Available online at http://jfpt.scik.org
J. Fixed Point Theory, 2023, 2023:4
ISSN: 2052-5338

# GENERALIZED WEAK CONTRACTION

JOSEPH FRANK GORDON[1,*], JERRY CARL DEKU[2], ANGELA ATUUMCHANA KUDEWURA[2], HARUNA ABDUL-KARIM[3], KENNEDY GYIMAH[1], THOMAS MENSAH-WONKYI[4]

[1]Department of Mathematics Education, Akenten Appiah-Menka University of Skills Training and Entrepreneurial Development, Kumasi, Ghana

[2]Department of Mathematics, Kwame Nkrumah University of Science and Technology, Kumasi, Ghana

[3]Department of Mathematics and ICT, Gambaga College of Education, Gambaga, Ghana

[4]Department of Mathematics Education,University of Education, Winneba, Ghana



**Abstract.** In this paper, we introduce one of the weakest conditions for a map to be a Picard operator which generalizes Berinde's weak contraction mappings. Under certain conditions, we also resolve a problem posed by Berinde in his paper for quasi-contractions to be weak contractions.

**Keywords:** generalized weak contractions; quasi-contractions; ultrametric space; metric space.

**2020 AMS Subject Classification:** 47H10, 54H25

## 1. Introduction

An important problem in the theory of self-maps $T : \mathscr{X} \to \mathscr{X}$ where $X$ is a set, is the fixed point problem. Now for $x \in \mathscr{X}$ is a fixed point of $T$ if

$$T(x) = x. \tag{1.1}$$

*Corresponding author
E-mail address: jfgordon@aamusted.edu.gh
Received May 06, 2023

For a solution to the fixed point equation to be meaningful, the set $\mathscr{X}$ must allow for continuity and separation of points: both notions are captured in the concept of a *complete metric space*. A metric $d : \mathscr{X} \times \mathscr{X} \to \mathbb{R}_{\geq 0}$ satisfies

(1) Non-negativity: $d(x,y) \geq 0$,

(2) Identity: $d(x,y) = 0$ if and only if $x = y$,

(3) Symmetry: $d(x,y) = d(y,x)$,

(4) Subadditivity: $d(x,y) \leq d(x,z) + d(z,y)$.

The set $\mathscr{X}$ together with the metric $d$ is termed a *metric space*. Convergence of a countable sequence of elements $\{x_n\}_{n\geq 1} = \{x_1, x_2, \ldots\}$ in $\mathscr{X}$ is then memorialized by saying that the sequence is *Cauchy*, to wit, for every real number $\varepsilon > 0$, there exists a sufficiently large natural number $N := N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m, n \geq N$. The element to which the sequence converges may or may not be an element of $\mathscr{X}$. A metric space containing the converging points of all its Cauchy sequence is termed *complete*. The set of real numbers is a typical example. Once a sequence $\{x_n\}_{n\geq 1}$ converges in a complete metric space to an element $x$ say, then the sequence of non-negative real numbers $\{d(x_n, x)\}_{n\geq 1}$ is a null sequence, that is, it converges to zero. A mapping $T$, termed a *self-map*, on a complete metric space $(\mathscr{X}, d)$ is then *continuous* at a point $x \in \mathscr{X}$ if and only if for every Cauchy sequence $\{d(Tx_n, x)\}_{n\geq 1}$ of non-negative real numbers is a null space. Thus the problem of finding a sequence of approximate solutions to equation (1.1) in a complete metric space boils down largely to finding a Cauchy sequence of approximate solutions where at least continuity at the solution point is assumed or guaranteed.

In practice, continuity at a point is a difficult notion to work with in general; one thus resorts to mappings satisfying a more rigorous form of continuity, that of *uniform continuity*: a self-map on a complete metric space $(\mathscr{X}, d)$ is uniformly continuous on the space if for every real number $\varepsilon > 0$, there exists real number $\delta > 0$ such that whenever $d(x,y) < \delta$ then $d(Tx, Ty) < \varepsilon$ for all $x, y \in \mathscr{X}$. Now given a complete metric space, the most well-studied examples of such mappings are those which are given by the metric inequality

$$d(Ty, Tx) \leq c \cdot d(y, x), \tag{1.2}$$

where $c > 0$ is a fixed real number usually referred to as the Lipschitz constant of $T$. Such mappings are called *Lipschitz continuous mappings* (or Lipschitz maps, for short). The most important property of Lipschitz maps is that they are *uniformly continuous* and *maps bounded sets to bounded sets*. Now the main problem in light of equation (1.1) and (1.2), is that

**Problem 1.1.** Given a Lipschtiz continuous mapping $T$ on a complete metric space $(\mathscr{X}, d)$, does there exists a fixed point of $T$? Moreover, if there is a fixed point, is it unique and can a sequence of approximate solutions be constructed?

To resolve Problem (1.1), Lipschitz mappings are generally classified into three categories: $T$ is a

- *contraction* mapping if $0 < c < 1$,
- non-expansive mapping if $c = 1$ and ,
- expansive mapping if $c > 1$.

Let $\mathscr{X}$ be any set and $T : \mathscr{X} \to \mathscr{X}$ a self-map. For any given $x \in \mathscr{X}$, we define $T^n(x)$ inductively by $T^0(x) = x$ and $T^{n+1}(x) = T(T^n(x))$; we call $T^n(x)$ the $n^{th}$ *iterate of $x$ under $T$*. The mapping $T^n (n \geq 1)$ is called the $n^{th}$ iterate of $T$. For any $x_0 \in \mathscr{X}$, the sequence $\{x_n\}_{n\geq 0} \subset \mathscr{X}$ given by

$$x_n = Tx_{n-1} = T^n x_0, \quad n = 1, 2, \ldots \tag{1.3}$$

is called the sequence of *successive approximations with the initial value* $x_0$. It is also called the *Picard iteration* starting at $x_0$. A mapping for which every Picard iteration converges to a fixed point is called a *Picard operator*.

The most basic result where existence and uniqueness are guaranteed is the Banach contraction mapping theorem [1], which is one of the most useful results in metric fixed point theory of Lipschitz maps, that is,

**Theorem 1.1** (Banach contraction mapping theorem)**.** Let $(\mathscr{X}, d)$ be a *complete metric space* and $T : \mathscr{X} \to \mathscr{X}$ be a contraction mapping. Then $T$ has a unique fixed point which can be obtained by the Picard iteration.

The Banach contraction mapping theorem has been applied in the determination of the existence and uniqueness of many results in analysis and economies (see for instance [2, 3]). Now let us turn to mappings that are technically non-expansive mappings but which admit, under a compactness hypothesis, a fixed point theorem as that of the classical Banach's contraction principle. Define a contractive mapping to be a non-expansive mapping such that inequality (1.2) (thus with $c = 1$) holds with equality if and only if $x = y$, that is $d(Ty,Tx) < d(y,x)$ if $x \neq y$; then the following result due to Edelstein [4] holds:

**Theorem 1.2** (Edelstein)**.** Let $T$ be a contractive mapping on a compact metric space. Then $T$ has a unique fixed point; moreover, the fixed point can be iteratively approximated by the Picard iteration $x_{n+1} = Tx_n$.

Due to the importance of these two results, many authors have sought to find contraction-like conditions that can guarantee existence and uniqueness of fixed points (see for instance [5]). One can easily see that the contractive condition in inequality (1.2) forces $T$ to be continuous on $\mathscr{X}$ and as a result it becomes natural to ask if there exists contractive conditions which do not imply the continuity of $T$. Several authors sought possibilities where the mapping $T$ would not be necessarily continuous and R. Kannan [6] was the first author who extended Theorem 1.1 to mappings that need not to be continuous by proving the following results:

**Theorem 1.3.** Let $(X,d)$ be a complete metric space and let $T : \mathscr{X} \to \mathscr{X}$ be a mapping such that there exists $b \in (0, \frac{1}{2})$ satisfying

$$d(Tx,Ty) \leq b \cdot [d(x,Tx) + d(y,Ty)] \text{ for all } x,y \in \mathscr{X}. \tag{1.4}$$

Then, $T$ has a unique fixed point $p \in \mathscr{X}$, and for any $x \in \mathscr{X}$, the sequence of iterates $\{T^n x\}$ converges to $p$ and $d(T^{n+1}x,p) \leq b \cdot \left(\frac{b}{1-b}\right)^n \cdot d(x,Tx)$, $n = 0,1,2,\ldots$

It is interesting that Kannan's theorem is independent of the Banach contraction principle (see for instance Rhoades [7]). Also, Kannan's fixed point theorem is very important because Subrahmanyam [8] proved that Kannan's theorem characterizes the metric completeness. That is, a metric space $\mathscr{X}$ is complete if and only if every Kannan mapping on $\mathscr{X}$ has a fixed point. Kannan's fixed point theorem and some of its generalizations for various classes of contractive

type conditions that do not require the continuity of $T$ are discussed in [9, 10, 11, 12, 13]. In particular, we have the following results due to Chatterjea [14].

**Theorem 1.4.** Let $(\mathscr{X},d)$ be a complete metric space and let $T:\mathscr{X}\to\mathscr{X}$ be a mapping such that there exists $c\in(0,\frac{1}{2})$ satisfying

$$d(Tx,Ty)\leq c\cdot[d(x,Ty)+d(y,Tx)],\ \forall x,y\in\mathscr{X}, \tag{1.5}$$

then $T$ has a unique fixed point.

Zamfirescu [15] obtained a very interesting fixed point theorem on complete metric spaces by combining Theorem 1.1, Theorem 1.3 and Theorem 1.4 by proving the following results.

**Theorem 1.5.** Let $(\mathscr{X},d)$ be a complete metric space and $T:\mathscr{X}\to\mathscr{X}$ a map for which there exist real numbers $\alpha,\beta$ and $\gamma$ satisfying $0\leq\alpha<1, 0\leq\beta<0.5$ and $0\leq\gamma<0.5$, such that, for each $x,y\in\mathscr{X}$, at least one of the following is true:

(1) $d(Tx,Ty)\leq\alpha\cdot d(x,y)$,

(2) $d(Tx,Ty)\leq\beta\cdot[d(x,Tx)+d(y,Ty)]$,

(3) $d(Tx,Ty)\leq\gamma\cdot[d(x,Ty)+d(y,Tx)]$.

Then $T$ is a Picard operator.

In a more sweeping manner, Ciric [16] introduced a generalization of Zamfirescu mappings called *quasi-contractions* which are defined below.

**Definition 1.1.** Let $(\mathscr{X},d)$ be a metric space. A map $T:\mathscr{X}\to\mathscr{X}$ is called a *quasi-contraction* if there exists a constant $h\in(0,1)$ such that

$$d(Tx,Ty)\leq h\cdot\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}\text{ for all }x,y\in\mathscr{X}. \tag{1.6}$$

Ciric also proved that quasi-contractions are Picard operators. It is obvious that Theorem 1.3, Theorem 1.4 and each of the conditions of Theorem 1.5 are quasi-contractions.

In 2004, Berinde [17] obtained a fixed point results based on a contraction condition more general than each of the conditions in Theorem 1.5 and that does not require the continuity

of the map as well. Berinde introduced the concept of *weak contractions* which unify large classes of contractive type operators and whose fixed points can be obtained by means of Picard iteration and for which both a prior and a posteriori error estimates can be computed. He defined the following mapping.

**Definition 1.2.** Let $(\mathscr{X},d)$ be a metric space. A map $T:\mathscr{X}\to\mathscr{X}$ is called *weak contraction* if there exists a constant $\delta\in(0,1)$ and for some $L\geq 0$ such that

$$d(Tx,Ty)\leq\delta\cdot d(x,y)+L\cdot d(y,Tx),\ \text{ for all } x,y\in\mathscr{X}. \tag{1.7}$$

Now due to the symmetry of the metric function, condition (1.7) in Definition 1.2 implicitly includes the following dual one

$$d(Tx,Ty)\leq\delta\cdot d(x,y)+L\cdot d(x,Ty),\ \text{ for all } x,y\in\mathscr{X}, \tag{1.8}$$

obtained from inequality (1.7) by replacing $d(Tx,Ty)$ and $d(x,y)$ by $d(Ty,Tx)$ and $d(y,x)$, respectively, and then interchanging $x$ and $y$. In order to check the weak contractiveness of the map $T$, it is necessary to check both inequality (1.7) and (1.8). The following results are examples of weak contraction.

**Proposition 1.1.** Any map $T$ satisfying the contractive condition in Theorem 1.3 is a weak contraction.

*Proof.* See Berinde [17]. □

**Proposition 1.2.** Any map $T$ satisfying the contractive condition in Theorem 1.4 is a weak contraction

*Proof.* See Berinde [17]. □

**Corollary 1.1.** Any Zamfirescu map, thus any map that satisfies the three conditions in Theorem 1.5 is a weak contraction.

*Proof.* See Berinde [17]. □

Berinde, however, was unable to establish that all quasi-contractions are weak contractions. He rather proved the following weaker result.

**Proposition 1.3.** Any quasi-contraction with $0 < h < \frac{1}{2}$ is a weak contraction.

In other words, Berinde wasn't able to establish if quasi-contractions with $h \in [\frac{1}{2}, 1)$ are also weak contractions and as a result, he left an open problem in his paper which is stated as follows.

**Problem 1.2.** Is any quasi-contraction a weak contraction?

The results of this paper are in two folds; first, we resolve Problem 1.2 of Berinde on certain quasi-contractions that are fully weak contractions. That is, we show that all quasi-contractions for any $h \in (0,1)$ and not just $h \in (0, \frac{1}{2})$ are weak contractions by considering a special class of metric spaces called *ultrametric spaces* which are defined as follows.

**Definition 1.3.** *Ultrametric space* is a metric space $(\mathscr{X}, d)$ in which the subadditivity of the metric $d$ satisfies the following stronger condition called the ultrametric inequality

$$d(x,z) \leq \max\{d(x,y), d(y,z)\} \text{ for all } x,y,z \in \mathscr{X}. \tag{1.9}$$

Berinde extended Theorem 1.1, Theorem 1.5 and other related results by proving the following theorem.

**Theorem 1.6.** Let $(\mathscr{X}, d)$ be a complete metric space and $T : \mathscr{X} \to \mathscr{X}$ a weak contraction, that is, a map satisfying inequality (1.7) with $\delta \in (0,1)$ and some $L \geq 0$. Then

(1) $\texttt{Fix}(T) = \{x \in \mathscr{X} : Tx = x\} \neq \emptyset$,

(2) For any $x_0 \in \mathscr{X}$, the Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by equation (1.3) converges to some $x^* \in \texttt{Fix}(T)$,

(3) The following estimates

$$d(x_n, x^*) \leq \frac{\delta^n}{1-\delta} d(x_0, x_1), \;\; n = 0, 1, 2, \ldots$$
$$d(x_n, x^*) \leq \frac{\delta}{1-\delta} d(x_{n-1}, x_n), \;\; n = 1, 2, \ldots$$

hold, where $\delta$ is the constant appearing in inequality (1.7).

*Proof.* See Berinde [17] □

Now note that, although the fixed point results in Theorem 1.1, Theorem 1.5 and other related results actually forces the uniqueness of the fixed points, the weak contractions need not to have a unique fixed point, see for instance Berinde [17, Example 1]. Berinde, however, made it possible to force the uniqueness of the fixed point of a weak contraction by imposing additional contractive condition, quite similar to inequality (1.7), by proving the following result.

**Theorem 1.7.** Let $(\mathscr{X},d)$ be a complete metric space and $T:\mathscr{X}\to\mathscr{X}$ a weak contraction for which there exists $\theta\in(0,1)$ and some $L_1\geq 0$ such that

$$d(Tx,Ty)\leq\theta\cdot d(x,y)+L_1\cdot d(x,Tx),\ \text{ for all } x,y\in\mathscr{X}. \tag{1.10}$$

Then

(1) $T$ has a unique fixed point, that is, $\texttt{Fix}(T)=\{x^*\}$,

(2) The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by equation (1.3) converges to $x^*$, for any $x_0\in\mathscr{X}$,

(3) The prior and posterior error estimates

$$d(x_n,x^*)\leq\frac{\delta^n}{1-\delta}d(x_0,x_1),\ \ n=0,1,2,\ldots$$
$$d(x_n,x^*)\leq\frac{\delta}{1-\delta}d(x_{n-1},x_n),\ \ n=1,2,\ldots$$

hold,

(4) The rate of convergence of the Picard iteration is given by

$$d(x_n,x^*)\leq\theta d(x_{n-1},x^*),\ \ n=1,2,\ldots$$

*Proof.* See Berinde [17] □

The second aspect of this paper is to introduce the concept of *generalized weak contractions* which extend Berinde's weak contractions and whose fixed point can also be obtained by Picard iteration and for which both the prior and posterior error estimates can be obtained just as in Berinde's case. The new mapping is defined as follows.

**Definition 1.4.** Let $(\mathscr{X},d)$ be a metric space. A map $\mathscr{T}:\mathscr{X}\to\mathscr{X}$ is called a *generalized weak contraction* if there exists a constant $c\in(0,1)$ such that

$$d(Tx,Ty)\leq c\cdot d(x,y)+f(d(Tx,y)), \tag{1.11}$$

where $f$ is any function $f:\mathbb{R}\to\mathbb{R}$ that *vanishes* at 0 and is *right upper semi-continuous* at the origin. That is,

$$\limsup_{t\to 0^+} f(t)\leq f(0)=0. \tag{1.12}$$

## 2. Main Results

We give the proof of the main result of this paper, which is accomplished in Proposition 2.1 and Theorem 2.1

**Proposition 2.1.** Any quasi-contraction with $0<h<1$ is a weak contraction when the metric space is an ultrametric space.

*Proof.* The proof follows similarly as in Berinde's case except for some few modifications where the ultrametric inequality is applied. Let $(\mathscr{X},d)$ be a metric space and $T:\mathscr{X}\to\mathscr{X}$ be a quasi-contraction, thus, a map for which there exists $0<h<1$ such that

$$d(Tx,Ty)\leq h\cdot M(x,y),\quad \forall x,y\in\mathscr{X}, \tag{2.1}$$

where

$$M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\}. \tag{2.2}$$

Just as in the proof of Berinde's case, there are five possible cases to consider.

**Case I:** $M(x,y)=d(x,y)$, when, in view of inequality (2.1), condition (1.7) and (1.8) are clearly satisfied (with $\delta=h$ and $L=0$).

**Case II:** $M(x,y)=d(x,Tx)$, when by inequality (2.1) and the triangle inequality, we have

$$d(Tx,Ty)\leq h\,d(x,Tx)\leq h\,[d(x,y)+d(y,Tx)], \tag{2.3}$$

and so condition (1.7) holds with $\delta = h$ and $L = h$. Now since $d(x,Tx) \leq d(x,Ty) + d(Ty,Tx)$, we have

$$d(Tx,Ty) \leq h\, d(x,Tx) \leq h\, [d(x,Ty) + d(Ty,Tx)], \tag{2.4}$$

from which it follows that

$$d(Tx,Ty) - h\, d(Tx,Ty) \leq h\, d(x,Ty) \Rightarrow (1-h)\, d(Tx,Ty) \leq h\, d(x,Ty), \tag{2.5}$$

and hence one obtains

$$d(Tx,Ty) \leq \frac{h}{1-h} d(x,Ty) \leq \delta\, d(x,y) + \frac{h}{1-h}\, d(x,Ty), \tag{2.6}$$

for all $\delta \in (0,1)$ and $L = \dfrac{h}{1-h} > 0$ and so condition (1.8) also holds.

**Case III:** $M(x,y) = d(y,Ty)$ when condition (1.7) and (1.8) are obviously true following Case II, in virtue of the symmetry of $M(x,y)$.

**Case IV:** $M(x,y) = d(x,Ty)$, one obtains the following

$$d(Tx,Ty) \leq h\, d(x,Ty) \leq \delta\, d(x,y) + h\, d(x,Ty), \tag{2.7}$$

for all $\delta \in (0,1)$ and $L = h$ and so condition (1.8) is true. Now condition (1.7) is obtained only if $h < \frac{1}{2}$. This was the result proved by Berinde in Proposition 1.3. Now in this paper, we prove for the case where $h \in (0,1)$ when the underlying space is an ultrametric space. Indeed, by inequality (2.1), $d(Tx,Ty) \leq h \cdot d(x,Ty)$ and by applying the ultrametric inequality twice to $d(x,Ty)$, the right-hand side can be bounded as follows:

$$d(x,Ty) \leq \max\{d(x,y), d(y,Tx), d(Tx,Ty)\}, \tag{2.8}$$

from which it is clear that if the maximum is attained by the first two (that is either $d(x,y)$ or $d(y,Tx)$), then $T$ is a weak contraction in the sense that one obtains

$$d(Tx,Ty) \leq h \cdot d(x,Ty) \leq \delta\, d(x,y) + L\, d(y,Tx), \tag{2.9}$$

which is condition (1.7) with $\delta = h$ and $L > 0$ if $d(x,Ty) \leq d(x,y)$ or $\delta \in (0,1)$ and $L = h$ if $d(x,Ty) \leq d(y,Tx)$. Finally assume the maximum occurs at $d(Tx,Ty)$, it follows that

$$d(Tx,Ty) \leq h\, d(x,Ty) \leq h\, d(Tx,Ty), \tag{2.10}$$

which cannot occur because it leads to the contradiction that $h \geq 1$ if $x \neq y$. But since the first two possible maximums do not restrict $h$, it follows that $T$ is a weak contraction for any $h \in (0,1)$.

**Case V:** $M(x,y) = d(y,Tx)$, reduces to Case IV and that completes the proof. □

**Theorem 2.1.** Let $(\mathscr{X},d)$ be a complete metric space and $T : \mathscr{X} \to \mathscr{X}$ be a generalized weak contraction, that is a map satisfying equation (1.11) with $c \in (0,1)$ and for any function $f : \mathbb{R} \to \mathbb{R}$ satisfying equation (1.12). Then

(1) $F(T) = \{x \in \mathscr{X} : Tx = x\} \neq 0$.

(2) For any $x_0 \in \mathscr{X}$, the picard iteration $\{x_n\}_{n\geq 1}$ given by

$$x_n = Tx_{n-1} = T^n x_0, \tag{2.11}$$

converges to some $x^* \in F(T)$.

*Proof.* We want to show that $T$ has at least a fixed point in $\mathscr{X}$. To prove existence of fixed point, we show that for any $x_0 \in \mathscr{X}$, the picard iteration $\{x_n\}_{n\geq 1}$ defined in equation (2.11) is a Cauchy sequence. Let $x_0 \in \mathscr{X}$. Take $x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, x_3 = Tx_2 = T^3x_0, \ldots, x_n = T^n x_0$. Then,

$$d(x_1,x_2) = d(Tx_0,Tx_1) \leq cd(x_0,x_1),$$

$$d(x_2,x_3) = d(Tx_1,Tx_2) \leq cd(x_1,x_2) \leq c^2 d(x_0,x_1).$$

In general, for any positive integer $n$, $d(x_n,x_{n+1}) \leq c^n d(x_0,x_1)$. Now for any positive integer $p$, we have

$$\begin{aligned} d(x_n,x_{n+p}) &\leq d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + \cdots + d(x_{n+p-1},x_{n+p}), \\ &\leq c^n d(x_0,x_1) + c^{n+1} d(x_0,x_1) + \cdots + c^{n+p-1} d(x_0,x_1), \\ &= (c^n + c^{n+1} + \cdots + c^{n+p-1}) d(x_0,x_1), \end{aligned}$$

$$
\begin{aligned}
&= \frac{c^n - c^{n+p}}{1-c} d(x_0, x_1), \\
&= \frac{c^n(1-c^p)}{1-c} d(x_0, x_1), \\
&< \frac{c^n}{1-c} d(x_0, x_1), \\
&\to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

So $d(x_n, x_{n+p}) \to 0$ as $n \to \infty$ for $p = 1, 2, \ldots$. Hence $\{x_n\}$ is a Cauchy sequence and as a result $x_n \to x^* \in \mathscr{X}$ since $\mathscr{X}$ is complete. That is $x^* = \lim_{n\to\infty} T^n x_0 \in \mathscr{X}$. Then,

$$
\begin{aligned}
d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*), \\
&= d(x^*, x_{n+1}) + d(Tx_n, Tx^*), \\
&\leq \limsup_{n\to\infty} d(x^*, x_{n+1}) + \limsup_{n\to\infty} c \cdot d(x_n, x^*) + \limsup_{n\to\infty} f(d(Tx_n, x^*)), \\
&\leq d(x^*, x^*) + c \cdot d(x^*, x^*) + f(0), \\
&= 0.
\end{aligned}
$$

Hence $d(x^*, Tx^*) = 0$ and that implies that $x^*$ is a fixed point of $T$. □

**Remark 2.1.**

(1) Clearly when $f$ is the zero map, then Theorem 2.1 reduces to Theorem 1.1 and when $f$ is a non-negative scalar function, Theorem 2.1 reduces to Theorem 1.6.

(2) Just like in Berinde's weak contraction, generalized weak contractions need also not to have a unique fixed point. However, it is possible to force the uniqueness of the fixed point of a generalized weak contraction by imposing additional contractive condition, quite similar to condition (1.11) as shown by the following theorem.

**Theorem 2.2.** Let $(\mathscr{X}, d)$ be a complete metric space and $T : \mathscr{X} \to \mathscr{X}$ a generalized weak contraction for which there exists $\lambda \in (0,1)$ and for any function $f : \mathbb{R} \to \mathbb{R}$ satisfying equation (1.12) such that

$$
d(Tx, Ty) \leq \lambda\, d(x, y) + f(d(Tx, x)), \quad \text{for all } x, y \in \mathscr{X}, \tag{2.12}
$$

Then

(1) $T$ has a unique fixed point, that is, $\texttt{Fix}(T) = \{p\}$,

(2) The Picard iteration $\{x_n\}_{n=0}^{\infty}$ given by equation (1.3) converges to $p$, for any $x_0 \in \mathscr{X}$.

*Proof.* To see this, suppose $x^*, y^* \in F(T), x^* \neq y^*$, then $Tx^* = x^*$ and $Ty^* = y^*$ and that implies that

$$\begin{aligned} d(x^*, y^*) = d(Tx^*, Ty^*) &\leq \lambda \cdot d(x^*, y^*) + f(d(Tx^*, x^*)), \\ &= \lambda \cdot d(x^*, y^*) + f(d(x^*, x^*)), \\ &= \lambda \cdot d(x^*, y^*) + f(0), \\ &= \lambda \cdot d(x^*, y^*). \end{aligned}$$

Since $x^* \neq y^*$, then $d(x^*, y^*) > 0$ and that implies $\lambda \geq 1$ which contradict the fact $\lambda \in (0,1)$. Hence $x^* = y^*$. The rest of the proof follows by Theorem 2.1. □

We support the novelty of this paper with the following illustrative example.

**Example 2.1.** Let $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ be the ceiling and floor functions respectively. It is well known, and in any case easy to prove that $\lceil \cdot \rceil$ is lower semicontinuous and that $\lfloor \cdot \rfloor$ is upper semicontinuous. Define the set

$$\mathscr{X} := \{x \in \mathbb{R} \backslash \mathbb{Z} : \lfloor x \rfloor \in 2\mathbb{Z}\}. \tag{2.13}$$

Here, $2\mathbb{Z}$ denotes the even numbers. In the definition of $\mathscr{X}$,thus equation (2.13), one may alternatively consider odd numbers instead of even numbers. Now define the map

$$T : \mathscr{X} \to \mathscr{X}, \ x \mapsto \lceil x \rceil, \tag{2.14}$$

then the following result holds.

**Theorem 2.3.** For any $\varepsilon \in (0,1)$, the following holds:

$$|Tx - Ty| \leq (1-\varepsilon)|x-y| + (1+\varepsilon)\lfloor |Tx - y| \rfloor; \tag{2.15}$$

i.e. $T$ is a *generalised weak contraction* under the usual absolute value metric $d(x,y) := |x-y|$, contraction constant $c = 1-\varepsilon$, and upper semicontinuous function $f(t) := (1+\varepsilon)\lfloor t \rfloor$.

*Proof.* Explicitly, we wish to show that

$$|\lceil x\rceil - \lceil y\rceil| \leq (1-\varepsilon)|x-y| + (1+\varepsilon)\lfloor |\lceil x\rceil - y|\rfloor \,. \tag{2.16}$$

To see this, let $\lceil x\rceil = m$ and $\lceil y\rceil = n$ for some integers $m,n$ and write

$$x = m - \{x\},\ \ y = n - \{y\} \tag{2.17}$$

where $0 \leq \{x\},\{y\} < 1$. Without loss of generality, we assume that $m \neq n$ (this is the case when the left hand side of equation (2.16) is just zero). It follows from the definition of $\mathscr{X}$ that $|m-n| \geq 2$. The left hand side of equation (2.16) evaluates to

$$|\lceil x\rceil - \lceil y\rceil| = |m-n| \,. \tag{2.18}$$

For the right hand side of equation (2.16), first we observe from the triangle inequality that the following inequalities hold

$$\begin{aligned} |x-y| &\geq |m-n| - |\{x\} - \{y\}| \\ &> |m-n| - 1 \,. \end{aligned} \tag{2.19}$$

$$\begin{aligned} \lfloor |\lceil x\rceil - y|\rfloor &= \lfloor |m-n+\{y\}|\rfloor \\ &\geq \lfloor |m-n| - |\{y\}|\rfloor \\ &\geq |m-n| - 1 \,. \end{aligned} \tag{2.20}$$

From equation (2.19) and (2.20), we obtain that

$$\begin{aligned} (1-\varepsilon)|x-y| + (1+\varepsilon)\lfloor |\lceil x\rceil - y|\rfloor &\geq (1-\varepsilon)(|m-n|-1) + (1+\varepsilon)(|m-n|-1) \\ &= 2(|m-n|-1) \\ &\geq |m-n| \,, \end{aligned} \tag{2.21}$$

and that completes the proof. Note that equation (2.21) holds because $|m-n| \geq 2$. □

## 3. Conclusion

The topic of this paper is precisely the extension of results from [17]. The novelty of the paper also lies in the generality of the mappings comprising our generalized weak contractions in Definition 1.4. For instance the example provided in Example 2.1 is not covered by Berinde's weak contraction mappings [17] and, to the best of our knowledge, is not covered either under any explicitly defined mappings that are also Picard operators. Indeed, the minimality of the hypothesis on the function $f$ makes our generalized weak contraction mappings into an all-encompassing class of mappings that are Picard operators. We also resolved a problem posed by Berinde by considering a special class of spaces called the ultrametric spaces.

## Conflict of Interests

The authors declare that there is no conflict of interests.

## References

[1] S. Banach, H. Steinhaus, Sur le principe de la condensation de singularités, Fund. Math. 1 (1927), 50–61.

[2] K.C. Border, Fixed point theorems with applications to economics and game theory, Cambridge University Press, 1985.

[3] M. Freeman, The inverse as a fixed point in function space, Amer. Math. Mon. 83 (1976), 344–348.

[4] M. Edelstein, On fixed and periodic points under contractive mappings, J. Lond. Math. Soc. 1 (1962), 74–79.

[5] J.F. Gordon, The monotone contraction mapping theorem, J. Math. 2020 (2020), 2879283.

[6] R. Kannan, Some results on fixed points, Bull. Cal. Math. Soc. 60 (1968), 71–76.

[7] B.E. Rhoades. A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. 226 (1977), 257–290.

[8] P.V. Subrahmanyam, Completeness and fixed-points, Monatsh. Math. 80 (1975), 325–330.

[9] G.E. Hardy, T.D. Rogers, A generalization of a fixed point theorem of Reich, Canadian Math. Bull. 16 (1973), 201–206.

[10] S. Reich, Some remarks concerning contraction mappings, Canadian Math. Bull. 14 (1971), 121–124.

[11] S. Reich, Fixed points of contractive functions, Boll. Unione Mat. Ital. 5 (1972), 26–42.

[12] I.A. Rus, Principles and applications of the fixed point theory, Editura Dacia, Cluj-Napoca, 1979.

[13] I.A. Rus, Generalized contractions and applications, Cluj University Press, 2001.

[14] S. Chatterjea, Fixed-point theorems, Dokl. Bolg. Akad. Nauk. 25 (1972), 727-735.

[15] T. Zamfirescu, Fix point theorems in metric spaces, Arch. Math. 23 (1972), 292-298.

[16] Lj. B. Ćirić, A generalization of banach's contraction principle, Proc. Amer. Math. Soc. 45 (1974), 267-273.

[17] V. Berinde, Approximating fixed points of weak contractions using the Picard iteration, Nonlinear Anal. Forum, 9 (2004), 43-54.