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STRONG CONVERGENCE OF AN ITERATIVE ALGORITHM FOR SUMS OF TWO MONOTONE OPERATORS

SUN YOUNG CHO

Department of Mathematics, Gyeongsang National University, Jinju 660-701, Korea

Abstract. In this work, an iterative algorithm for treating zero points of the sum of two monotone operators is investigated. Strong convergence of the algorithm is obtained in the framework of Hilbert spaces.

Keywords: variational inequality; monotone operator; nonexpansive mapping; solution.

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1. Introduction

Splitting methods have recently received much attention due to the fact that many nonlinear problems arising in applied areas such as image recovery, signal processing, and machine learning are mathematically modeled as a nonlinear operator equation and this operator is decomposed as the sum of two nonlinear operators; see [1-10] and the references therein.

In this paper, we study a splitting iterative algorithm for treating zero points of the sum of an inverse-strongly monotone and a maximal monotone operator. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a splitting

E-mail address: ooly61@yahoo.co.kr

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iterative algorithm is investigated. A strong convergence of the algorithm is obtained in the framework of Hilbert spaces.

2. Preliminaries

Let C be a nonempty, closed and convex subset of a real Hilbert space H. Let $A : C \to H$ be a mapping. Recall that A is said to be monotone iff

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C.$$

Recall that A is said to be inverse-strongly monotone iff there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

For such a case, A is also said to be α-inverse-strongly monotone.

Recall that a set-valued mapping $B : H \to 2^H$ is said to be monotone iff, for all $x, y \in H$, $f \in Bx$ and $g \in By$ imply $\langle x - y, f - g\rangle > 0$. In this paper, we use $B^{-1}(0)$ to stand for the zero point of B. A monotone mapping $B : H \to 2^H$ is maximal iff the graph $Graph(B)$ of B is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping B is maximal if and only if, for any $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$, for all $(y, g) \in Graph(B)$ implies $f \in Bx$. For a maximal monotone operator B on H, and $r > 0$, we may define the single-valued resolvent $J_r : H \to Dom(B)$, where $Dom(B)$ denote the domain of B. It is known that J_r is firmly nonexpansive, and $B^{-1}(0) = F(J_r)$.

Let $S : C \to C$ be a mapping. $F(S)$ stands for the fixed point set of S; that is, $F(S) := \{x \in C : x = Sx\}$. Recall that S is said to be nonexpansive iff

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C.$$

In order to prove our main results, we also need the following lemmas.

Lemma 2.1 [11] *Let $\{a_n\}$ be a sequence of nonnegative numbers satisfying the condition $a_{n+1} \leq (1 - t_n)a_n + t_n b_n + c_n, \forall n \geq 0$, where $\{t_n\}$ is a number sequence in $(0, 1)$ such that $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=0}^{\infty} t_n = \infty$, $\{b_n\}$ is a number sequence such that $\limsup_{n\to\infty} b_n \leq 0$, and $\{c_n\}$ is a positive number sequence such that $\sum_{n=0}^{\infty} c_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.*

Lemma 2.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a mapping, and $B : H \to 2^H$ a maximal monotone operator. Then $F(J_r(I - rA)) = (A + B)^{-1}(0)$.*

Proof. Notice that

$$p \in F(J_r(I - rA)) \Longleftrightarrow p - rAp \in p + rBp \Longleftrightarrow p \in (A + B)^{-1}(0).$$

This completes the proof.

Lemma 2.3 [12] *Let E be a Banach space, and A an m-accretive operator. For $\lambda > 0$, $\mu > 0$, and $x \in E$, we have*

$$J_\lambda x = J_\mu \Big(\frac{\mu}{\lambda} x + \Big(1 - \frac{\mu}{\lambda}\Big) J_\lambda x\Big),$$

where $J_\lambda = (I + \lambda A)^{-1}$ and $J_\mu = (I + \mu A)^{-1}$.

3. Main results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping and let B be a maximal monotone operator on H. Assume that $Dom(B) \subset C$ and $(A + B)^{-1}(0)$ is not empty. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$ and*

$$x_{n+1} = J_{r_n}(\alpha_n u + (1 - \alpha_n)x_n - r_n A(\alpha_n u + (1 - \alpha_n)x_n)), \quad \forall n \geq 0,$$

where $J_{r_n} = (I + r_n B)^{-1}$, u is a fixed element in C, $\{\alpha_n\}$ is a real number sequence in $(0,1)$ and $\{r_n\}$ is a positive real number sequence in $(0,2\alpha)$. If the control sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;
(b) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

then $\{x_n\}$ converges strongly to a point $\bar{x} \in (A + B)^{-1}(0)$, where $\bar{x} = Proj_{(A+B)^{-1}(0)}u$.

Proof. First, we prove that $\{x_n\}$ is bounded. Since A is inverse-strongly monotone, we find that $I-r_nA$ is nonexpansive. Indeed, we have

$$\begin{aligned}
&\|(I-r_nA)x-(I-r_nA)y\|^2\\
&=\|x-y\|^2-2r_n\langle x-y,Ax-Ay\rangle+{r_n}^2\|Ax-Ay\|^2\\
&\le\|x-y\|^2-r_n(2\alpha-r_n)\|Ax-Ay\|^2.
\end{aligned}$$

In view of the restriction (b), we find that $I-r_nA$ is nonexpansive. Putting $y_n=\alpha_nu+(1-\alpha_n)x_n$ and fixing $p\in(A+B)^{-1}(0)$, we find that

$$\|y_n-p\|\le\alpha_n\|u-p\|+(1-\alpha_n)\|x_n-p\|.$$

It follows that

$$\begin{aligned}
\|x_{n+1}-p\|&\le\|(y_n-r_nAy_n)-(p-r_nAp)\|\\
&\le\|y_n-p\|\\
&\le\alpha_n\|u-p\|+(1-\alpha_n)\|x_n-p\|.
\end{aligned}$$

This proves that the sequence $\{x_n\}$ is bounded, so is $\{y_n\}$. Notice that

$$\|y_n-y_{n-1}\|\le(1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|u-x_{n-1}\|.$$

Putting $z_n=y_n-r_nAy_n$, we find that

$$\begin{aligned}
\|z_n-z_{n-1}\|&\le\|y_n-y_{n-1}\|+\|r_n-r_{n-1}\|\|Ay_{n-1}\|\\
&\le(1-\alpha_n)\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|u-x_{n-1}\|+\|r_n-r_{n-1}\|\|Ay_{n-1}\|
\end{aligned}$$

In view of Lemma 2.3, we find that

$$
\begin{aligned}
\|x_{n+1}-x_n\| &= \|J_{r_n}z_n - J_{r_{n-1}}z_{n-1}\| \\
&= \|J_{r_{n-1}}\big(\frac{r_{n-1}}{r_n}z_n + (1-\frac{r_{n-1}}{r_n})J_{r_n}z_n\big) - J_{r_{n-1}}z_{n-1}\| \\
&\le \|\frac{r_{n-1}}{r_n}(z_n - z_{n-1}) + (1-\frac{r_{n-1}}{r_n})(J_{r_n}z_n - z_{n-1})\| \\
&\le \|(z_n - z_{n-1}) + (1-\frac{r_{n-1}}{r_n})(J_{r_n}z_n - z_n)\| \\
&\le \|z_n - z_{n-1}\| + \frac{|r_n - r_{n-1}|}{a}\|J_{r_n}z_n - z_n\| \\
&\le (1-\alpha_n)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|u - x_{n-1}\| \\
&\quad + \|r_n - r_{n-1}\|\|Ay_{n-1}\| + \frac{|r_n - r_{n-1}|}{a}\|J_{r_n}z_n - z_n\|.
\end{aligned}
$$

It follows from Lemma 2.1 that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.1}$$

Since $\|y_n - x_n\| \le \alpha_n\|u - x_n\|$, we find from the above that

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{3.2}$$

Notice that

$$\|y_n - x_{n+1}\| \le \|y_n - x_n\| + \|x_n - x_{n+1}\|.$$

Combining (3.1) with (3.2), we find that

$$\lim_{n\to\infty}\|y_n - x_{n+1}\| = 0. \tag{3.3}$$

Next, we show that

$$\limsup_{n\to\infty}\langle u-\bar{x}, y_n - \bar{x}\rangle \le 0,$$

where $\bar{x} = Proj_{(A+B)^{-1}(0)}u$ (notice that $Proj_{(A+B)^{-1}(0)}f$ is contractive). To show this inequality, we choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$ such that

$$\limsup_{n\to\infty}\langle f(\bar{x})-\bar{x}, y_n - \bar{x}\rangle = \lim_{i\to\infty}\langle f(\bar{x})-\bar{x}, y_{n_i} - \bar{x}\rangle \le 0,$$

Since $\{y_{n_i}\}$ is bounded, we find that there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converges weakly to $\hat{x}$. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup \hat{x}$.

Now, we show that $\hat{x} \in (A+B)^{-1}(0)$. Notice that $y_n - r_n A y_n \in x_{n+1} + r_n B x_{n+1}$; that is,

$$\frac{y_n - r_n A y_n - x_{n+1}}{r_n} \in Bx_{n+1}.$$

Let $\mu \in B\nu$. Since B is monotone, we find that

$$\left\langle \frac{y_n - x_{n+1}}{r_n} - Ay_n - \mu, x_{n+1} - \nu \right\rangle \geq 0.$$

It follows from the restriction (b) that $\langle -A\hat{x} - \mu, \hat{x} - \nu \rangle \geq 0$. This implies that $-A\hat{x} \in B\hat{x}$, that is, $\hat{x} \in (A+B)^{-1}(0)$. This proves that

$$\limsup_{n\to\infty} \langle u - \bar{x}, y_n - \bar{x} \rangle \leq 0. \tag{3.4}$$

Notice that

$$\begin{aligned} \|y_n - \bar{x}\|^2 &\leq (1-\alpha_n)\|x_n - \bar{x}\|\|y_n - \bar{x}\| + \alpha_n \langle u - \bar{x}, y_n - \bar{x} \rangle \\ &\leq \frac{1-\alpha_n}{2}(\|x_n - \bar{x}\|^2 + \|y_n - \bar{x}\|^2) + \alpha_n \langle u - \bar{x}, y_n - \bar{x} \rangle. \end{aligned}$$

This implies that

$$\|y_n - \bar{x}\|^2 \leq (1-\alpha_n)\|x_n - \bar{x}\|^2 + \alpha_n \langle u - \bar{x}, y_n - \bar{x} \rangle.$$

Hence, we have

$$\begin{aligned} \|x_{n+1} - \bar{x}\|^2 &\leq \|J_{r_n}(y_n - r_n A y_n) - \bar{x}\|^2 \\ &\leq \|(y_n - r_n A y_n) - (I - r_n A)\bar{x}\|^2 \\ &\leq \|y_n - \bar{x}\|^2 \\ &\leq (1-\alpha_n)\|x_n - \bar{x}\|^2 + \alpha_n \langle u - \bar{x}, y_n - \bar{x} \rangle. \end{aligned}$$

An application of Lemma 2.1 to the above inequality yields that $\lim_{n\to\infty} \|x_n - \bar{x}\| = 0$. This completes the proof.

Let $A : C \to H$ be a monotone operator. Recall that the classical variational inequality, denoted by $VI(C,A)$, is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C.$$

One can see that the variational inequality is equivalent to a fixed point problem. The element $u \in C$ is a solution of the variational inequality iff $u \in C$ satisfies the equation $u = P_C(u - rAu)$, where $r > 0$ is a constant.

Corollary 3.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be an α-inverse-strongly monotone mapping such that $VI(C,A) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C in the following process: $x_0 \in C$ and*

$$x_{n+1} = Proj_C(\alpha_n u + (1-\alpha_n)x_n - r_n A(\alpha_n u + (1-\alpha_n)x_n)), \quad \forall n \geq 0,$$

where u is a fixed element in C, $\{\alpha_n\}$ is a real number sequence in $(0,1)$ and $\{r_n\}$ is a positive real number sequence in $(0,2\alpha)$. If the control sequences satisfy the following restrictions:

(a) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;

(b) $0 < a \leq r_n \leq b < 2\alpha$ *and* $\sum_{n=1}^{\infty} |r_n - r_{n-1}| < \infty$,

then $\{x_n\}$ converges strongly to a point $\bar{x} \in VI(C,A)$, where $\bar{x} = Proj_{VI(C,A)} u$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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