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REGULARITY THEOREM FOR QUASILINEAR PARABOLIC SYSTEMS WITH SINGULAR COEFFICIENTS IN THE SPECIFIC DIVERGENT FORM

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Abstract. We consider the quasilinear parabolic system with the measurable coefficients in the specific divergent form $\partial_t \vec{u} = \nabla_i \left(a_{ij} \left(x,\ t,\ \vec{u} \right) \nabla_j \vec{u} \right) + \vec{b} \left(x,\ t,\ \vec{u},\ \nabla \vec{u} \right)$, where the function $\vec{u}$ is unknown N-dimensional vector-function defined on $\Omega \times [0,\ T]$, $\Omega \subset R^l$, $\quad l > 2$. Under minimal restrictions on the structural coefficients, we obtain a priori estimations of the weak solutions of the system and establish the solvability of the system in the Holder functional classes. To prove the existence of the solution we apply the Leray-Schauder fixed point method, for establishing uniqueness we employ the argument of contraction, which follows from Lipschitz conditions.

Keywords: quasilinear partial differential equation; Holder solution; regularity theory; form-bounded; heat kernel; weak solution; a priori estimation.

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1. Introduction

The subject of this article is the regularity of solutions to the boundary problems for the quasilinear parabolic systems of partial differential equations. The parabolic linear and quasi-linear equations and systems have been studied for several decades, in the linear case the fundamental results were obtained by E. DeGiorgi, J. Nash, and J. Moser, whose works provided a possibility to establish a sufficient quantity of a priori estimations for parabolic equations with rough

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coefficients to establish solvability in certain functional class. In the linear case, DeGiorgi-Nash-Moser results [11, 12] were developed by A.O. Ladyzenskaja, V.A. Solonnikov, Q. S. Zhang, J. A. Goldstein, and others [5, 6, 8, 37 - 40], in the quasilinear case, the situation is more complex, some results were obtained by A.O. Ladyzenskaja, V.A. Solonnikov, and N. N. Uralceva and viscosity solutions are studied by K. Ishii, M. Pierre, and T. Suzuki. For further studies see the list of references [1 - 40].

We consider a quasilinear parabolic system in the specific divergent form

$$\frac{\partial}{\partial t}\vec{u} = \sum_{i,j=1,\dots,l} \nabla_i \left(a_{ij}\left(x,\, t,\, \vec{u}\right)\nabla_j \vec{u}\right) + \vec{b}\left(x,\, t,\, \vec{u},\, \nabla\vec{u}\right), \tag{1}$$

where $(x,\, t) \in D_T = \Omega \times (0,\, T)$, vector-function $\vec{u}(x,\, t) = \left(u^1(x,\, t),\, \dots,\, u^N(x,\, t)\right)$ is an unknown N-dimensional vector in $clos\,(D_T)$, $l \geq 3$; $\vec{b}\ :\ \Omega \times [0,\, T] \times R^N \times R^l \times R^N \to R^N$ is a known vector-function. Functions a_{ij} comprise a symmetric $l \times l$-matrix uniformly elliptic, namely,

$$\nu\left(\vec{u}\right)\xi^2 \leq a_{ij}\xi_i\xi_j \leq \mu\left(\vec{u}\right)\xi^2 \tag{2}$$

for all $(x,\, t) \in R^l \times [0,\, T]$ and all $\xi \in R^l$.

Quasilinear parabolic systems of a more general type

$$\frac{\partial}{\partial t}\vec{u} = \sum_{i,j=1,\dots,l} \tilde{a}_{ij}\left(x,\, t,\, \vec{u}\right)\nabla_i\nabla_j\vec{u} + \vec{\tilde{b}}\left(x,\, t,\, \vec{u},\, \nabla\vec{u}\right) \tag{3}$$

can be presented in the divergent form (1) by taking

$$\begin{aligned} &b^k(x,\, t) = \tilde{b}^k\left(x,\, t,\, \vec{u}(x,\, t),\, \nabla\vec{u}(x,\, t)\right) + \\ &+ \tfrac{\partial \tilde{a}_{ij}(x,\, t,\, \vec{u}(x,\, t))}{\partial u^m}\nabla_i u^m \nabla_j u^k + \tfrac{\partial \tilde{a}_{ij}(x,\, t,\, \vec{u}(x,\, t))}{\partial x_i}\nabla_j u^k, \end{aligned} \tag{4}$$

where functions $\tilde{a}_{ij}$ and $\vec{\tilde{b}}$ are smooth enough, since we can calculate the full derivatives

$$\frac{da_{ij}}{dx_d} = \frac{\partial \tilde{a}_{ij}}{\partial u^m}\nabla_d u^m + \frac{\partial \tilde{a}_{ij}}{\partial x_d}. \tag{5}$$

The first boundary problem for system (1) is formulated as follows: to find the function $\vec{u}(x,\, t)$ that satisfies system (1) and the boundary condition $\vec{u}|_{\partial\Omega_T = \{(x,\, t)\,:\, x\in\Omega,\, t\in[0,\, T]\}} = 0$ and $\vec{u}|_{t=0} = \vec{u}(0,\, t) = \vec{\phi}(x)$ where $\vec{\phi}$ is given function. We have to establish the dependence properties of solutions on the properties of structural coefficients of the system (1), namely, we have to specify classes to which must belong a_{ij} and $\vec{b}$. Now, we formalize more general conditions on the structural coefficients of the system (1)

$$a_{ij}(x,\,t,\,\vec{u})\vec{k}_i\vec{k}_j \geq \nu\left(|\vec{u}|\right)\left|\vec{k}\right|^2 - \gamma_0(x,\,t) \tag{6}$$

$$\left|a_{ij}(x,\,t,\,\vec{u})\vec{k}_j\right| \leq \mu\left(|\vec{u}|\right)\left|\vec{k}\right| + \gamma_1(x,\,t) \tag{7}$$

$$\left|\vec{b}\left(x,\,t,\,\vec{u},\,\vec{k}\right)\right| \leq \tilde{\mu}\left(|\vec{u}|\right)\left|\vec{k}\right|^2 + \gamma_2(x,\,t) \tag{8}$$

where functions $\nu(\tau)$, $\mu(\tau)$ and $\tilde{\mu}(\tau)$ are positive, continuous, and restrict the growth of structural coefficients of the system (1); function $\nu(\tau)$ monotone decreases; functions $\mu(\tau)$ and $\tilde{\mu}(\tau)$ monotone increase; functions γ_i control the singularities of the coefficients, $\gamma_0,\ {\gamma_1}^2,\ \gamma_2 \in PK(\beta)$.

A function $\vec{f}\,:\,D_T \to R^N$ is said to be form-boundary or belongs to the class $PK(\beta)$ if the following condition

$$\int_{[0,\,T]}\int_\Omega \left|\vec{f}\vec{\varphi}\right|^2 dxdt \leq \beta\int_{[0,\,T]}\int_\Omega |\nabla\vec{\varphi}|^2\,dxdt + c(\beta)\int_{[0,\,T]}\int_\Omega |\vec{\varphi}|^2\,dxdt \tag{9}$$

holds with some positive constants β and $c(\beta)$, and all functions $\vec{\varphi}\,:\,R^l \times [0,\,T] \to R^N$, $\vec{\varphi} \in C_0^\infty$.

Definition. *A real-valued vector-function $\vec{u}(x,\,t)$ is called a weak solution to system* (1) *if $\vec{u} \in V_{1,0}^2(D_T)$, $\operatorname*{ess\,max}_{(x,\,t)\in D_T}|\vec{u}(x,\,t)| < \infty$ and satisfies the identity*

$$\begin{aligned} &\textstyle\int_\Omega \vec{u}(x,\,t)\,\vec{\varphi}(x,\,t)\,dx\big|_0^T = \\ &= \textstyle\int_{[0,\,T]}\int_\Omega \vec{u}\partial_t\vec{\varphi}dxdt - \\ &- \textstyle\int_{[0,\,T]}\int_\Omega a_{ij}(x,\,t,\,\vec{u})\,\nabla_j\vec{u}\nabla_i\vec{\varphi}dxdt + \\ &+ \textstyle\int_{[0,\,T]}\int_\Omega \vec{b}\vec{\varphi}dxdt \end{aligned} \tag{10}$$

for all $\vec{\varphi} \in C_0^\infty$.

A weak solution is called a weak solution to a first boundary problem for system (1) if there exists a limit

$$\lim_{t\to 0}\int_\Omega \vec{u}(x,\,t)\,\vec{\varphi}(x)\,dx = \int_\Omega \vec{\phi}(x)\,\vec{\varphi}(x)\,dx \tag{11}$$

for all $\vec{\varphi} \in C_0^\infty$.

For system (1), we consider the general and Holder classes of solutions and establish the uniqueness of such solutions in a certain class of functions. The growth restriction on the structural coefficients is essential and cannot be weakened as we can see from the Heinz example

$$\begin{aligned} \partial_t u^1 - \partial_{xx} u^1 &= u^1 \left(\left(\partial_x u^1 \right)^2 + \left(\partial_x u^2 \right)^2 \right) \\ \partial_t u^2 - \partial_{xx} u^2 &= u^2 \left(\left(\partial_x u^1 \right)^2 + \left(\partial_x u^2 \right)^2 \right), \end{aligned}$$

where functions $u^1 = \cos(mx)$ and $u^2 = \sin(mx)$ are solutions, however, these functions do not satisfy the estimation $\max\limits_{[0,\,2\pi]} |\nabla \vec{u}|$. The Heinz example shows that we need additional growth restrictions for the quasilinear systems.

For operators $(\lambda - \Delta)^{-\frac{1}{2}} \gamma^{\frac{1}{2}} : L^2 \to L^2$ and $(\lambda - \Delta)^{-1} \gamma : L^2 \to L^\infty$ the estimation

$$\left\| (\lambda - \Delta)^{-\frac{1}{2}} \gamma^{\frac{1}{2}} \right\|_2^2 \leq \left\| (\lambda - \Delta)^{-1} \gamma \right\|_\infty$$

holds for some $\lambda = \lambda(\nu) \geq 0$, thus for $\gamma \in PK(\beta)$ sufficient that $|\gamma|^2$ belongs to the Kato class K_ν^l, which consists of all potentials $\gamma \in L^1_{loc}$ that satisfy $\left\| (\lambda - \Delta)^{-1} |\gamma| \right\|_\infty \leq \nu$. Therefore, our assumptions $\gamma_0,\ \gamma_1{}^2,\ \gamma_2 \in PK(\beta)$ on structural coefficients include potentials with singularities of form-boundary of the orders $\sqrt{\beta}\frac{l-2}{2}\frac{x}{|x|^2} \in PK(\beta)$ since $\left(\frac{l-2}{2}\right)^2 \left\| |x|^{-1} \right\|_2^2 \leq \|\nabla\|_2^2$.

2. The Holder Continuity of Weak Solutions

We denote $ess\max\limits_{(x,\,t)\in D_T} |\vec{u}(x,\,t)| = M_1$, $\nu = \nu(M_1)$ and $\tilde{\mu} = \tilde{\mu}(M_1)$.

From the definition of the weak solution, we obtain the inequality

$$\begin{aligned} &\textstyle\int_\Omega \vec{u}(x,\,t)\,\vec{\varphi}(x,\,t)\,dx\big|_0^t + \\ &+ \textstyle\int_{[0,\,t]}\int_\Omega a_{ij}(x,\,t,\,\vec{u})\,\nabla_j\vec{u}\nabla_i\vec{\varphi}\,dxdt \leq \\ &\leq \textstyle\int_{[0,\,t]}\int_\Omega \vec{u}\partial_t\vec{\varphi}\,dxdt + \\ &+ \textstyle\int_{[0,\,t]}\int_\Omega \left(\tilde{\mu}\,|\nabla\vec{u}|^2 + \gamma_2(x,\,t)\right)|\vec{\varphi}|\,dxdt \end{aligned} \tag{12}$$

for all $t \in [0,\,T]$.

Theorem 1. *Let functions a_{ij} and $\vec{b}$ satisfy conditions* (6)-(8) *with $\gamma_0,\ \gamma_1{}^2,\ \gamma_3 \in PK(\beta)$ and $\gamma_0, \in L^1$. Let function $\vec{u}$ be a weak solution to the system* (1) *such that $ess\max\limits_{(x,\,t)\in D_T} |\vec{u}(x,\,t)| = M_1 < \infty$. Then, there is a positive number α such that $\vec{u} \in H^{\alpha,\,\frac{\alpha}{2}}(D_T)$.*

Proof. We compose the identity

$$\begin{aligned}&\int_{[0,\,T]}\int_{\Omega}\partial_t\vec{u}\vec{\varphi}dxdt+\\&+\int_{[0,\,T]}\int_{\Omega}a_{ij}(x,\,t,\,\vec{u})\nabla_j\vec{u}\nabla_i\vec{\varphi}dxdt=\\&=\int_{[0,\,T]}\int_{\Omega}\vec{b}\vec{\varphi}dxdt\end{aligned}$$

and we take $\vec{\varphi}(x,\,t)=\xi^2(x,\,t)\max\{\vec{u}_h(x,\,t)-n,\quad 0\}$, where we denote $\vec{u}_h$ the averaging at the time variable. Applying standard technically and taking the limit as $h\to 0$, we obtain

$$\begin{aligned}&\tfrac{1}{2}\int_{\Omega}\xi^2(x,\,t)\,|\vec{u}_n(x,\,t)|^2dx\Big|_0^T-\\&-\int_{[0,\,T]}\int_{\Omega}|\vec{u}_n|^2\,\xi\,\partial_t\xi\,dxdt+\\&+\int_{[0,\,T]}\int_{\Omega}a_{ij}\nabla_j\vec{u}\nabla_i\left(\xi^2\vec{u}_n\right)dxdt=\\&=\int_{[0,\,T]}\int_{\Omega}\vec{b}\xi^2\vec{u}_ndxdt.\end{aligned}$$

We calculate

$$\begin{aligned}&\int_{[0,\,T]}\int_{\Omega}a_{ij}\nabla_j\vec{u}_n\nabla_i\left(\xi^2\vec{u}_n\right)dxdt=\\&=\int_{[0,\,T]}\int_{\Omega}\xi^2a_{ij}\nabla_j\vec{u}_n\nabla_i\vec{u}_ndxdt+2\int_{[0,\,T]}\int_{\Omega}a_{ij}\xi\vec{u}_n\nabla_j\vec{u}_n\nabla_i\xi dxdt\end{aligned}$$

by applying conditions, we obtain the estimation

$$\begin{aligned}&\tfrac{1}{2}\int_{\Omega}\xi^2(x,\,t)\,|\vec{u}_n(x,\,t)|^2dx\Big|_0^T+\\&+\nu\int_{[0,\,T]}\int_{\Omega}\xi^2\,|\nabla\vec{u}_n|^2\,dxdt\le\\&\le\int_{[0,\,T]}\int_{\Omega}|\vec{u}_n|^2\,\xi\,|\partial_t\xi|\,dxdt+\\&+\int_{[0,\,T]}\int_{\Omega}\gamma_0\xi^2dxdt\\&+2\int_{[0,\,T]}\int_{\Omega}(\mu\,|\nabla\vec{u}_n|+\gamma_1)\,|\vec{u}_n|\,\xi\,|\nabla\xi|\,dxdt+\\&+\int_{[0,\,T]}\int_{\Omega}\left(\tilde{\mu}\,|\nabla\vec{u}_n|^2+\gamma_2\right)\xi^2\,|\vec{u}_n|\,dxdt.\end{aligned}$$

Next, we write

$$\begin{aligned}&2\int_{[0,\,T]}\int_{\Omega}\gamma_1\,|\vec{u}_n|\,\xi\,|\nabla\xi|\,dxdt\le\\&\le\varepsilon\int_{[0,\,T]}\int_{\Omega}\gamma_1{}^2\xi^2dxdt+\tfrac{1}{\varepsilon}\int_{[0,\,T]}\int_{\Omega}|\vec{u}_n|^2\,|\nabla\xi|^2\,dxdt\end{aligned}$$

and

$$\int_{[0,\,T]}\int_{\Omega}\gamma_2\,|\vec{u}_n|\,\xi^2dxdt\le M_1\int_{[0,\,T]}\int_{\Omega}\gamma_2\xi^2dxdt.$$

Let $\Lambda_{n,\rho}$ denote a set of all $x\in B(\rho)$ such that $\min\limits_{k=1,\dots,N} u^k(x)>n$. Applying $\gamma_0,\ {\gamma_1}^2,\ \gamma_2\in PK(\beta)$, we have

$$\begin{aligned}
&\tfrac{1}{2}\left\|\xi^2(x,\,t)\,|\vec{u}_n(x,\,t)|^2\right\|^2_{2,\quad B(\rho)}\Big|_{t_1}^{t_2}+\\
&+\nu\int_{[t_1,\,t_2]}\int_{B(\rho)}\xi^2\,|\nabla\vec{u}_n|^2\,dxdt\le\\
&\le\int_{[t_1,\,t_2]}\int_{\Lambda_{n,\rho}}|\vec{u}_n|^2\,\xi\,\partial_t\xi\,dxdt+\\
&+2\mu M_1\int_{[t_1,\,t_2]}\int_{\Lambda_{n,\rho}}|\nabla\vec{u}_n|\,\xi\,|\nabla\xi|\,dxdt+\\
&+\tilde{\mu}M_1\int_{[t_1,\,t_2]}\int_{\Lambda_{n,\rho}}|\nabla\vec{u}_n|^2\,\xi^2dxdt+\\
&+\hat{c}\left(\beta\int_{[t_1,\,t_2]}\int_{\Lambda_{n,\rho}}|\nabla\xi|^2\,dxdt+c(\beta)\int_{[t_1,\,t_2]}\int_{\Lambda_{n,\rho}}|\xi|^2\,dxdt\right)
\end{aligned}$$

where $t_1,\,t_2\in[0,\,T]$ and $B(\rho)\subset\Omega$ is a ball of radius ρ. Thus, we obtain

$$\begin{aligned}
&\left\|\xi^2(x,\,t_2)\,|\vec{u}_n(x,\,t_2)|^2\right\|^2_{2,\quad B(\rho)}+\\
&+\nu\int_{[t_1,\,t_2]}\int_{B(\rho)}\xi^2\,|\nabla\vec{u}_n|^2\,dxdt\le\\
&\le\left\|\xi^2(x,\,t_1)\,|\vec{u}_n(x,\,t_1)|^2\right\|^2_{2,\quad B(\rho)}+\\
&+\ \widehat{c}\ \int_{[t_1,\,t_2]}\int_{B(\rho)}|\vec{u}_n|^2\left(\xi\,|\partial_t\xi|+|\nabla\xi|^2\right)dxdt+\\
&+\hat{c}\left(\beta\int_{[t_1,\,t_2]}\int_{\Lambda_{n,\rho}}|\nabla\xi|^2\,dxdt+c(\beta)\int_{[t_1,\,t_2]}\int_{\Lambda_{n,\rho}}|\xi|^2\,dxdt\right),
\end{aligned}$$

where we chose n so that

$$\max_{B(\rho)\times[t_1,\,t_2]}\max_k u^k(x,\,t)-n\le\delta=\frac{\nu}{\upsilon\tilde{\mu}}.$$

The class $B_2(D_T)$ consists of all functions $\vec{u}(x,\,t)$ such that there is a system $\left\{\phi^i\left(u^1,\dots,u^N\right)\right\}_{i=1,\dots,\tilde{N}}$ of $\tilde{N}$-functions $\phi^i:R^N\to R$, $i=1,\dots,\tilde{N}$, which are continuously differentiable in their domains and such that functions $\omega^i(x,\,t)=\phi^i\left(u^1(x,\,t),\dots,u^N(x,\,t)\right)$, $\quad\omega^i\in V^2_{1,0}(D_T)$, $i=1,\dots,\tilde{N}$ satisfy conditions:
1)$ess\max\limits_{D_T}\left|\omega^i(x,\,t)\right|\le\tilde{M},i=1,\dots,\tilde{N}$;
2) for any cylinder $S=B(2\rho)\times[\tilde{t},\,\tilde{t}+\tau]\subset D_T$ and for any $t_0\in(\tilde{t},\,\tilde{t}+\tau)$, there is a number m such that

$$osc\left\{\omega^m(x,\,t),\quad D_T\right\}\ge\delta_1\max_{k=1,\dots,N}osc\left\{u^k(x,\,t),\quad D_T\right\}$$

and

$$\mu\left(\left\{\begin{array}{l} x\in B(\rho)\ :\ \omega^m(x,\,t_0)\le \\ \le ess\max\limits_{D_T}\omega^m(x,\,t)-\delta_2 osc\{\omega^m(x,\,t),\quad D_T\} \end{array}\right\}\right)\ge$$
$$\ge(1-\delta_3)\,c_l\rho^l,$$

where we denote μ *is a standard Lebesgue measure, and we assume that balls* $B(\rho)$ *and* $B(2\rho)$ *are concentric and* $\delta_1,\,\delta_2,\,\delta_3>0,\quad \delta_2,\,\delta_3<1$*;*

3) there are $\vartheta_1,\,\vartheta_2\in(0,\,1)$ *and number n, so that each function* ω^i*,* $i=1,\ldots,\tilde N$ *satisfy the inequalities*

$$\max_{t_0\in[\tilde t,\,\tilde t+\tau]}\|\omega^m{}_n(\cdot,\,t)\|^2_{2,B(\rho-\vartheta_1\rho)}\le$$
$$\le\max_{t_0\in[\tilde t,\,\tilde t+\tau]}\|\omega^m{}_n(\cdot,\,\tilde t)\|^2_{2,B(\rho)}+$$
$$+\breve c\left(\tfrac{1}{(\vartheta_1\rho)^2}\|\omega^m{}_n(\cdot,\,\tilde t)\|^2_{2,B(\rho)\times[\tilde t,\,\tilde t+\tau]}+\hat c\left(\mu\left(\Lambda_{n,\rho}\right)\right)\right),$$

and

$$\|\omega^m{}_n(\cdot,\,t)\|_{H,\,\overset{2}{2},\,B(\rho-\vartheta_1\rho)\times[\tilde t,\,\tilde t+\vartheta_2\tau]}\le$$
$$\le\breve c\left(\tfrac{1}{(\vartheta_1\rho)^2}+\tfrac{1}{(\vartheta_2\tau)^2}\right)\|\omega^m{}_n(\cdot,\,\tilde t)\|^2_{2,B(\rho)\times[\tilde t,\,\tilde t+\tau]}+$$
$$+\hat c\left(\mu\left(\Lambda_{n,\rho}\right)\right),$$

where we denote the Holder norm by $\|\cdot\|_H$*.*

From the analysis, we have the following theorem.

Theorem (estimation of the oscillation). *Let* $\vec u\in B_2(D_T)$*. Then, there is a positive* α *such that the oscillation of the function* $\vec u$ *estimates as*

$$\max_{k=1,\ldots,N} osc\left\{u^k,\quad B(\tilde\rho)\times\left[\tilde t,\,\tilde t+\vartheta\tilde\rho^2\right]\right\}\le c\tilde\rho^{-\alpha}\rho^\alpha$$

for some positive c, ϑ*, where balls* $B(\rho)$ *and* $B(\tilde\rho)$ *are concentric.*

Thus, we have that if $\vec u(x,\,t)$ satisfies the conditions of theorem 1 then $\vec u\in B_2(D_T)$ so there exists $\alpha>0$ such that $\vec u\in H^{\alpha,\,2^{-1}\alpha}(D_T)$.

3. Solvability of the First Boundary Problem

We define $S=\left\{(x,\,t)\in R^l\times R_+\ :\ \{x\in\partial\Omega,\,t\in[0,\,T]\}\cup\{(x,\,t)\ :\ x\in\Omega,\,t=0\}\right\}$.

Theorem 2. *Let functions* a_{ij} *and* $\vec b$ *satisfy* (6)-(8) *and conditions*

$$\nu(M_1)\,\xi^2\le a_{ij}\xi_i\xi_j\le\mu(M_1)\,\xi^2\tag{13}$$

for all $\xi \in R^l$*;*

$$\left|a_{ij}(x,\,t,\,\vec{u})\vec{k}_j\right| + \left|\frac{\partial a_{ij}(x,\,t,\,\vec{u})\vec{k}_j}{\partial u^k}u^k\right| \le \mu\left|\vec{k}\right| + \gamma_1(x,\,t)\,, \tag{14}$$

$$\left|\frac{\partial a_{ij}(x,\,t,\,\vec{u})}{\partial x_j}\vec{k}_j\right| \le \mu\left|\vec{k}\right|^2 + \gamma_2(x,\,t)\,, \tag{15}$$

$$\left|\vec{b}\left(x,\,t,\,\vec{u},\,\vec{k}\right)\right| \le \tilde{\mu}\left|\vec{k}\right|^2 + \gamma_3(x,\,t)\,, \tag{16}$$

$$\begin{aligned}&\sum\nolimits_i\left(\left|a_{ij}(x,\,t,\,\vec{u})\vec{k}_j\right| + \left|\tfrac{\partial a_{ij}(x,\,t,\,\vec{u})\vec{k}_j}{\partial u^k}u^k\right|\right)\left(1+\left|\vec{k}\right|\right) + \\ &+\sum\nolimits_{i,j}\left|\tfrac{\partial a_{ij}(x,\,t,\,\vec{u})\vec{k}_j}{\partial x_j}\right| + \left|\vec{b}\right| \le \mu\left(\left|\vec{k}\right|+1\right)^2,\end{aligned} \tag{17}$$

$$\left|\vec{b}\left(x,\,t,\,\vec{u},\,\vec{k}\right)\right| \le \omega\left(\left|\vec{k}\right|,\,|\vec{u}|\right)\left(\left|\vec{k}\right|+c_1\right)^2 + c_2(|\vec{u}|)\,, \tag{18}$$

where $\omega\left(\left|\vec{k}\right|,\,|\vec{u}|\right) \overset{|\vec{k}|\to\infty}{\longrightarrow} 0$ *and* $c_2(|\vec{u}|)$ *is a small constant.*

Let function $\vec{\phi} \in C^{3,2}(clos(D_T))$ *satisfies the identity*

$$\frac{\partial}{\partial t}\vec{\phi} = \sum_{i,j=1,\dots,l}\nabla_i\left(a_{ij}\left(x,\,t,\,\vec{\phi}\right)\nabla_j\vec{\phi}\right) + \vec{b}\left(x,\,t,\,\vec{\phi},\,\nabla\vec{\phi}\right) \tag{19}$$

on the subset

$$S = \left\{(x,\,t) \in R^l \times R_+ \;:\; \{x \in \partial\Omega,\, t \in [0,\,T]\} \cup \{(x,\,t) \;:\; x \in \Omega,\, t = 0\}\right\};$$

$\max\limits_{x\in\Omega}|\nabla\phi(x,\,0)| < \infty$ *and* $\vec{\phi} \in H^{\tilde{\alpha},2^{-1}\tilde{\alpha}}(clos(D_T))$, $\gamma_0,\ \gamma_1{}^2,\ \gamma_2 \in PK(\beta)$*. Then, there exists a solution* $\vec{u} \in H^{\alpha,\,2^{-1}\alpha}(clos(D_T))$ *to the boundary problem* $\vec{u}\big|_S = \vec{\phi}\Big|_S$ *for the system* (1)*, if the function* $\vec{b}\left(x,\,t,\,\vec{u},\,\vec{k}\right)$ *is uniformly Lipschitz continuous at* $\vec{u}$ *and* $\vec{k}$*, then such a solution is unique.*

Proof. Here, we provide a scheme of proof, which is based on the Leray-Schauder fixed point principle.

We denote the quasi-linear operator

$$\begin{aligned}&L\vec{u} = \tfrac{\partial}{\partial t}\vec{u} - \sum\nolimits_{i,j=1,\dots,l}a_{ij}(x,\,t,\,\vec{u})\nabla_i\nabla_j\vec{u} - \\ &-\sum\nolimits_{i,j=1,\dots,l}\left(\nabla_j\vec{u}\right)\nabla_i\left(a_{ij}(x,\,t,\,\vec{u})\right) - \vec{b}(x,\,t,\,\vec{u},\,\nabla\vec{u}) = 0,\end{aligned} \tag{20}$$

we put $B(x,\,t,\,\vec{u},\,\nabla\vec{u}) = -\sum_{i,j=1,\ldots,l}\left(\nabla_j\vec{u}\right)\nabla_i\left(a_{ij}(x,\,t,\,\vec{u})\right) - \vec{b}(x,\,t,\,\vec{u},\,\nabla\vec{u})$, and compose a set of linear problems

$$\begin{aligned} &\tfrac{\partial}{\partial t}\vec{v} - \left(\tau a_{ij}(x,\,t,\,\vec{w}) + (1-\tau)\,\delta_{ij}\right)\nabla_i\nabla_j\vec{v} + \\ &+\tau B(x,\,t,\,\vec{w},\,\nabla\vec{w}) - (1-\tau)\left(\tfrac{\partial}{\partial t}\vec{\phi} - \Delta\vec{\phi}\right) = 0, \end{aligned} \tag{21}$$

under condition $\vec{v}|_S = \vec{\phi}\Big|_S$ for each $\tau \in [0,\,1]$.

Set of problems (20) is a linear system, where $\vec{v}(x,\,t)$ is unknown function and functions $\vec{w}(x,\,t)$ are considered given.

We consider a system of linear problems

$$\begin{aligned} &L(\tau)\vec{u} = \tfrac{\partial}{\partial t}\vec{u} - \sum_{i,j=1,\ldots,l}\nabla_i\left(\left(a_{ij}(x,\,t,\,\vec{u})\nabla_j\vec{u}\right) + (1-\tau)\nabla_i\vec{u}\right) + \\ &-\tau\vec{b}(x,\,t,\,\vec{u},\,\nabla\vec{u}) - (1-\tau)\left(\tfrac{\partial}{\partial t}\vec{\phi} - \Delta\vec{\phi}\right) = 0, \end{aligned} \tag{22}$$

and boundary conditions $\vec{u}|_S = \vec{\phi}\Big|_S$ for each $\tau \in [0,\,1]$.
The solution $\vec{u}_\tau = \vec{u}(\tau)$ to the boundary problem for system (22) is a fixed point of the mapping $\Upsilon(\vec{w},\,\tau)$ so that $\Upsilon(\vec{u}_\tau,\,\tau) = \vec{u}_\tau$, the operator Υ defines the correspondence between functions $\vec{w}$ and solutions of linear problems for system (21) by $\vec{v} = \Upsilon(\vec{w},\,\tau)$. The operator Υ maps linear functional space $\Theta_\delta, \quad \delta > 0$, which consists of all continuous functions $\vec{w}(x,\,t)$ and with the norm given by

$$\|\vec{w}\|_{H,\,\Theta_\delta} = \|\vec{w}\|_{H(\delta),\,D_T} + \|\nabla\vec{w}\|_{H(\delta),\,D_T}\,.$$

Under our conditions, the operator $\Upsilon(\tau)\,:\,\vec{w} \mapsto \vec{v}$ depends on the parameter $\tau \in [0,\,1]$, the fixed point of the operator $\Upsilon(1)\,:\,\vec{w} \mapsto \vec{v}$ is a solution to the boundary problem for the system (1).

Let Θ_δ be a space, where $\delta > 0$ is a sufficiently small number then the theory of linear parabolic equations guarantees the existence of $\vec{u}_\tau$ such that

$$\max_{D_T}|\vec{u}_\tau| \le M_1,$$

$$\max_{D_T}|\nabla\vec{u}_\tau| \le M_2,$$

$$\|\vec{u}_\tau\|_{H,\,\Theta_\alpha} = \|\vec{u}_\tau\|_{H(\alpha),\,D_T} + \|\nabla\vec{u}_\tau\|_{H(\alpha),\,D_T} \le M_3$$

with some $\alpha \in (0,\,1)$ and for all solutions $\vec{u}_\tau$ to the problems (22).

Let ε be a strictly positive number, we consider the operator Υ on the subspace $\mathrm{E} \subset \Theta_\delta$ that consists of all function $\vec{w} \in \Theta_\delta$ such that

$$\max_{D_T} |\vec{w}| \le M_1 + \varepsilon,$$

$$\max_{D_T} |\nabla \vec{w}| \le M_2 + \varepsilon,$$

$$\|\vec{w}\|_{H,\,\Theta_\alpha} \le M_3 + \varepsilon.$$

All fixed points $\vec{u}_\tau$ of mapping Υ belong to E.

Straightforward calculations yield that $\Upsilon(\vec{w},\,\tau)$ is a set of uniformly equicontinuous at $\vec{w}$ and τ, and uniformly compact operators. Thus, for each $\tau \in [0,\,1]$, there exists a fixed point $\vec{u}_\tau = \Upsilon(\vec{u}_\tau,\,\tau)$, which is a solution to the problem (22) that belongs to $H^{\lambda,\,2^{-1}\lambda}(clos(D_T))$. The solution to (22) is a solution to (21) under the condition $\vec{w} = \vec{u}_\tau$ therefore $\vec{v} = \vec{u}_\tau$ for all $\tau \in [0,\,1]$ and $\tau = 1$ proves the existence of the solution to the boundary problem for (1).

The uniqueness can be proven by contradiction assuming that there are two distinct solutions $\vec{u}_1$ and $\vec{u}_2$ to the boundary problem $\vec{u}|_S = \vec{\varphi}\big|_S$ for the system (1). Then, functions $\vec{u}_1$ and $\vec{u}_2$ must satisfy the integral identity

$$\begin{aligned}&\int_{[0,\,T]}\int_\Omega \left(\partial_t \vec{u}\vec{\varphi} + a_{ij}(x,\,t,\,\vec{u})\nabla_j \vec{u}\nabla_i \vec{\varphi}\right) dxdt = \\ &= \int_{[0,\,T]}\int_\Omega \vec{b}\vec{\varphi} dxdt\end{aligned}$$

for all functions $\vec{\varphi} \in C_0^\infty$ which equal zero on S. We subtract from the first identity the second and obtain

$$\begin{aligned}&\int_{[0,\,T]}\int_\Omega \left(\partial_t \vec{w}\vec{\varphi} + \left(\tilde{a}_{ij}\nabla_j \vec{w} + A_i \vec{w}\right)\nabla_i \vec{\varphi}\right) dxdt = \\ &= \int_{[0,\,T]}\int_\Omega \left(B_i \nabla_i \vec{w} + B\vec{w}\right)\vec{\varphi} dxdt\end{aligned} \tag{23}$$

where $\vec{w} \equiv \vec{u}_1 - \vec{u}_2$, and we denote

$$\begin{aligned}&a_{ij}(x,\,t,\,\vec{u}_1)\nabla_j \vec{u}_1 - a_{ij}(x,\,t,\,\vec{u}_2)\nabla_j \vec{u}_2 = \\ &= \nabla_j \vec{w}\int_{[0,\,1]} a_{ij}(x,\,t,\,\tau\vec{u}_1 + (1-\tau)\vec{u}_2)\,d\tau + \\ &+ w^k \int_{[0,\,1]} \tfrac{\partial a_{ij}(x,\,t,\,\tau\vec{u}_1 + (1-\tau)\vec{u}_2)}{\partial u^k} d\tau = \\ &= \tilde{a}_{ij}\nabla_j \vec{w} + A_i \vec{w}\end{aligned}$$

and

$$\vec{b}(x,\,t,\,\vec{u}_1,\,\nabla\vec{u}_1)-\vec{b}(x,\,t,\,\vec{u}_2,\,\nabla\vec{u}_2)=$$
$$=\nabla_i w^k\int_{[0,\,1]}\frac{\partial\vec{b}(x,\,t,\,\tau\vec{u}_1+(1-\tau)\vec{u}_2,\,\tau\nabla\vec{u}_1+(1-\tau)\nabla\vec{u}_2)}{\partial\nabla_i u^k}+$$
$$+w^k\int_{[0,\,1]}\frac{\partial\vec{b}(x,\,t,\,\tau\vec{u}_1+(1-\tau)\vec{u}_2,\,\tau\nabla\vec{u}_1+(1-\tau)\nabla\vec{u}_2)}{\partial u^k}d\tau=$$
$$=B_i\nabla_i\vec{w}+B\vec{w}.$$

Applying the linear theory to (23), we obtain that there exists a unique solution to (23) $\vec{w}\equiv 0$ so that the uniqueness is proven.

Now, we have to estimate the $\max\limits_{D_T}|\nabla\vec{u}(x,\,t)|$ by $\max\limits_{D_T}|\vec{u}(x,\,t)|=M_1$ and functions of the coefficients. We consider an integral identity

$$\int_{[0,\,t]}\int_\Omega\left(\partial_t\vec{u}\vec{\varphi}+a_{ij}(x,\,t,\,\vec{u})\nabla_j\vec{u}\nabla_i\vec{\varphi}\right)dxdt= \tag{24}$$
$$=\int_{[0,\,T]}\int_\Omega\vec{b}\vec{\varphi}dxdt$$

for all $t\in[0,\,T]$. We take $\varphi^k(x,\,t)=u^k(x,\,t)\,\xi^2(x)\exp\left(\lambda\,|\vec{u}(x,\,t)|^2\right)$ where ξ is cutoff for Ω. We have

$$\frac{1}{2\lambda}\int_\Omega\xi^2\exp\left(\lambda\,|\vec{u}|^2\right)dx\Big|_0^T+$$
$$+\frac{\lambda}{2}\int_{[0,\,t]}\int_\Omega a_{ij}\xi^2\exp\left(\lambda\,|\vec{u}|^2\right)\nabla_i\left(|\vec{u}|^2\right)\nabla_j\left(|\vec{u}|^2\right)dxdt+$$
$$+\int_{[0,\,t]}\int_\Omega a_{ij}\xi\exp\left(\lambda\,|\vec{u}|^2\right)\nabla_i\xi\nabla_j\left(|\vec{u}|^2\right)dxdt+$$
$$+\int_{[0,\,t]}\int_\Omega a_{ij}\xi^2\exp\left(\lambda\,|\vec{u}|^2\right)\nabla_i\vec{u}\nabla_j\vec{u}dxdt=$$
$$=\int_{[0,\,T]}\int_\Omega\vec{b}\vec{u}\xi^2\exp\left(\lambda\,|\vec{u}|^2\right)dxdt,$$

applying conditions and $\gamma_0,\,\gamma_1^2,\,\gamma_3\in PK(\beta)$, we obtain an estimation

$$\int_{[0,\,T]}\int_\Omega\xi^2\sum_{k=1,\dots,N}\sum_{i=1,\dots,l}\left(\nabla_i u^k\right)^2dxdt\le const.$$

Furthermore, in (24), we take $\varphi^k=\nabla_m\left(\xi\nabla_m u^k\right)$ and obtain

$$\frac{1}{2}\int_{[0,\,t]}\int_\Omega\xi\,\partial_t\left(\sum_{k=1,\dots,N}\sum_{i=1,\dots,l}\left(\nabla_i u^k\right)^2\right)dxdt+$$
$$+\int_{[0,\,t]}\int_\Omega a_{ij}\xi\left(\nabla_m\nabla_i\vec{u}\right)\left(\nabla_m\nabla_j\vec{u}\right)dxdt+$$
$$+\frac{1}{2}\int_{[0,\,t]}\int_\Omega a_{ij}\left(\nabla_i\xi\right)\nabla_j\left(\sum_{k=1,\dots,N}\sum_{i=1,\dots,l}\left(\nabla_i u^k\right)^2\right)dxdt+$$
$$+\int_{[0,\,t]}\int_\Omega\left(\nabla_m a_{ij}\right)\left(\nabla_i\xi\right)\left(\nabla_j\vec{u}\right)\left(\nabla_m\vec{u}\right)dxdt+$$
$$+\int_{[0,\,t]}\int_\Omega\left(\nabla_m a_{ij}\right)\xi\left(\nabla_j\vec{u}\right)\left(\nabla_i\nabla_m\vec{u}\right)dxdt+$$
$$+\int_{[0,\,T]}\int_\Omega\vec{b}\left(\xi\Delta\vec{u}+\left(\nabla_m\vec{u}\right)\left(\nabla_m\xi\right)\right)dxdt=0,$$

where, as always,

$$\vec{b}\left(\xi\Delta\vec{u}+\left(\nabla_m\vec{u}\right)\left(\nabla_m\xi\right)\right)\equiv\sum_{k=1,\ldots,N}b^k\left(\xi\Delta u^k+\sum_{m=1,\ldots,l}\left(\nabla_m u^k\right)\left(\nabla_m\xi\right)\right).$$

Next, we assume $\xi=2\left(\sum_{k=1,\ldots,N}\sum_{i=1,\ldots,l}\left(\nabla_i u^k\right)^2\right)^{\upsilon}\eta^2$ where the function $\eta\left(x\right)$ is cutoff for the ball $B\left(\rho\right)\subset\Omega$, $\upsilon\geq 0$. Applying conditions, we estimate

$$\begin{aligned}
&\tfrac{1}{1+\upsilon}\int_\Omega\eta^2\left(\sum_{k=1,\ldots,N}\sum_{i=1,\ldots,l}\left(\nabla_i u^k\right)^2\right)^{\upsilon+1}dx\Big|_0^t+\\
&+2\nu\int_{[0,\,t]}\int_\Omega\eta^2\left(\sum_{k=1,\ldots,N}\sum_{i=1,\ldots,l}\left(\nabla_i u^k\right)^2\right)^{\upsilon}\left|\nabla\nabla\vec{u}\right|^2dxdt+\\
&+\nu\upsilon\int_{[0,\,t]}\int_\Omega\eta^2\left(\sum_{k=1,\ldots,N}\sum_{i=1,\ldots,l}\left(\nabla_i u^k\right)^2\right)^{\upsilon-1}\left|\nabla\left(\sum_{k=1,\ldots,N}\sum_{i=1,\ldots,l}\left(\nabla_i u^k\right)^2\right)\right|^2dxdt\leq\\
&\leq c\left(\upsilon\right)\left(\int_{[0,\,t]}\int_\Omega\left(\sum_{k=1,\ldots,N}\sum_{i=1,\ldots,l}\left(\nabla_i u^k\right)^2\right)^{\upsilon+1}\left|\nabla\eta\right|^2dxdt+\right.\\
&+\int_{[0,\,t]}\int_\Omega\left(\sum_{k=1,\ldots,N}\sum_{i=1,\ldots,l}\left(\nabla_i u^k\right)^2\right)^{\upsilon+2}\eta^2dxdt+\\
&\left.+c\int_{[0,\,t]}\int_\Omega\left(\eta^2+\beta\left|\nabla\eta\right|^2\right)dxdt\right).
\end{aligned}$$

The inequality

$$\begin{aligned}
&\int_{B(\rho)}\eta^2\left(\sum_{k=1,\ldots,N}\sum_{i=1,\ldots,l}\left(\nabla_i u^k\right)^2\right)^{\upsilon+2}dx\leq\\
&\leq c\rho^\alpha\left(\int_{B(\rho)}\eta^2\left(\sum_{k=1,\ldots,N}\sum_{i=1,\ldots,l}\left(\nabla_i u^k\right)^2\right)^{\upsilon}\left|\nabla\nabla\vec{u}\right|^2dx+\right.\\
&\left.+\int_{B(\rho)}\left(\sum_{k=1,\ldots,N}\sum_{i=1,\ldots,l}\left(\nabla_i u^k\right)^2\right)^{\upsilon+1}\left|\nabla\eta\right|^2dxdt\right)
\end{aligned}$$

holds for small enough ρ and $\upsilon>0$. Therefore, we have

$$\begin{aligned}
&\tfrac{1}{1+\upsilon}\int_{B(\rho)}\eta^2\left(\sum_{k=1,\ldots,N}\sum_{i=1,\ldots,l}\left(\nabla_i u^k\right)^2\right)^{\upsilon+1}dx\Big|_0^t+\\
&+2^{-1}\nu\int_{[0,\,t]}\int_{B(\rho)}\eta^2\left(\sum_{k=1,\ldots,N}\sum_{i=1,\ldots,l}\left(\nabla_i u^k\right)^2\right)^{\upsilon}\left|\nabla\nabla\vec{u}\right|^2dxdt+\\
&+2^{-1}\nu\int_{[0,\,t]}\int_{B(\rho)}\eta^2\left(\sum_{k=1,\ldots,N}\sum_{i=1,\ldots,l}\left(\nabla_i u^k\right)^2\right)^{\upsilon+2}dxdt\leq\\
&\leq c\left(\upsilon\right)\left(\int_{[0,\,t]}\int_{B(\rho)}\left(\sum_{k=1,\ldots,N}\sum_{i=1,\ldots,l}\left(\nabla_i u^k\right)^2\right)^{\upsilon+1}\left|\nabla\eta\right|^2dxdt+\right.\\
&\left.+c\int_{[0,\,t]}\int_{B(\rho)}\left(\eta^2+\beta\left|\nabla\eta\right|^2\right)dxdt\right),
\end{aligned}$$

thus $\max\limits_{t\in[0,\,T]}\int_{\tilde{\Omega}}\left(\sum_{k=1,\ldots,N}\sum_{i=1,\ldots,l}\left(\nabla_i u^k\right)^2\right)^{\upsilon+1}dxdt\leq c\left(\upsilon,\,\tilde{\Omega}\right).$

By taking $\vec{\varphi}(x,\, t) = \nabla_m \vec{\xi}\,(x,\, t)$ where the function $\vec{\xi}$ equals zero near S, we have

$$\begin{aligned}&\int_{[0,\, t]} \int_\Omega \vec{\xi}\,\partial_t \nabla_m \vec{u} dxdt + \\ &+ \int_{[0,\, t]} \int_\Omega a_{ij} \left(\nabla_m \nabla_j \vec{u}\right) \left(\nabla_i \vec{\xi}\right) dxdt = \\ &= \int_{[0,\, t]} \int_\Omega \vec{\Psi}_{i,m} \left(\nabla_i \vec{\xi}\right) dxdt\end{aligned}$$

where we denote

$$\vec{\Psi}_{i,m} = -\frac{\partial a_{ij}}{\partial x_m} \nabla_i \vec{u} - \frac{\partial a_{ij}}{\partial u^k} \nabla_m u^k \nabla_i \vec{u} + \vec{b}\delta_{im},$$

denoting $v = \nabla_m u^k, \quad k = 1, ..., N; \quad m = 1, ..., l$, we obtain that functions $v \in V_{1,0}^2\,(D_T)$ as solutions to the linear equations

$$\partial_t v - \nabla_j \left(a_{ij} \nabla_i v - {\Psi_{i,m}}^k\right) = 0.$$

Applying conditions theorem 2, from linear theory, we finally obtain

$$\max_{D_T} |\nabla \vec{u}\,(x,\, t)| \le const\,(\nu,\, \mu,\, \tilde{\mu}),$$

where we applied standard argument extending estimation to the boundary which follows from previous inequalities.

Conflict of Interests

The author declares that there is no conflict of interests.

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