrow \infty} S(gy_n, gy_n, gy_{n+1}) = 0$$

and

$$\lim_{n \rightarrow \infty} S(gz_n, gz_n, gz_{n+1}) = 0.$$

Thus $\{gx_n\}, \{gy_n\}$ and $\{gz_n\}$ are S -Cauchy sequences in $g(X)$.

Since $g(X)$ is S -complete, we get $\{gx_n\}, \{gy_n\}$ and $\{gz_n\}$ are converges to some $x \in X, y \in X$ and $z \in X$ respectively. Since g is S -continuous, we have $\{g(gx_n)\}, \{g(gy_n)\}$ and $\{g(gz_n)\}$ are converges to gx, gy and gz respectively. that is

$$(3.15) \quad \lim_{n \rightarrow \infty} g(gx_n) = gx,$$

$$(3.16) \quad \lim_{n \rightarrow \infty} g(gy_n) = gy$$

and

$$(3.17) \quad \lim_{n \rightarrow \infty} g(gz_n) = gz.$$

Also from commutativity of F and g , we have

$$(3.18) \quad F((gx_n), g(y_n), g(z_n)) = gF(x_n, y_n, z_n) = g(gx_{n+1}),$$

$$(3.19) \quad F((gy_n), g(x_n), g(y_n)) = gF(y_n, x_n, y_n) = g(gy_{n+1})$$

and

$$(3.20) \quad F((gz_n), g(y_n), g(x_n)) = gF(z_n, y_n, x_n) = g(gz_{n+1}),$$

Next we claim that (x, y, z) is a tripled coincidence point of F and g .

Now from the condition 3.1, we have

$$\begin{aligned}
S(g(gx_{n+1}), g(gx_{n+1}), F(x, y, z)) &= S(F(gx_n, gy_n), F(gx_n, gy_n), F(x, y, z)) \\
&\leq k[S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy)]
\end{aligned}$$

Letting $n \rightarrow \infty$ and using the fact that S is continuous on its variables, we get that

$$S(gx, gx, F(x, y, z)) \leq \frac{k}{3}[S(gx, gx, gx) + S(gy, gy, gy) + S(gz, gz, gz)] = 0$$

Hence $gx = F(x, y, z)$. Similarly, we may show that $gy = F(y, x, y)$ and $gz = F(z, y, x)$.

Finally, we claim that x is common fixed point of F and g .

Since (x, y, z) is a tripled coincidence point of the mappings F and g , we have $gx = F(x, y, z)$, $gy = F(y, x, y)$ and $gz = F(z, y, x)$. Assume that $gx \neq gy \neq gz$. Then by 3.1, we get

$$\begin{aligned}
(3.21) \quad S(gx, gx, gy) &= S(F(x, y, z), F(x, y, z), F(y, x, y)) \\
&\leq k[S(gx, gx, gy) + S(gy, gy, gx) + S(gz, gz, gy)]
\end{aligned}$$

also

$$\begin{aligned}
(3.22) \quad S(gy, gy, gx) &= S(F(y, x, y), F(y, x, y), F(x, y, z)) \\
&\leq k[S(gx, gx, gy) + S(gy, gy, gx) + S(gy, gy, gz)]
\end{aligned}$$

and

$$\begin{aligned}
(3.23) \quad S(gz, gz, gy) &= S(F(z, y, x), F(z, y, x), F(y, x, y)) \\
&\leq k[S(gz, gz, gy) + S(gy, gy, gx) + S(gx, gx, gy)]
\end{aligned}$$

by adding 3.21, 3.22 and 3.28 we have

$$S(gx, gx, gy) + S(gy, gy, gx) + S(gz, gz, gy) \leq 3k[S(gx, gx, gy) + S(gy, gy, gx) + S(gz, gz, gy)]$$

since $k < 1$, we get

$$S(gx, gx, gy) + S(gy, gy, gx) + S(gz, gz, gy) \leq [S(gx, gx, gy) + S(gy, gy, gx) + S(gz, gz, gy)]$$

Which contradiction. So $gx = gy = gz$, and hence

$$F(x, y, z) = gx = gy = gz = F(y, x, y) = F(z, y, x).$$

Since $\{gx_{n+1}\}$ is subsequence of $\{gy_n\}$ we have $\{gy_{n+1}\}$ is S -convergent to x . Thus

$$\begin{aligned} S(gx_{n+1}, gx_{n+1}, gx) &= S(gx_{n+1}, gx_{n+1}, F(x, y, z)) \\ &= S(F(x_n, y_n, z_n), F(x_n, y_n, z_n), F(x, y, z)) \\ &\leq k[S(gx_n, gx_n, gx) + S(gy_n, gy_n, gy) + S(gz_n, gz_n, gz)] \end{aligned}$$

Letting $n \rightarrow \infty$ and use the fact that S is continuous on its variables, we get

$$S(x, x, gx) \leq k[S(x, x, gx) + S(y, y, gy) + S(z, z, gz)]$$

Similarly, we may show that

$$S(y, y, gy) \leq k[S(x, x, gx) + S(y, y, gy) + S(z, z, gz)]$$

and

$$S(z, z, gz) \leq k[S(x, x, gx) + S(y, y, gy) + S(z, z, gz)]$$

Thus

$$S(x, x, gx) + S(y, y, gy) + S(z, z, gz) \leq 3k[S(x, x, gx) + S(y, y, gy) + S(z, z, gz)].$$

Since $3k < 1$, the last inequality happens only if $S(x, x, gx) = 0$, $S(y, y, gy) = 0$ and $S(z, z, gz) = 0$.

Hence $x = gx$, $y = gy$ and $z = gz$. Thus we get $gx = F(x, y, z) = x$, $gy = F(y, x, y) = y$ and $gz = F(z, y, x) = z$. Thus F and g have a common fixed point. This completes the proof of the theorem. \square

The following example show that Theorem 23 is more general then Theorem 13.

Example 24. Let $X = [0, 1]$, with the usual partial ordered \leq . Let a S -metric on X be defined by

$$S(x, y, z) = |x - z| + |y - z|.$$

Then (X, S) is a complete regular ordered S -metric space.

Let $g : X \rightarrow X$ and $F : X \times X \times X \rightarrow X$ be defined by

$$(3.24) \quad F(x, y, z) = \frac{1}{9}[\sin x + \sin y + \sin z] \text{ and } g(x) = \frac{x}{3}$$

for all $x, y, z \in X$. Since $|\sin x - \sin y| \leq |x - y|$ and $|\sin z - \sin y| \leq |z - y|$ holds for all $x, y, z \in X$. Then we have $k = \frac{1}{3} \in [0, 1)$. So all the conditions of Theorem 23 and Theorem 26 are satisfied. Then there exists a tripled fixed point of F . In this case $(0, 0, 0)$ is tripled fixed point of F .

Example 25. Let X be the set $[0, \infty)$ and

$$S(x, y, z) = |x - z| + |y - z|.$$

We set $g : X \rightarrow X$ and $F : X \times X \rightarrow X$ be defined as $g(x) = x^2$ and

$$F(x, y) = \begin{cases} \frac{3x^2 - 3y^2 - 3z^2 + 1}{10} & \text{if } x \geq y \geq z \\ 0 & \text{if otherwise} \end{cases}$$

Then for $k = \frac{3}{5} \in [0, 1)$ we have tripled fixed point $(0, 0, 0)$ in X .

Theorem 26. In the above Theorem 23, in the place of condition (ii), if we assume the following conditions in the complete S -metric space X , namely

(3.25) if $\{x_n\}, \{z_n\} \subset X$ is a nondecreasing sequence with $x_n \rightarrow x \in X$ and $z_n \rightarrow z \in X$ then

$$(3.26) \quad z_n \leq z, \quad \forall n,$$

and

(3.27) if $\{y_n\} \subset X$ is a nonincreasing sequence with $y_n \rightarrow y \in X$, then $y_n \geq y, \quad \forall n$.

Then, we have the conclusion of Theorem 23, provided g is non decreasing.

Proof. Proceeding exactly as in Theorem 23, we have $\{gx_n\}, \{gy_n\}$ and $\{gz_n\}$ are S -Cauchy sequences in X . Since (X, S) is a complete, there exists $(x, y, z) \in X \times X \times X$ such that

$$(3.28) \quad \lim_{n \rightarrow \infty} F(x_n, y_n, z_n) = \lim_{n \rightarrow \infty} g(x_n) = x$$

$$(3.29) \quad \lim_{n \rightarrow \infty} F(y_n, x_n, y_n) = \lim_{n \rightarrow \infty} g(y_n) = y$$

and

$$(3.30) \quad \lim_{n \rightarrow \infty} F(z_n, y_n, x_n) = \lim_{n \rightarrow \infty} g(z_n) = z.$$

Therefore, from (iii) we arrive at 3.15, 3.16 3.20 and 3.19. Since $\{gx_n\}$, $\{gz_n\}$ are a non decreasing sequences such that $gx_n \rightarrow x$, $gz_n \rightarrow z$ and as $\{gy_n\}$ is non increasing sequence and $gy_n \rightarrow y$, by assumption 3.25 and 3.27 we have, $g(gx_n) \leq gx$, $g(gy_n) \geq gy$ and $g(gz_n) \leq gz$ for all $n \geq 0$. If $g(gx_n) = gx$, $g(gy_n) = gy$ and $g(gz_n) = gz$ for some n , then by construction $g(gx_{n+1}) = gx$, $g(gy_{n+1}) = gy$ and $g(gz_{n+1}) = gz$ and (x, y, z) is tripled fixed point. So we assume either $g(gx_n) \neq gx$ or $g(gy_n) \neq gy$ or $g(gz_n) \neq gz$. Applying the contractive condition 3.1, we have

$$\begin{aligned} (3.31) \quad & S(F(x, y, z), F(x, y, z), gx) \\ & \leq 3[S(F(x, y, z), F(x, y, z), F(gx_n, gy_n, gz_n)) + S(F(x, y, z), F(x, y, z), F(gx_n, gy_n, gz_n)) \\ & \quad + S(gx, gx, F(gx_n, gy_n, gz_n))] \\ & = 3[S(F(x, y, z), F(x, y, z), F(gx_n, gy_n, gz_n)) + S(F(x, y, z), F(x, y, z), F(gx_n, gy_n, gz_n)) \\ & \quad + S(gF(x_n, y_n, z_n), gF(x_n, y_n, z_n), gx)] \\ & \leq k[2S(x, x, gx_n) + S(g(gx_{n+1}), g(gx_{n+1}), gx)] \end{aligned}$$

Taking $n \rightarrow \infty$ in the above inequality we obtain $S(F(x, y, z), F(x, y, z), gx) = 0$, that is, $F(x, y, z) = g(x)$. Similarly we have that $F(y, x, y) = g(y)$ and $F(z, y, x) = g(z)$. Remaining part of the proof follows from Theorem 23. Hence we have $g(x) = F(x, x, x) = x$. This completes the proof of the theorem. □

Corollary 27. *Let (X, S, \leq) be a partially ordered S - metric space. Let $F : X \times X \times X \rightarrow X$ and $g : X \rightarrow X$ be mappings such that F has the mixed g - monotone property on X and there exists a $k \in [0, 1)$*

$$(3.32) \quad S(F(x, y, z), F(x, y, z), F(u, v, w)) \leq \frac{k}{3}[S(gx, gx, gu) + S(gy, gy, gv) + S(gz, gz, gw)]$$

for all $x, y, z, u, v, w \in X$ for which $gx \geq gu$, $gy \leq gv$ and $gz \geq gw$. If there exists $x_0, y_0, z_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0, z_0), \quad gy_0 \geq F(y_0, x_0, y_0) \quad \text{and} \quad gz_0 \leq F(z_0, y_0, x_0).$$

We assume the following hypotheses,

- (i). $F : (X \times X \times X) \subseteq g(X)$,
- (ii). $g(X)$ is S -complete,
- (iii). g is S -continuous and commutes with F .

Then F and g have a tripled coincidence point. If $gx = gu$, $gy = gv$ and $gz = gw$, then F and g have common fixed point, that is, there exist $x \in X$ such that

$$g(x) = F(x, x, x) = x.$$

Proof. It follows from Theorem 23 if we take $x = u$, $y = v$ and $z = w$. □

Corollary 28. Let (X, S, \leq) be a partially ordered S -metric space. Let $F : X \times X \times X \rightarrow X$ be mapping such that F has the mixed monotone property on X and there exists a $k \in [0, 1)$

$$(3.33) \quad S(F(x, y, z), F(u, v, w), F(a, b, c)) \leq \frac{k}{3}[S(x, u, a) + S(y, v, b) + S(z, w, c)]$$

for all $x, y, z, u, v, w, a, b, c \in X$ for which $x \geq u \geq a$, $y \leq v \leq b$ and $z \geq w \geq c$ where either $u \neq a$ or $v \neq b$ or $w \neq c$. If there exists $x_0, y_0, z_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0, z_0), \quad gy_0 \geq F(y_0, x_0, y_0) \quad \text{and} \quad gz_0 \leq F(z_0, y_0, x_0).$$

Then F has tripled fixed point in X . If $u = a$, $v = b$ and $w = c$, then F has fixed point, that is, there exist $x \in X$ such that

$$x = F(x, x, x).$$

Proof. If we define $g : X \rightarrow X$ be an identity mapping in Theorem 23 then result is follows. □

Corollary 29. Let (X, S, \leq) be a partially ordered S -metric space. Let $F : X \times X \rightarrow X$ be mapping such that F has the mixed monotone property on X and there exists a $k \in [0, 1)$

$$(3.34) \quad S(F(x, y, z), F(x, y, z), F(u, v, w)) \leq \frac{k}{3}[S(gx, gx, gu) + S(gy, gy, gv) + S(gz, gz, gw)]$$

for all $x, y, z, u, v, w \in X$ for which $gx \geq gu$, $gy \leq gv$ and $gz \geq gw$. If there exists $x_0, y_0, z_0 \in X$ such that

$$gx_0 \leq F(x_0, y_0, z_0), \quad gy_0 \geq F(y_0, x_0, y_0) \quad \text{and} \quad gz_0 \leq F(z_0, y_0, x_0).$$

Then F has tripled fixed point in X . If $u = a$, $v = b$ and $w = c$, then F has fixed point, that is, there exist $x \in X$ such that

$$x = F(x, x, x).$$

Proof. If we define $g : X \rightarrow X$ be an identity mapping in Corollary 28 then result is follows. \square

4. CONCLUSION

In this paper one thing is observed that Theorem 13 implies Theorem 23 but converges may not be true. Example 24, 25 and 22 are in support of this fact. This fact is obvious because of the property of $[0, \frac{1}{3}) \subset [0, 1)$ but converges is not true.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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