Available online at http://jfpt.scik.org
J. Fixed Point Theory, 2025, 2025:1
ISSN: 2052-5338

# RANDOM FIXED POINT THEOREMS AND APPLICATION IN $S_b$ METRIC SPACES

VISHAKHA LODHI, ANIMESH GUPTA*

Department of Mathematics & Computer Sciences, Govt. Science College Jabalpur (M.P.)-India



**Abstract.** In this paper we give some applications to integral equations as well as homotopy theory via random fixed point theorems in partially ordered complete $S_b-$metric spaces by using generalized contractive conditions. We also give some example in supports of our main result. Our obtained results generalized some previous known results.

**Keywords:** random fixed point; $\sigma-$algebra; measurable space; $S_b-$metric space; w-compatible pairs; $S_b-$completeness.

**2020 AMS Subject Classification:** 45H10; 54H25.

## 1. Introduction and Preliminaries

Banach contraction principle in metric spaces is one of the most important results in fixed theory and nonlinear analysis in general. Since 1922, when Stefan Banach [1] formulated the concept of contraction and posted a famous theorem, scientists around the world have published new results related to the generalization of a metric space or with contractive mappings. Banach contraction principle is considered to be the initial result of the study of fixed point theory in metric spaces.

*Corresponding author

E-mail address: dranimeshgupta10@gmail.com

Received March 13, 2024

In the year 1989, Bakhtin introduced the concept of $b-$metric spaces as a generalization of metric spaces [6]. Later several authors proved so many results on b-metric spaces (see [13, 14, 15, 16]. Mustafa and Sims defined the concept of a generalized metric space which is called a $G-$metric space [12]. Sedghi, Shobe and Aliouche gave the notion of an S-metric space and proved some fixed point theorems for a self-mapping on a complete S-metric space [22]. Aghajani, Abbas and Roshan presented a new type of metric which is called $G_b-$metric and studied some properties of this metric [2]. Recently Sedghi et al. [20] defined $S_b-$metric spaces using the concept of $S-$metric spaces [22].

The aim of this paper is to prove some unique fixed point theorems for generalized contractive conditions in complete $S_b-$metric spaces. Also, we give applications to integral equations as well as homotopy theory. Throughout this paper $R$, $R^+$ and $N$ denote the sets of all real numbers, non-negative real numbers and positive integers, respectively. First we recall some definitions, lemmas and examples.

**Definition 1.** [22] *Let $X$ be a non-empty set. An $S-$metric on $X$ is a function $S : X^3 \to [0,+\infty)$ that satisfies the following conditions for each $x,y,z,a \in X$ :*

*(S1):* $0 < S(x,y,z)$ *for all* $x,y,z \in X$ *with* $x \neq y \neq z \neq x$,

*(S2):* $S(x,y,z) = 0$ *if and only if* $x = y = z$,

*(S3):* $S(x,y,z) \leq S(x,x,a)+S(y,y,a)+S(z,z,a)$ *for all* $x,y,z,a \in X$.

*Then the pair $(X,S)$ is called an S-metric space.*

**Definition 2.** [20] *Let $X$ be a non-empty set and $b \geq 1$ be a given real number. Suppose that a mapping $S_b : X^3 \to [0,\infty)$ is a function satisfying the following properties:*

$(S_b1)$ : $0 < S_b(x,y,z)$ *for all* $x,y,z \in X$ *with* $x \neq y \neq z \neq x$,

$(S_b2)$ : $S_b(x,y,z) = 0$ *if and only if* $x = y = z$,

$(S_b3)$ : $S_b(x,y,z) \leq b(S_b(x,x,a)+S_b(y,y,a)+S_b(z,z,a))$ *for all* $x,y,z,a \in X$.

*Then the function $S_b$ is called an $S_b-$ metric on $X$ and the pair $(X,S_b)$ is called an $S_b-$ metric space.*

**Remark 3.** [20] *It should be noted that the class of $S_b-$ metric spaces is effectively larger than that of $S-$ metric spaces. Indeed each $S-$ metric space is an $S_b-$ metric space with $b=1$. The following example shows that an $S_b-$ metric on $X$ need not be an S-metric on $X$.*

**Example 4.** [20] *Let $(X,S)$ be an S-metric space and $S_*(x,y,z)=S(x,y,z)^p$, where $p>1$ is a real number. Note that $S_*$ is an $S_b-$ metric with $b=2^{2(p-1)}$. Also, $(X,S_*)$ is not necessarily an $S-$ metric space.*

**Definition 5.** [20] *Let $(X,S_b)$ be an $S_b-$ metric space. Then, for $x\in X, r>0$, we define the open ball $B_{S_b}(x,r)$ and the closed ball $B_{S_b}[x,r]$ with center $x$ and radius $r$ as follows, respectively:*

$$B_{S_b}(x,r)=\left\{y\in X: S_b(y,y,x)<r\right\},$$

$$B_{S_b}[x,r]=\left\{y\in X: S_b(y,y,x)\leq r\right\}.$$

**Lemma 6.** [20] *In an $S_b-$ metric space, we have*

$$S_b(x,x,y)\leq bS_b(y,y,x)$$

*and*

$$S_b(y,y,x)\leq bS_b(x,x,y).$$

**Lemma 7.** [20] *In an $S_b-$ metric space, we have*

$$S_b(x,x,z)\leq 2bS_b(x,x,y)+b^2S_b(y,y,z).$$

**Definition 8.** [20] *If $(X,S_b)$ is an $S_b-$ metric space, a sequence $\{x_n\}$ in $X$ is said to be:*

*(1) $S_b-$ Cauchy sequence if, for each $\varepsilon>0$, there exists $n_0\in\mathcal{N}$ such that $S_b(x_n,x_n,x_m)<\varepsilon$ for each $m,n\geq n_0$.*

*(2) $S_b-$ convergent to a point $x\in X$ if, for each $\varepsilon>0$, there exists a positive integer $n_0$ such that $S_b(x_n,x_n,x)<\varepsilon$ or $S_b(x,x,x_n)<\varepsilon$ for all $n\geq n_0$, and we denote $\lim_{n\to\infty}x_n=x$.*

**Definition 9.** [20] *An $S_b-$ metric space $(X,S_b)$ is called complete if every $S_b-$ Cauchy sequence is $S_b-$ convergent in $X$.*

**Lemma 10.** [20] *If $(X,S_b)$ is an $S_b-$ metric space with $b\geq 1$, and suppose that $\{x_n\}$ is $S_b-$ convergent to $x$, then we have*

$$\text{(i)} \quad \frac{1}{2b}S_b(y,x,x) \leq \lim_{n\to\infty}\inf S_b(y,y,x_n) \leq \lim_{n\to\infty}\sup S_b(y,y,x_n) \leq 2bS_b(y,y,x)$$

*and*

$$\text{(ii)} \quad \frac{1}{b^2}S_b(x,x,y) \leq \lim_{n\to\infty}\inf S_b(x_n,x_n,y) \leq \lim_{n\to\infty}\sup S_b(x_n,x_n,y) \leq b^2 S_b(x,x,y)$$

*for all $y \in X$. In particular, if $x = y$, then we have* $\lim_{n\to\infty} S_b(x_n,x_n,y) = 0$.

## 2. Random Variable

If $X$ is a metric space, we shall use $B(X)$ to denote the Borel $\sigma-$algebra on $X$. Let $(\Omega,\mathscr{F})$ be a measurable space. The expression $f \otimes B(X)$ denotes the smallest $\sigma-$algebra on $\Omega \times X$ which contains all the sets $Q \times S$, where $Q \in f$ and $S \in B(X)$. Given a topological space $E$, we denote

$$\mathscr{P}(E) = \{Y \subset E : Y \neq \emptyset\}, \mathscr{P}_{cl}(E) = \{Y \in \mathscr{P}(E) : Y \text{ closed}\}.$$

Let $f : X \to \mathscr{P}(Y)$ be a multivalued map. A single-valued map $\overline{f} : X \to Y$ is said to be a selection of $f$ (and we write $\overline{f} \subset f$) whenever $\overline{f}(x) \in f(x)$ for every $x \in X$. Let $(\Omega,\Sigma)$ be a measurable space and $f : \Omega \to \mathscr{P}(X)$ a multivalued mapping, $f$ is called measurable if

$$f_+^{-1}(Q) = \{\omega \in \Omega : f(\omega) \subset Q\}$$

is measurable for every $Q \in \mathscr{P}_{cl}(X)$; equivalently, for every $U$ open subset of $X$, the set

$$f_-^{-1}(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\}$$

is measurable.

**Proposition 11.** [22] *Let $(\Omega,\mathscr{U})$ be a measurable space and $f : \Omega \to \mathscr{P}_{cl}(X)$ be a multivalued map. If, for every $C \in \mathscr{P}(X)$, we have $f_-^{-1}(C) \in \mathscr{U}$, then $f$ is measurable.*

**Definition 12.** *Recall that a mapping $f : \Omega \times X \to X$ is said to be a random operator if, for any $x \in X, f(\cdot,x)$ is measurable.*

**Definition 13.** *A random fixed point of $f$ is a measurable function $y : \Omega \to X$ such that*

$$y(\omega) = f\big(\omega, y(\omega)\big) \quad \textit{for all } \omega \in \Omega.$$

Equivalently, it is a measurable selection for the multivalued map $\operatorname{Fix} f : \Omega \to \mathscr{P}(X)$ defined by

$$\operatorname{Fix} f(\omega) = \{x \in X : x = f(\omega, x)\}, \quad \omega \in \Omega.$$

**Theorem 14.** *([23], Theorem 6.1) Let $X$ be a separable metric space, $Y$ a metric space and $f : \Omega \times X \to Y$ a continuous random operator. Then $f$ is a measurable random operator.*

**Theorem 15.** *Let $(\Omega, \Sigma)$ be a measurable space, $Y$ be a separable metric space, and $\phi : \Omega \to \mathscr{P}_{cl}(Y)$ be a measurable multivalued function. Then $\phi$ has a measurable selection.*

Let $X, Y$ be two locally compact metric spaces and $f : \Omega \times X \to Y$. By $C(X,Y)$, we denote the space of continuous functions from $X$ into $Y$ endowed with the compact-open topology.

**Lemma 16.** [24] *$f$ is a Carathéodory function if and only if $\omega \to r(\omega)(\cdot) = f(\omega, \cdot)$ is a measurable function from $\Omega$ to $C(X,Y)$.*

Now we prove our main results.

## 3. Results and Discussions

**Definition 17.** *Let $(\Omega, f)$ be a measurable space, $(X, S_b, \preceq)$ be a separable partially ordered complete $S_b-$ metric space which is said to be regular if every two random variables $x, y : \Omega \to X$ are comparable,*

$$\text{either } x(\omega) \preceq y(\omega) \text{ or } y(\omega) \preceq x(\omega).$$

**Definition 18.** *Let $(\Omega, f)$ be a measurable space, $(X, S_b, \preceq)$ be a separable partially ordered complete $S_b-$ metric space which is also regular; let $f : \Omega \times X \to X$ be a mapping. We say that $f$ satisfies $(\psi, \phi)-$ contraction if there exist $\psi, \phi : [0, \infty) \to [0, \infty)$ such that*

*(3.2.1): $f$ is non-decreasing,*

*(3.2.2): $\Psi$ is continuous, monotonically non-decreasing and $\phi$ is lower semi-continuous,*

*(3.2.3): $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$,*

*(3.2.4):*

$$\psi(4b^4 S_b(f(\omega,x), f(\omega,x), f(\omega,y))) \leq \psi(M_f^i(x(\omega), y(\omega))) - \phi(M_f^i(x(\omega), y(\omega))),$$

*for all* $(\omega,x),(\omega,y)\in(\Omega\times X), x(\omega)\preceq y(\omega), i=3,4,5$ *and*

$$M_f^5((\omega,x),(\omega,y))=\max\Big\{S_b(x(\omega),x(\omega),y(\omega)),S_b(x(\omega),x(\omega),f(\omega,x)),S_b(y(\omega),y(\omega),f(\omega,y)),$$
$$S_b(x(\omega),x(\omega),f(\omega,y)),S_b(y(\omega),y(\omega),f(\omega,x))\Big\},$$
$$M_f^4(x(\omega),y(\omega))=\max\Big\{S_b(x(\omega),x(\omega),y(\omega)),S_b(x(\omega),x(\omega),f(\omega,x)),S_b(y(\omega),y(\omega),f(\omega,y)),$$
$$\frac{1}{4b^4}\left[S_b(x(\omega),x(\omega),f(\omega,y))+S_b(y(\omega),y(\omega),f(\omega x))\right]\Big\},$$
$$M_f^3(x(\omega),y(\omega))=\max\Big\{S_b(x(\omega),x(\omega),y(\omega)),\frac{1}{4b^4}\left[S_b(x(\omega),x(\omega),f(\omega,x))+S_b(y(\omega),y(\omega),f(\omega,y))\right],$$
$$\frac{1}{4b^4}\left[S_b(x(\omega),x(\omega),f(\omega,y))+S_b(y(\omega),y(\omega),f(\omega,x))\right]\Big\}.$$

**Definition 19.** *Let* $(\Omega,f)$ *be a measurable space,* $(X,S_b,\preceq)$ *be a separable partially ordered set and* $f$ *is a mapping from* $\Omega\times X$ *into* $X$*. We say that* $f$ *is non-decreasing if for every* $x,y:\Omega\to X$,

$$x(\omega)\preceq y(\omega) \text{ implies that } f(\omega,x)\preceq f(\omega,y). \tag{3.1}$$

*for all* $\omega\in\Omega$.

**Theorem 20.** *Let* $(\Omega,f)$ *be a measurable space,* $(X,S_b,\preceq)$ *be a separable partially ordered* $S_b$ *metric space, which is also regular, and let* $f:\Omega\times X\to X$ *satisfy* $(\psi,\phi)-$ *contraction with* $i=5$*. If there exists* $x_0:\Omega\to X$ *with* $x_0(\omega)\preceq f(\omega,x_0)$*, then* $f$ *has a unique random fixed point.*

*Proof.* If for each $\omega\in\Omega$, since $f$ is a mapping from $\Omega\times X$ into $X$, there exists a sequence $\{x_n(\omega)\}$ of elements of $X$ such that

$$x_{n+1}(\omega)=f(\omega,x_n),\quad n=0,1,2,3,\ldots.$$

Case (i): If $x_n(\omega)=x_{n+1}(\omega)$, then $x_n(\omega)$ is a random fixed point of $f$.

Case (ii): Suppose $x_n(\omega)\neq x_{n+1}(\omega)$ $\forall n$.

Since $x_0(\omega)\preceq f(\omega,x_0)=x_1(\omega)$ and $f$ is non-decreasing, it follows that

$$x_0(\omega)\preceq f(\omega,x_0)\preceq f^2(\omega,x_0)\preceq f^3(\omega,x_0)\preceq\cdots\preceq f^n(\omega,x_0)\preceq f^{n+1}(\omega,x_0)\preceq\cdots.$$

Now

$$\begin{aligned}\psi\big(4b^4 S_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big)\big) &= \psi\big(4b^4 S_b(f(\omega,x_0),f(\omega,x_0),fx_1)\big)\\ &\le \psi\big(M_f^5(x_0(\omega),x_1(\omega))\big)-\phi\big(M_f^5(x_0(\omega),x_1(\omega))\big),\end{aligned}$$

where

$$\begin{aligned}&M_f^5(x_0(\omega),x_1(\omega))\\ &= \max\left\{\begin{matrix} S_b(x_0(\omega),x_0(\omega),x_1(\omega)),S_b(x_0(\omega),x_0(\omega),f(\omega,x_0)),S_b(x_1(\omega),x_1(\omega),f(\omega,x_1))\\ S_b(x_0(\omega),x_0(\omega),f^2(\omega,x_0)),S_b(f(\omega,x_0),f(\omega,x_0),f(\omega,x_0))\end{matrix}\right\}\\ &= \max\left\{S_b(x_0(\omega),x_0(\omega),f(\omega,x_0)),S_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big),S_b\big(x_0(\omega),x_0(\omega),f^2(\omega,x_0)\big)\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\psi\big(4b^4 S_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big)\big)\\ &\le \psi\left(\max\left\{S_b(x_0(\omega),x_0(\omega),f(\omega,x_0)),S_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big),S_b\big(x_0(\omega),x_0(\omega),f^2(\omega,x_0)\big)\right\}\right)\\ &\quad-\phi\left(\max\left\{S_b(x_0(\omega),x_0(\omega),f(\omega,x_0)),S_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big),S_b\big(x_0(\omega),x_0(\omega),f^2(\omega,x_0)\big)\right\}\right)\\ &\le \psi\left(\max\left\{S_b(x_0(\omega),x_0(\omega),f(\omega,x_0)),S_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big),S_b\big(x_0(\omega),x_0(\omega),f^2(\omega,x_0)\big)\right\}\right).\end{aligned}$$

By the definition of $\Psi$, we have that

$$S_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big) \le \max\left\{\begin{matrix}\frac{1}{4b^4}S_b(x_0(\omega),x_0,f(\omega,x_0))\\ \frac{1}{4b^4}S_b(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0))\\ \frac{1}{4b^4}S_b(x_0(\omega),x_0(\omega),f^2(\omega,x_0))\end{matrix}\right\}. \tag{3.2}$$

But

$$\begin{aligned}&\frac{1}{4b^4}S_b\big(x_0(\omega),x_0(\omega),f^2(\omega,x_0)\big)\\ &\le \frac{1}{4b^4}\left[2bS_b(x_0(\omega),x_0(\omega),f(\omega,x_0))+b^2S_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big)\right]\\ &\le \max\left\{\frac{1}{b^3}S_b(x_0(\omega),x_0(\omega),f(\omega,x_0),\frac{1}{2b^2}S_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big)\right\}.\end{aligned}$$

From 3.2 we have that

$$\begin{aligned}&S_b\big(f(\omega,x_0),f(\omega,x_0)),f^2(\omega,x_0)\big)\\ &\le \max\left\{\frac{1}{b^3}S_b(x_0(\omega),x_0(\omega),f(\omega,x_0)),\frac{1}{2b^2}S_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big)\right\}.\end{aligned} \tag{3.3}$$

If $\frac{1}{2b^2}S_b(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0))$ is maximum, we get a contradiction. Hence

$$S_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big) \le \frac{1}{b^3}S_b(x_0(\omega),x_0(\omega),f(\omega,x_0)). \tag{3.4}$$

Also

$$\begin{aligned}\psi\big(4b^4S_b\big(f^2(\omega,x_0),f^2(\omega,x_0),f^3(\omega,x_0)\big)\big) &= \psi\big(4b^4S_b(f(\omega,x_1),f(\omega,x_1),f(\omega,x_2))\big)\\ &\le \psi\big(M_f^5(x_1(\omega),x_2(\omega))\big)-\phi\big(M_f^5(x_1(\omega),x_2(\omega))\big),\end{aligned}$$

where

$$\begin{aligned}M_f^5(x_1(\omega),x_2(\omega)) &= \max\left\{\begin{matrix} S_b(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)),S_b(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)),\\ S_b(f^2(\omega,x_0),f^2(\omega,x_0),f^3(\omega,x_0))S_b(f(\omega,x_0),f(\omega,x_0),f^3(\omega,x_0)),\\ S_b(f^2(\omega,x_0),f^2(\omega,x_0),f^2(\omega,x_0))\end{matrix}\right\}\\ &= \max\left\{\begin{matrix} S_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big),S_b\big(f^2(\omega,x_0),f^2(\omega,x_0),f^3(\omega,x_0)\big),\\ S_b\big(f(\omega,x_0),f(\omega,x_0),f^3(\omega,x_0)\big)\end{matrix}\right\}.\end{aligned}$$

Therefore

$$\begin{aligned}&\psi\big(4b^4S_b\big(f^2(\omega,x_0),f^2(\omega,x_0),f^3(\omega,x_0)\big)\big)\\ &\quad\le \psi\left(\max\left\{\begin{matrix} S_b(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)),S_b(f^2(\omega,x_0),f^2(\omega,x_0),f^3(\omega,x_0))\\ S_b(f(\omega,x_0),f(\omega,x_0),f^3(\omega,x_0))\end{matrix}\right\}\right)\\ &\qquad-\phi\left(\max\left\{\begin{matrix} S_b(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)),S_b(f^2(\omega,x_0),f^2(\omega,x_0),f^3(\omega,x_0))\\ S_b(f(\omega,x_0),f(\omega,x_0),f^3(\omega,x_0))\end{matrix}\right\}\right)\\ &\quad\le \psi\left(\max\left\{\begin{matrix} S_b(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)),S_b(f^2(\omega,x_0),f^2(\omega,x_0),f^3(\omega,x_0))\\ S_b(f(\omega,x_0),f(\omega,x_0),f^3(\omega,x_0))\end{matrix}\right\}\right).\end{aligned}$$

By the definition of $\Psi$, we have that

$$S_b\big(f^2(\omega,x_0),f^2(\omega,x_0),f^3(\omega,x_0)\big)\le\max\left\{\begin{matrix}\frac{1}{4b^4}S_b(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0))\\ \frac{1}{4b^4}S_b(f^2(\omega,x_0),f^2(\omega,x_0),f^3(\omega,x_0))\\ \frac{1}{4b^4}S_b(f(\omega,x_0),f(\omega,x_0),f^3(\omega,x_0))\end{matrix}\right\}. \tag{3.5}$$

But

$$\begin{aligned}&\frac{1}{4b^4}S_b\big(f(\omega,x_0),f(\omega,x_0),f^3(\omega,x_0)\big)\\ &\quad\le\frac{1}{4b^4}\left[2bS_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big)+b^2S_b\big(f^2(\omega,x_0),f^2(\omega,x_0),f^3(\omega,x_0)\big)\right]\\ &\quad\le\max\left\{\frac{1}{b^3}S_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big),\frac{1}{2b^2}S_b\big(f^2(\omega,x_0),f^2(\omega,x_0),f^3(\omega,x_0)\big)\right\}.\end{aligned}$$

From 3.5 we have that

$$\begin{aligned} & S_b\big(f^2(\omega,x_0),f^2(\omega,x_0),f^3(\omega,x_0)\big) \\ & \leq \max\Big\{\frac{1}{b^3}S_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big),\frac{1}{2b^2}S_b\big(f^2(\omega,x_0),f^2(\omega,x_0),f^3(\omega,x_0)\big)\Big\}. \end{aligned}$$

If $\frac{1}{2b^2}S_b(f^2(\omega,x_0),f^2(\omega,x_0),f^3(\omega,x_0))$ is maximum, we get a contradiction. Hence

$$\begin{aligned} S_b\big(f^2(\omega,x_0),f^2(\omega,x_0),f^3(\omega,x_0)\big) \leq & \frac{1}{b^3}S_b\big(f(\omega,x_0),f(\omega,x_0),f^2(\omega,x_0)\big) \\ \leq & \frac{1}{(b^3)^2}S_b(x_0(\omega),x_0(\omega),f(\omega,x_0)). \end{aligned}$$

Continuing this process, we can conclude that

$$\begin{aligned} S_b\big(f^n(\omega,x_0),f^n(\omega,x_0),f^{n+1}(\omega,x_0)\big) \leq & \frac{1}{(b^3)^n}S_b(x_0(\omega),x_0(\omega),f(\omega,x_0)) \\ & \to 0 \quad \text{as } n\to\infty. \end{aligned}$$

That is,

$$\lim_{n\to\infty} S_b\big(f^n(\omega,x_0),f^n(\omega,x_0),f^{n+1}(\omega,x_0)\big)=0. \tag{3.6}$$

Now we prove that $\{f^n(\omega,x_0)\}$ is an $S_b-$ Cauchy sequence in $(X,S_b,\Omega)$. On the contrary, we suppose that $\{f^n(\omega,x_0)\}$ is not $S_b-$ Cauchy. Then there exist $\varepsilon>0$ and monotonically increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k>m_k$.

$$S_b\big(f^{m_k}(\omega,x_0),f^{m_k}(\omega,x_0),f^{n_k}(\omega,x_0)\big)\geq\varepsilon \tag{3.7}$$

and

$$S_b\big(f^{m_k}(\omega,x_0),f^{m_k}(\omega,x_0),f^{n_k-1}(\omega,x_0)\big)<\varepsilon. \tag{3.8}$$

From 3.7 and 3.8, we have

$$\begin{aligned} \varepsilon & \leq S_b\big(f^{m_k}(\omega,x_0),f^{m_k}(\omega,x_0),f^{n_k}(\omega,x_0)\big) \\ & \leq 2bS_b\big(f^{m_k}(\omega,x_0),f^{m_k}(\omega,x_0),f^{m_k+1}(\omega,x_0)\big) \\ & \quad +b^2S_b\big(f^{m_k+1}(\omega,x_0),f^{m_k+1}(\omega,x_0),f^{n_k}(\omega,x_0)\big). \end{aligned}$$

So that

$$\begin{aligned} 4b^2\varepsilon \leq & 8b^3S_b\big(f^{m_k}(\omega,x_0),f^{m_k}(\omega,x_0),f^{m_k+1}(\omega,x_0)\big) \\ & +4b^4S_b\big(f^{m_k+1}(\omega,x_0),f^{m_k+1}(\omega,x_0),f^{n_k}(\omega,x_0)\big). \end{aligned}$$

Letting $k \to \infty$ and applying $\Psi$ on both sides, we have that

$$\begin{aligned}\psi\big(4b^2\varepsilon\big) &\leq \lim_{k\to\infty}\psi\big(4b^4 S_b\big(f^{m_k+1}(\omega,x_0),f^{m_k+1}(\omega,x_0),f^{n_k}(\omega,x_0)\big)\big)\\ &= \lim_{k\to\infty}\psi\big(4b^4 S_b(f(\omega,x_{m_k}),f(\omega,x_{m_k}),f(\omega,x_{n_k-1}))\big) \quad (3.9)\\ &\leq \lim_{k\to\infty}\psi\big(M_f^5(x_{m_k}(\omega),x_{n_k-1}(\omega))\big)-\lim_{k\to\infty}\phi\left(M_f^5(x_{m_k}(\omega),x_{n_k-1}(\omega))\right),\end{aligned}$$

where

$$\begin{aligned}&\lim_{k\to\infty} M_f^5(x_{m_k}(\omega),x_{n_k-1}(\omega))\\ &= \lim_{k\to\infty}\max\left\{\begin{array}{c} S_b(f^{m_k}(\omega,x_0),f^{m_k}(\omega,x_0),f^{n_k-1}(\omega,x_0)),S_b(f^{m_k}(\omega,x_0),f^{m_k}(\omega,x_0),f^{m_k+1}(\omega,x_0))\\ S_b(f^{n_k-1}(\omega,x_0),f^{n_k-1}(\omega,x_0),f^{n_k}(\omega,x_0)),S_b(f^{m_k}(\omega,x_0),f^{m_k}(\omega,x_0),f^{n_k}(\omega,x_0))\\ S_b(f^{n_k-1}(\omega,x_0),f^{n_k-1}(\omega,x_0),f^{m_k+1}(\omega,x_0))\end{array}\right\}\\ &< \lim_{k\to\infty}\max\left\{\varepsilon,0,0,S_b\big(f^{m_k}(\omega,x_0),f^{m_k}(\omega,x_0),f^{n_k}(\omega,x_0)\big),S_b\big(f^{n_k-1}(\omega,x_0),f^{n_k-1}(\omega,x_0),f^{m_k+1}(\omega,x_0)\big)\right\}.\end{aligned}$$

But

$$\lim_{k\to\infty} S_b\big(f^{m_k}(\omega,x_0),f^{m_k}(\omega,x_0),f^{n_k}(\omega,x_0)\big) \leq \lim_{k\to\infty}\left[\begin{array}{c} 2bS_b(f^{m_k}(\omega,x_0),f^{m_k}(\omega,x_0),f^{n_k-1}(\omega,x_0))\\ +b^2 S_b(f^{n_k-1}(\omega,x_0),f^{n_k-1}(\omega,x_0),f^{n_k}(\omega,x_0))\end{array}\right] < 2b\varepsilon.$$

Also

$$\lim_{k\to\infty} S_b\big(f^{n_k-1}(\omega,x_0),f^{n_k-1}(\omega,x_0),f^{m_k+1}(\omega,x_0)\big) \leq \lim_{k\to\infty}\left[\begin{array}{c} 2bS_b(f^{n_k-1}(\omega,x_0),f^{n_k-1}(\omega,x_0),f^{m_k}(\omega,x_0))\\ +b^2 S_b(f^{m_k}(\omega,x_0),f^{m_k}(\omega,x_0),f^{m_k+1}(\omega,x_0))\end{array}\right] < 2b^2\varepsilon.$$

Therefore

$$\begin{aligned}\lim_{k\to\infty} M_f^5(x_{m_k}(\omega),x_{n_k-1}(\omega)) &\leq \max\left\{\varepsilon,2b\varepsilon,2b^2\varepsilon\right\}\\ &= 2b^2\varepsilon.\end{aligned}$$

From 3.9, by the definition of $\Psi$, we have that

$$4b^2\varepsilon \leq 2b^2\varepsilon,$$

which is a contradiction. Hence $\{f^n(\omega,x_0)\}$ is an $S_b-$ Cauchy sequence in complete regular $S_b-$ metric spaces $(X,S_b,\preceq)$. By the completeness of $(X,S_b)$, it follows that the sequence $\{f^n x_0\}$ converges to $\approx$ in $(X,S_b)$. Thus

$$\lim_{k\to\infty} f^n(\omega,x_0) = \alpha(\omega) = \lim_{k\to\infty} f^{n+1}(\omega,x_0).$$

Since $x_n(\omega)$, $\alpha : \omega \to X$ and $X$ is regular, it follows that either $x_n(\omega) \preceq \alpha(\omega)$ or $\alpha(\omega) \preceq x_n(\omega)$.

Now we have to prove that $\alpha(\omega)$ is a random fixed point of $f$.

Suppose $f(\omega,\alpha)\neq\alpha(\omega)$, by Lemma 10, we have that

$$\frac{1}{2b}S_b(f(\omega,\alpha),f(\omega,\alpha),\alpha(\omega))\leq\liminf_{n\to\infty}S_b(f(\omega,\alpha),f(\omega,\alpha),f^{n+1}(\omega,x_0)).$$

Now from 3.5 and applying $\Psi$ on both sides, we have that

$$\begin{aligned}\psi\big(2b^3S_b(f(\omega,\alpha),f(\omega,\alpha),\alpha(\omega))\big)&\leq\liminf_{n\to\infty}\psi\big(4b^4S_b\big(f(\omega,\alpha),f(\omega,\alpha),f^{n+1}(\omega,x_0)\big)\big)\\ &\leq\liminf_{n\to\infty}\psi\big(M_f^5(\alpha(\omega),x_n(\omega))\big)-\liminf_{n\to\infty}\phi\big(M_f^5(\alpha(\omega),x_n(\omega))\big).\end{aligned}\tag{3.10}$$

Here

$$\begin{aligned}&\liminf_{n\to\infty}M_f^5(\alpha(\omega),x_n(\omega))\\ &=\liminf_{n\to\infty}\max\left\{\begin{matrix}S_b(\alpha(\omega),\alpha(\omega),x_n(\omega)),S_b(\alpha(\omega),\alpha(\omega),f(\omega,\alpha)),S_b(x_n(\omega),x_n(\omega),f(\omega,x_n))\\ S_b(\alpha(\omega),\alpha(\omega),f(\omega,x_n)),S_b(x_n(\omega),x_n(\omega),f(\omega,\alpha))\end{matrix}\right\}\\ &\leq\limsup_{n\to\infty}\max\{0,S_b(\alpha(\omega),\alpha(\omega),f(\omega,\alpha)),0,0,S_b(x_n(\omega),x_n(\omega),f(\omega,\alpha))\}\\ &\leq\max\big\{S_b(\alpha(\omega),\alpha(\omega),f(\omega,\alpha)),b^2S_b(\alpha(\omega),\alpha(\omega),f(\omega,\alpha))\big\}\\ &\leq b^3S_b(f(\omega,\alpha),f(\omega,\alpha),\alpha(\omega)).\end{aligned}$$

Hence from 3.10 we have that

$$\begin{aligned}&\psi\big(2b^3S_b(f(\omega,\alpha),f(\omega,\alpha),\alpha(\omega))\big)\\ &\leq\psi\big(b^3S_b(\alpha(\omega),\alpha(\omega),f(\omega,\alpha))\big)-\liminf_{n\to\infty}\phi\big(M_f^5(\alpha(\omega),x_n(\omega))\big)\\ &\leq\psi\big(b^3S_b(f(\omega,\alpha),f(\omega,\alpha),\alpha(\omega))\big),\end{aligned}$$

which is a contradiction. So that $\alpha(\omega)$ is a random fixed point of $f$.

Suppose that $\alpha^*(\omega)$ is another random fixed point of $f$ such that $\alpha(\omega)\neq\alpha^*(\omega)$.

Consider

$$\begin{aligned}\psi\big(4b^4S_b\big(\alpha(\omega),\alpha(\omega),\alpha^*(\omega)\big)\big)&\leq\psi\big(M_f^5\big(\alpha(\omega),\alpha^*(\omega)\big)\big)-\phi\big(M_f^5\big(\alpha(\omega),\alpha^*(\omega)\big)\big)\\ &=\psi\big(\max\big\{S_b\big(\alpha(\omega),\alpha(\omega),\alpha^*(\omega)\big),S_b\big(\alpha^*(\omega),\alpha^*(\omega),\alpha(\omega)\big)\big\}\big)\\ &\quad-\phi\big(\max\big\{S_b\big(\alpha(\omega),\alpha(\omega),\alpha^*(\omega)\big),S_b\big(\alpha^*(\omega),\alpha^*(\omega),\alpha(\omega)\big)\big\}\big)\\ &\leq\psi\big(bS_b\big(\alpha(\omega),\alpha(\omega),\alpha^*(\omega)\big)\big),\end{aligned}$$

which is a contradiction. Hence $\alpha$ is a unique random fixed point of $f$ in $(X,S_b)$. □

**Example 21.** *Let $X=[0,1]$ and $S:X\times X\times X\to\mathbb{R}^+$ by $S_b(x,y,z)=(|y+z-2x|+|y-z|)^2$ and by $a\preceq b\iff a\leq b$, then $(X,S_b,\preceq)$ is a complete ordered $S_b-$ metric space with $b=4$.*

*Define* $f:\Omega\times X\to X$ *by* $f(\omega,x)=\frac{x}{32\sqrt{2}}(1-\omega)$. *Also define* $\psi,\phi:\mathbb{R}^+\to\mathbb{R}^+ by \psi(t)=t$ *and* $\phi(t)=\frac{t}{2}$.

$$\begin{aligned}
\psi\big(4b^4S_b(f(\omega,x),f(\omega,x),f(\omega,y))\big) &= 4b^4\big(|f(\omega,x)+f(\omega,y)-2f(\omega,x)|+|f(\omega,x)-f(\omega,y)|\big)^2\\
&= 4b^4\left(2\left|\frac{x}{32\sqrt{2}}(1-\omega)-\frac{y}{32\sqrt{2}}(1-\omega)\right|\right)^2\\
&= \frac{4b^4}{8b^4}S_b((\omega,x),(\omega,x),(\omega,y))\\
&\leq \frac{1}{2}M_f^5((\omega,x),(\omega,y))\\
&\leq \psi\big(M_f^5((\omega,x),(\omega,y))\big)-\phi\big(M_f^5((\omega,x),(\omega,y))\big),
\end{aligned}$$

*Hence, all the conditions of Theorem 20 are satisfied and* 0 *is a unique random fixed point of* $f$.

**Theorem 22.** *Let* $(\Omega,f)$ *be a measurable space,* $(X,S_b,\preceq)$ *be a separable partially ordered* $S_b$ *metric space, which is also regular, and let* $f:\Omega\times X\to X$ *satisfy* $(\psi,\phi)-$ *contraction with* $i=3$ *or* 4. *If there exists* $x_0:\Omega\to X$ *with* $x_0(\omega)\preceq f(\omega,x_0)$, *then* $f$ *has a unique random fixed point.*

*Proof.* Follows along similar lines of Theorem 20 if we take $M_f^3(x(\omega),y(\omega))$ or $M_f^4(x(\omega),y(\omega))$ in place of $M_f^5(x(\omega),y(\omega))$ in Theorem 20. □

**Theorem 23.** *Let* $(\Omega,f)$ *be a measurable space,* $(X,S_b,\preceq)$ *be a separable partially ordered* $S_b$ *metric space, which is also regular, and let* $f:\Omega\times X\to X$ *satisfy*

$$4b^4S_b(f(\omega,x),f(\omega,x),f(\omega,y))\leq M_f^i(x(\omega),y(\omega))-\varphi\big(M_f^i(x(\omega),y(\omega))\big),$$

*where* $\varphi:[0,\infty)\to[0,\infty)$ *and* $i=3$ *or* 4 *or* 5. *If there exists* $x_0(\omega)\in X$ *with* $x_0(\omega)\preceq f(\omega,x_0)$, *then* $f$ *has a unique random fixed point in* $X$.

*Proof.* The proof follows from Theorems 20 and 22 by taking $\psi(t)=t$ and $\phi(t)=\varphi(t)$. □

**Theorem 24.** *Let* $(\Omega,f)$ *be a measurable space,* $(X,S_b,\preceq)$ *be a separable partially ordered* $S_b$ *metric space, which is also regular, and let* $f:\Omega\times X\to X$ *satisfy*

$$S_b(f(\omega,x),f(\omega,x),f(\omega,y))\leq \lambda M_f^i(x(\omega),y(\omega)),$$

*where* $\lambda \in [0, \frac{1}{4b^4})$ *and* $i = 3,4,5$. *If there exists* $x_0(\omega) \in X$ *with* $x_0(\omega) \preceq f(\omega, x_0)$, *then* $f$ *has a unique random fixed point in* $X$.

## 4. Application to Integral Equations

In this section, we study the existence of a unique solution to an initial value problem as an application to Theorem 20.

**Theorem 25.** *Consider the initial value problem*

$$x^1(t) = T\big(t, x(t)\big), \quad t \in I = [0,1], x(0) = x_0, \tag{4.1}$$

*where* $T : I \times [\frac{x_0}{4}, \infty) \to [\frac{x_0}{4}, \infty)$ *and* $x_0 \in \mathbb{R}$. *Then there exists a unique solution in* $C(I, [\frac{x_0}{4}, \infty))$ *for initial value problem 4.1.*

*Proof.* The integral equation corresponding to initial value problem 4.1 is

$$x(t) = x_0 + 3b^2 \int_0^t T\big(s, x(s)\big)\, ds.$$

Let $X = C(I, [\frac{x_0}{4}, \infty))$ and $S_b(x,y,z) = (|y+z-2x| + |y-z|)^2$ for $x, y \in X$. Define $\psi, \phi : [0,\infty) \to [0,\infty)$ by $\psi(t) = t, \phi(t) = \frac{5t}{9}$. Define $f : X \to X$ by

$$f(x)(t) = \frac{x_0}{3b^2} + \int_0^t T\big(s, x(s)\big)\, ds. \tag{4.2}$$

Now

$$\begin{aligned}
&\psi\big(4b^4 S_b\big(fx(t), fx(t), fy(t)\big)\big) \\
&\quad = 4b^4 \big\{\big|fx(t) + fy(t) - 2fx(t)\big| + \big|fx(t) - fy(t)\big|\big\}^2 \\
&\quad = 16b^4 \big|fx(t) - fy(t)\big|^2 \\
&\quad = \frac{16b^4}{9b^4} \left| x_0 + 3b^2 \int_0^t T\big(s, x(s)\big)\, ds - y_0 - 3b^2 \int_0^t T\big(s, y(s)\big)\, ds \right|^2 \\
&\quad = \frac{16}{9} \big|x(t) - y(t)\big|^2 \\
&\quad = \frac{4}{9} S(x,x,y) \\
&\quad \leq \frac{4}{9} M_f^5(x,y) \\
&\quad = \psi\big(M_f^5(x,y)\big) - \phi\big(M_f^5(x,y)\big),
\end{aligned}$$

where

$$M_f^5(x,y)=\max\left\{S_b(x,x,y),S_b(x,x,fx),S_b(y,y,fy),S_b(x,x,fy),S_b(y,y,fx)\right\}.$$

It follows from Theorem 20 that $f$ has a unique fixed point in $X$. □

## 5. Application to Homotopy

In this section, we study the existence of a unique solution to homotopy theory.

**Theorem 26.** *Let $(X,S_b)$ be a complete $S_b-$ metric space, $U$ be an open subset of $X$ and $\overline{U}$ be a closed subset of $X$ such that $U\subseteq\overline{U}$. Suppose that $H:\Omega\times\overline{U}\times[0,1]\to X$ is an operator such that the following conditions are satisfied:*

*(i) $x(\omega)\neq H(\omega,x,\lambda)$ for each $x:\Omega\to\partial U$ and $\lambda\in[0,1]$ (here $\partial U$ denotes the boundary of $U$ in $X$ ),*

*(ii)*

$$\psi(4b^4S_b(H(\omega,x,\lambda),H(\omega,x,\lambda),H(\omega,y,\lambda)))\leq\psi(S_b(x(\omega),x(\omega),y(\omega)))-\phi(S_b(x(\omega),x(\omega),y(\omega)))$$

*for all $x,y:\Omega\to\overline{U}$ and $\lambda\in[0,1]$, where $\psi:[0,\infty)\to[0,\infty)$ is continuous, non-decreasing and $\phi:[0,\infty)\to[0,\infty)$ is lower semi-continuous with $\phi(t)>0 fort>0$,*

*(iii) there exists $M\geq0$ such that*

$$S_b\big(H(\omega,x,\lambda),H(\omega,x,\lambda),H(\omega,x,\mu)\big)\leq M|\lambda-\mu|$$

*for every $x\Omega\to\overline{U}$ and $\lambda,\mu\in[0,1]$. Then $H(\omega,\cdot,0)$ has a random fixed point if and only if $H(\omega,\cdot,1)$ has a random fixed point.*

*Proof.* Consider the set

$$A=\big\{\lambda\in[0,1]:x=H(\omega,x,\lambda)\text{ for some }x:\Omega\to U\big\}.$$

Since $H(\omega,\cdot,0)$ has a random fixed point in $U$, we have that $0\in A$. So that $A$ is a non-empty set. We will show that $A$ is both open and closed in $[0,1]$, and so, by the connectedness, we have that $A=[0,1]$. As a result, $H(\omega,\cdot,1)$ has a random fixed point in $U$. First we show that $A$ is closed in $[0,1]$. To see this, let $\{\lambda_n\}_{n=1}^{\infty} \subseteq A$ with $\lambda_n \to \lambda \in [0,1]$ as $n \to \infty$. We must show that $\lambda \in A$. Since $\lambda_n \in A$ for $n = 1,2,3,\dots$, there exists $x_n \in U$ with $x_n(\omega) = H(\omega, x_n, \lambda_n)$. Consider

$$\begin{aligned} S_b(x_n(\omega), x_n(\omega), x_{n+1}(\omega)) &= S_b\big(H(\omega,x_n,\lambda_n), H(\omega,x_n,\lambda_n), H(\omega,x_{n+1},\lambda_{n+1})\big) \\ &\leq 2bS_b\big(H(\omega,x_n,\lambda_n), H(\omega,x_n,\lambda_n), H(\omega,x_{n+1},\lambda_n)\big) \\ &\quad + b^2 S_b\big(H(\omega,x_{n+1},\lambda_n), H(\omega,x_{n+1},\lambda_n), H(\omega,x_{n+1},\lambda_{n+1})\big) \\ &\leq S_b\big(H(\omega,x_n,\lambda_n), H(\omega,x_n,\lambda_n), H(\omega,x_{n+1},\lambda_n)\big) + M|\lambda_n - \lambda_{n+1}|. \end{aligned}$$

Letting $n \to \infty$, we get

$$\lim_{n\to\infty} S_b(x_n(\omega), x_n(\omega), x_{n+1}(\omega)) \leq \lim_{n\to\infty} S_b\big(H(\omega,x_n,\lambda_n), H(\omega,x_n,\lambda_n), H(\omega,x_{n+1},\lambda_n)\big) + 0.$$

Since $\Psi$ is continuous and non-decreasing, we obtain

$$\begin{aligned} &\lim_{n\to\infty} \psi\big(4b^4 S_b(x_n(\omega), x_n(\omega), x_{n+1}(\omega))\big) \\ &\leq \lim_{n\to\infty} \psi\big(4b^4 S_b\big(H(\omega,x_n,\lambda_n), H(\omega,x_n,\lambda_n), H(\omega,x_{n+1},\lambda_n)\big)\big) \\ &\leq \lim_{n\to\infty} \big[\psi\big(S_b(x_n(\omega), x_n(\omega), x_{n+1}(\omega))\big) - \phi\big(S_b(x_n(\omega), x_n(\omega), x_{n+1}(\omega))\big)\big]. \end{aligned}$$

By the definition of$\Psi$, it follows that

$$\lim_{n\to\infty} \big(4b^4 - 1\big) S_b(x_n(\omega), x_n(\omega), x_{n+1}(\omega)) \leq 0.$$

So that

$$\lim_{n\to\infty} S_b(x_n(\omega), x_n(\omega), x_{n+1}(\omega)) = 0. \tag{5.1}$$

Now we prove that $\{x_n(\omega)\}$ is an $S_b-$ Cauchy sequence in $(X, d_p)$. On the contrary, suppose that $\{x_n(\omega)\}$ is not $S_b-$ Cauchy.

There exists $\varepsilon > 0$ and monotone increasing sequences of natural numbers $\{m_k\}$ and $\{n_k\}$ such that $n_k > m_k$,

$$S_b(x_{m_k}(\omega), x_{m_k}(\omega), x_{n_k}(\omega)) \geq \varepsilon \tag{5.2}$$

and

$$S_b(x_{m_k}(\omega), x_{m_k}(\omega), x_{n_k-1}(\omega)) < \varepsilon. \tag{5.3}$$

From 5.2 and 5.3, we obtain

$$\begin{aligned}\varepsilon &\leq S_b(x_{m_k}(\omega), x_{m_k}(\omega), x_{n_k}(\omega)) \\ &\leq 2bS_b(x_{m_k}(\omega), x_{m_k}, x_{m_k+1}(\omega)) + b^2 S_b(x_{m_k+1}(\omega), x_{m_k+1}(\omega), x_{n_k}(\omega)).\end{aligned}$$

Letting $k \to \infty$ and applying $\Psi$ on both sides, we have that

$$\psi\big(2b^2\varepsilon\big) \leq \lim_{n\to\infty} \psi\big(4b^4 S_b(x_{m_k+1}(\omega), x_{m_k+1}(\omega), x_{n_k}(\omega))\big). \tag{5.4}$$

But

$$\begin{aligned}&\lim_{n\to\infty} \psi\big(4b^4 S_b(x_{m_k+1}(\omega), x_{m_k+1}(\omega), x_{n_k}(\omega))\big) \\ &\quad= \lim_{n\to\infty} \psi\big(S_b\big(4b^4 H(\omega, x_{m_k+1}, \lambda_{m_k+1}), H(\omega, x_{m_k+1}, \lambda_{m_k+1}), H(\omega, x_{n_k}, \lambda_{n_k})\big)\big) \\ &\quad\leq \lim_{n\to\infty} \big[\psi\big(S_b(x_{m_k+1}(\omega), x_{m_k+1}(\omega), x_{n_k}(\omega))\big) - \phi\big(S_b(x_{m_k+1}(\omega), x_{m_k+1}(\omega), x_{n_k}(\omega))\big)\big].\end{aligned}$$

It follows that

$$\lim_{n\to\infty} \big(4b^4 - 1\big) S_b(x_{m_k+1}(\omega), x_{m_k+1}(\omega), x_{n_k}(\omega)) \leq 0.$$

Thus

$$\lim_{n\to\infty} S_b(x_{m_k+1}(\omega), x_{m_k+1}(\omega), x_{n_k}(\omega)) = 0.$$

Hence from 5.4 and the definition of $\Psi$, we have that

$$\varepsilon \leq 0,$$

which is a contradiction. Hence $\{x_n\}$ is an $S_b-$ Cauchy sequence in $(X, S_b)$ and, by the completeness of $(X, S_b)$, there exists $\alpha \in U$ with

$$\lim_{n\to\infty} x_n = \alpha = \lim_{n\to\infty} x_{n+1},$$

$$\begin{aligned}\psi\big(2b^3 S_b\big(H(\alpha,\lambda), H(\alpha,\lambda), \alpha\big)\big) &\leq \liminf_{n\to\infty} \psi\big(4b^4 S_b\big(H(\alpha,\lambda), H(\alpha,\lambda), H(x_n,\lambda)\big)\big) \\ &\leq \liminf_{n\to\infty} \big[\psi\big(S_b(\alpha,\alpha,x_n)\big) - \phi\big(S_b(\alpha,\alpha,x_n)\big)\big] \\ &= 0.\end{aligned} \tag{5.5}$$

It follows that $\alpha = H(\alpha, \lambda)$. Thus $\lambda \in A$. Hence $A$ is closed in $[0,1]$. Let $\lambda_0 \in A$. Then there exists $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$. Since $U$ is open, there exists $r > 0$ such that $B_{S_b}(x_0, r) \subseteq U$. Choose $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$ such that $|\lambda - \lambda_0| \leq \frac{1}{M^n} < \varepsilon$. Then, for

$$x \in \overline{B_p(x_0, r)} = \{x \in X / S_b(x, x, x_0) \leq r + b^2 S_b(x_0, x_0, x_0)\},$$

$$\begin{aligned}
&S_b\big(H(x,\lambda),H(x,\lambda),x_0\big)\\
&\quad= S_b\big(H(x,\lambda),H(x,\lambda),H(x_0,\lambda_0)\big)\\
&\quad\le 2bS_b\big(H(x,\lambda),H(x,\lambda),H(x,\lambda_0)\big)+b^2S_b\big(H(x,\lambda_0),H(x,\lambda_0),H(x_0,\lambda_0)\big)\\
&\quad\le 2bM|\lambda-\lambda_0|+b^2S_b\big(H(x,\lambda_0),H(x,\lambda_0),H(x_0,\lambda_0)\big)\\
&\quad\le \frac{2b}{M^{n-1}}+b^2S_b\big(H(x,\lambda_0),H(x,\lambda_0),H(x_0,\lambda_0)\big).
\end{aligned}$$

Letting $n\to\infty$, we obtain

$$S_b\big(H(x,\lambda),H(x,\lambda),x_0\big)\le b^2S_b\big(H(x,\lambda_0),H(x,\lambda_0),H(x_0,\lambda_0)\big).$$

Since $\Psi$ is continuous and non-decreasing, we have

$$\begin{aligned}
\psi\big(S_b\big(H(x,\lambda),H(x,\lambda),x_0\big)\big)&\le \psi\big(4b^2S_b\big(H(x,\lambda),H(x,\lambda),x_0\big)\big)\\
&\le \psi\big(4b^4S_b\big(H(x,\lambda_0),H(x,\lambda_0),H(x_0,\lambda_0)\big)\big)\\
&\le \psi\big(S_b(x,x,x_0)\big)-\phi\big(S_b(x,x,x_0)\big)\\
&\le \psi\big(S_b(x,x,x_0)\big).
\end{aligned}$$

Since $\Psi$ is non-decreasing, we have

$$\begin{aligned}
S_b\big(H(x,\lambda),H(x,\lambda),x_0\big)&\le S_b(x,x,x_0)\\
&\le r+b^2S_b(x_0,x_0,x_0).
\end{aligned}$$

Thus, for each fixed

$$\lambda\in(\lambda_0-\varepsilon,\lambda_0+\varepsilon), H(\cdot,\lambda):\overline{B_p(x_0,r)}\to\overline{B_p(x_0,r)}.$$

Since also (ii) holds and $\Psi$ is continuous and non-decreasing and $\phi$ is continuous with $\phi(t)>0$ for $t>0$, then all the conditions of Theorem 26 are satisfied. Thus we deduce that $H(\cdot,\lambda)$ has a fixed point in $\overline{U}$. But this fixed point must be in $U$ since (i) holds. Thus $\lambda\in A$ for any $\lambda\in(\lambda_0-\varepsilon,\lambda_0+\varepsilon)$. Hence $(\lambda_0-\varepsilon,\lambda_0+\varepsilon)\subseteq A$ and therefore $A$ is open in $[0,1]$. For the reverse implication, we use the same strategy. □

**Corollary 27.** *Let $(X,p)$ be a complete partial metric space, $U$ be an open subset of $X$ and $H:\overline{U}\times[0,1]\to X$ with the following properties:*

*(1) $x \neq H(x,t)$ for each $x \in \partial U$ and each $\lambda \in [0,1]$ (here $\partial U$ denotes the boundary of $U$ in $X$ ),*

*(2) there exist $x, y \in \overline{U}$ and $\lambda \in [0,1], L \in [0, \frac{1}{4b^4})$ such that*

$$S_b\big(H(x,\lambda), H(x,\lambda), H(y,\mu)\big) \leq LS_b(x,x,y),$$

*(3) there exists $M \geq 0$ such that*

$$S_b\big(H(x,\lambda), H(x,\lambda), H(x,\mu)\big) \leq M|\lambda - \mu|$$

*for all $x \in \overline{U}$ and $\lambda, \mu \in [0,1]$. If $H(\cdot,0)$ has a fixed point in $U$, then $H(\cdot,1)$ has a fixed point in $U$.*

*Proof.* Proof follows by taking $\psi(x) = x, \phi(x) = x - Lx$ with $L \in [0, \frac{1}{4b^4})$ in Theorem (26). □

## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

## REFERENCES


[1] S. Banach, Theorie des Operations Lineaires, Monografie Matematyczne, Warsaw, 1932.

[2] A. Aghajani, M. Abbas, J.R. Roshan, Common fixed point of generalized weak contractive mappings in partially ordered $G_b$-metric spaces, Filomat 28 (2014), 1087-1101.

[3] A.H. Ansari, O. Ege, S. Radenović, Some fixed point results on complex valued $G_b$-metric spaces, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 111 (2017), 1-12.

[4] A.H. Ansari, P. Vetro, S. Radenović, Integration of type pair (H, F) upclass in fixed point result for $GP_{(\Lambda,\Theta)}$ contractive mappings, Filomat 31 (2017), 2211-2218.

[5] M. Arshad, Z. Kadelburg, S. Radenović, A. Shoaib, S. Shukla, $\alpha$-dominated mappings on dislocated metric spaces and fixed point results, Filomat 31 (2017), 3041-3056.

[6] I.A. Bakhtin, The contraction mapping principle in quasimetric spaces, Funct. Anal. Unianowsk Gos. Ped. Inst. 30 (1989), 26-37.

[7] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inform. Univ. Ostrav. 1 (1993), 5-11.

[8] T. Došenović, S. Radenović, Some critical remarks on the paper: "An essential remark on fixed point results on multiplicative metric spaces", J. Adv. Math. Stud. 10 (2017), 20-24.

[9] M.M.M. Jaradat, Z. Mustafa, A.H. Ansari, P.S. Kumari, D.D. Djekić, H.M. Jaradat, Some fixed point results for $F_{\alpha-\omega,\varphi}$-generalized cyclic contractions on metric-like space with applications to graphs and integral equations, J. Math. Anal. 8 (2017), 28-45.

[10] G.N.V. Kishore, K.P.R. Rao, V.M.L. Hima Bindu, Suzuki type unique common fixed point theorem in partial metric spaces by using (C): condition with rational expressions, Afr. Mat. 28 (2017), 1127-1138.

[11] V. Lakshmikantham, L. Ćirić, Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal. 70 (2009), 4341-4349.

[12] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7 (2006), 289-297.

[13] Z. Mustafa, J.R. Roshan, V. Parvaneh, Z. Kadelburg, Some common fixed point results in ordered partial b-metric space, J. Inequal. Appl. 2013 (2013), 562.

[14] Z. Mustafa, J.R. Roshan, V. Parvaneh, Z. Kadelburg, Fixed point theorems for weakly T-Chatterjea and weakly T-Kannan contractions in b-metric spaces, J. Inequal. Appl. 2014 (2014), 46.

[15] Z. Mustafa, M. Jaradat, A.H. Ansari, B.Z. Popović, H. Jaradat, C-class functions with new approach on coincidence point results for generalized $(\psi,\phi)$-weakly contractions in ordered b-metric spaces, SpringerPlus 5 (2016), 802.

[16] Z. Mustafa, M.M.M. Jaradat, H.M. Jaradat, Some common fixed point results of graphs on b-metric space, J. Nonlinear Sci. Appl. 9 (2016), 4838-4851.

[17] Z. Mustafa, H. Huang, S. Radenović, Some remarks on the paper "Some fixed point generalizations are not real generalizations", J. Adv. Math. Stud. 9 (2016), 110-116.

[18] S. Sedghi, I. Altun, N. Shobe, M. Salahshour, Some properties of S-metric space and fixed point results, Kyungpook Math. J. 54 (2014), 113-122.

[19] Y. Rohen, T. Došenović, S. Radenović, A note on the paper "A fixed point theorems in $S_b$-metric spaces", Filomat 31 (2017), 3335-3346.

[20] S. Sedghi, A. Gholidahneh, T. Došenović, J. Esfahani, S. Radenović, Common fixed point of four maps in $S_b$-metric spaces, J. Linear Topol. Algebra 5 (2016), 93-104.

[21] S. Sedghi, M.M. Rezaee, T. Došenović, R. Sü, Common fixed point theorems for contractive mappings satisfying $\phi$-maps in S-metric spaces, Acta Univ. Sapientiae Math. 8 (2016), 298-311.

[22] S. Sedghi, N. Shobe, A. Aliouche, A generalization of fixed point theorem in S-metric spaces, Mat. Vesn. 64 (2012), 258-266.

[23] S. Sedghi, N. Shobe, T. Došenović, Fixed point results in S-metric spaces, Nonlinear Funct. Anal. Appl. 20 (2015), 55-67.

[24] M. Zhou, X.L. Liu, S. Randenović, S-$\gamma-\phi$-contractive type mappings in S-metric spaces, J. Nonlinear Sci. Appl. 10 (2017), 1613-1639.