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SUZUKI-TYPE APPROXIMATE FIXED POINTS IN G-METRIC SPACES AND APPLICATIONS

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Abstract. In this paper, we will first introduce Suzuki-type fixed point property on G-metric spaces. Also, we give a sufficient condition on metric spaces possessing the Banach fixed point property (BFPP). Furthermore, we give a sufficient condition for not having BFPP on the G-metric.

Keywords: Suzuki-type fixed point; Banach fixed point property; η-chainable; $G_\alpha-$contraction.

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1. Introduction

Fixed point theory is a very popular tool in solving existence problems in many branches of Mathematical Analysis and its applications. In physics and engineering fixed point technique has been used in areas like image retrieval, signal processing and the study of existence and uniqueness of solutions for a class of nonlinear integral equations. Some functional equations arise in multistage decision processes where the origin of the theory of dynamic programming lies [3, 4]. The existence and uniqueness of the solutions of these functional equations have been studied by several authors [5, 15, 16, 25, 26] using fixed point theorems. Some recent work

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on fixed point theorems of integral type in G-metric spaces, stability of functional difference equation can be found in [24, 28] and the references therein.

In 1968, Kannan (see [14]) proved a fixed point theorem for operators which need not be continuous. Further, Chatterjea (see[11]), in 1972, also proved a fixed point theorem for discontinuous mapping, which is actually a kind of dual of Kannan mapping. In 1972, by combining the above three independent contraction conditions above, Zamfirescu (see [31]) obtained another fixed point result for operators which satisfy the following. In 2001, Rus (see [29]) defined $\alpha-$contraction. In [6], the author obtained a different contraction condition, also he formulated a corresponding fixed point theorem. In 2006, Berinde (see [7]) obtained some result on $\alpha-$contraction for approximate fixed point in metric space. In 2008, Suzuki [27] introduced a new type of mapping and obtained the following interesting and simple generalization of Banach contraction principle.

In [10], Chandok et al. obtained some result on Coupled common fixed point in partially ordered G-metric spaces. On the other hand, in 2006, Mustafa and Sims [22, 23] introduced the notion of generalized metric spaces or simply G-metric spaces. Many researchers have obtained fixd point, coupled fixed point, coupled common fixed point results on G-metric spaces (see [2, 10, 24]). Recently, Mohsenalhosseini et al in [21] introduced the approximate fixed point in G-metric spaces for various types of operators.

The aim of this paper is to introduce the new classes of operators and contraction maps regarding approximate fixed point and diameter approximate fixed point for a cyclic map $T : A\cup B\cup C \to A\cup B\cup C$ i.e. $T(A) \subseteq B$, $T(B) \subseteq C$ and $T(C) \subseteq A$ on G-metric spaces. Also, we give some illustrative example of our main results.

2. Preliminaries

This section recalls the following notations and the ones that will be used in what follows.

Definition 2.1. [22] Let X be a nonempty set and let $G : X\times X\times X \longrightarrow R^+$ be a function satisfying the following properties:

(G1) $G(x,y,z) = 0$ *if and only if* $x = y = z$;

(G2) $0 < G(x,x,y)$ *for all* $x,y \in X$ *with* $x \neq y$;

(G3) $G(x,x,y) \leq G(x,y,z) for\ all\ x,y,z \in X with\ z \neq y$;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots (symmetry\ in\ all\ three\ variables)$;

(G5) $G(x,y,z) \leq G(x,a,a) + G(a,y,z) for\ all\ x,y,z,a \in X(\ rectangle\ inequality)$.

Then, the function G is called generalized metric or, more specifically G-metric on X, and the pair (X,G) is called a G-metric space.

Proposition 2.2. [22] *Every G-metric (X,G) defines a metric space (X,d_G) by*

1) $d_G(x,y) = G(x,y,y) + G(y,x,x)$.

if (X,G) is a symmetric G- metric space. Then

2) $d_G(x,y) = 2G(x,y,y)$.

Definition 2.3. [22] *Let (X,G) be a G-metric space, then a sequence $(x_n) \in X$ is said to be G-Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $n,m,l \geq N$.*

Definition 2.4. [20] *Let $T : X \to X$, $\varepsilon > 0$, $x_0 \in X$. Then $x_0 \in X$ is an $\varepsilon-$fixed point for T if $\|Tx_0 - x_0\| < \varepsilon$.*

Remark 2.5. [20] *In this paper we will denote the set of all $\varepsilon-$ fixed points of T, for a given ε, by :*

$$AF(T) = \{x \in X \mid x\ is\ an\ \varepsilon - fixed\ point\ of\ T\}.$$

Definition 2.6. [20] Let $T : X \to X$. Then T has the approximate fixed point property (a.f.p.p) if

$$\forall \varepsilon > 0,\ AF(T) \neq \varnothing.$$

Theorem 2.7. [20] *Let $(X,\|.\|)$ be a complete norm space, $T : X \to X$, $x_0 \in X$ and $\varepsilon > 0$. If $\|T^n(x_0) - T^{n+k}(x_0)\| \to 0$ as $n \to \infty$ for some $k > 0$, then T^k has an $\varepsilon-$ fixed point.*

Theorem 2.8. [9] *Let (X,d) be a complete metric space and let T be a contraction on X, that is, there exists $\alpha \in (0,1)$ such that $d(Tx,Ty) \leq \alpha d(x,y)$ for all $x,y \in X$. Then T has a unique fixed point z and $\{T^n x\}$ converges to z for any $x \in X$.*

Definition 2.9. [13] A metric space (X,d) is said to possess the Banach fixed point property (BFPP) if every contraction on X has a approximate fixed point.

Remark 2.10. [13] *Let* $\{x_n\} \in CauS(X)$*. Then it is well known that a function* θ *from* X *into* $[0,\infty)$ *defined by*

$$\theta(x) = lim_{n\to\infty} d(x,x_n) \tag{2.1}$$

for $x \in X$ *is well defined. Also it is well known that*

$$|\theta(x) - \theta(y)| \leq d(x,y) \leq \theta(x) + \theta(y) \tag{2.2}$$

for any $x,y \in X$*.*

3. Main Results

We begin with two lemmas which will be used in order to prove all the results given in third section. Let (X,G) be a G-metric space. Throughout this paper we denote by G-CauS(X) the set of all G-Cauchy sequences in X

Definition 3.1. [21] Let A,B,C are closed subsets of a G-metric space X and $T : A\cup B\cup C \to A\cup B\cup C$ be a cyclic map. Let $\varepsilon > 0$ and $x_0 \in A\cup B\cup C$. Then x_0 is an ε- fixed point of T if

$$[G(x_0,Tx_0,Tx_0) + G(Tx_0,x_0,x_0)] < \varepsilon.$$

Definition 3.2. [21] Let A,B and C be non-empty subsets of a $G-$metric space X. A mapping $T : A\cup B\cup C \to A\cup B\cup C$ is a $G_\alpha-$contraction if there exists $\alpha \in (0,1)$

$$G(Tx,Ty,Ty) + G(Ty,Tx,Tx) \leq \alpha[G(x,y,y) + G(y,x,x)], \forall x,y \in A\cup B\cup C.$$

Remark 3.3. In this paper we will denote the set of all ε-fixed points of T, for a given ε, by:

$$F_G^\varepsilon(T) = \{x \in A\cup B\cup C \mid x\ is\ an\ \varepsilon - fixed\ point\ of\ T\}.$$

Definition 3.4. [21] Let A,B,C are closed subsets of a G-metric space X, $T : A\cup B\cup C \to A\cup B\cup C$ be a cyclic map and $\varepsilon > 0$. We define diameter of the set $F_G^\varepsilon(T)$, i.e.,

$$\delta(F_G^\varepsilon(T)) = \sup\{G(x,y,z) :\ x,y,z \in F_G^\varepsilon(T)\}.$$

Definition 3.5. [21] Let A,B,C are closed subsets of a G-metric space X and $T : A\cup B\cup C \to A\cup B\cup C$ be a cyclic map. Then T has the approximate fixed point property (a.f.p.p) if $\forall\varepsilon > 0$,

$$F_G^\varepsilon(T) \neq \emptyset.$$

Definition 3.6. [21] Let A,B,C are closed subsets of a G-metric space X. A cyclic map $T: A\cup B\cup C \to A\cup B\cup C$ is said to be asymptotically regular at a point $x \in A\cup B\cup C$, if

$$\lim_{n\to\infty}\{G(T^n x, T^{n+1}x, T^{n+1}x) + G(T^{n+1}x, T^n x, T^n x)\} = 0,$$

where T^n denotes the nth iterate of T at x.

Lemma 3.7. [21] *Let A,B,C are closed subsets of a G-metric space X. If $T: A\cup B\cup C \to A\cup B\cup C$ is asymptotically regular at a point $x \in A\cup B\cup C$, Then T has an approximate fixed point.*

Lemma 3.8. [21] *Let A,B,C are closed subsets of a G-metric space X, $T: A\cup B\cup C \to A\cup B\cup C$ a cyclic map and $\varepsilon > 0$. We assume that:*
a) $F_G^{\varepsilon}(T) \neq \emptyset$;
b) $\forall \xi > 0\ \exists \psi(\xi) > 0$ *such that*

$$[G(x,y,y) + G(y,x,x)] - [G(Tx,Ty,Ty) + G(Ty,Tx,Tx)] < \xi \Longrightarrow$$

$$G(x,y,y) + G(y,x,x) \leq \psi(\xi), \quad \forall x,y \in F_G^{\varepsilon}(T).$$

Then:

$$\delta(F_G^{\varepsilon}(T)) \leq \psi(2\varepsilon).$$

Definition 3.9. Let A,B,C are closed subsets of a G-metric space X and $T: A\cup B\cup C \to A\cup B\cup C$ be a cyclic map. Let $\varepsilon > 0$ and $x_0 \in A\cup B\cup C$. Then x_0 is an Banach ε- fixed point of T if

$$[G(x_0, Tx_0, Tx_0) + G(Tx_0, x_0, x_0)] < \varepsilon.$$

Remark 3.10. In this paper we will denote the set of all Banach ε-fixed points of T, for a given ε, by:

$$BF_G^{\varepsilon}(T) = \{x \in A\cup B\cup C \mid x \ is\ a\ Banach\ \varepsilon - fixed\ point\ of\ T\}.$$

Definition 3.11. A G-metric space (X,G) is said to possess the Banach ε-fixed point property if every contraction on X has a Banach ε-fixed point.

Theorem 3.12. *Let A, B and C be non-empty subsets of a $G-$metric space X. Suppose that the mapping $T : A \cup B \cup C \to A \cup B \cup C$ satisfying $T(A) \subset B, T(B) \subset C$ and $T(C) \subset A$ is a $G_\alpha-$contraction. Then for every $\varepsilon > 0$, $F_G^\varepsilon(T) \neq \emptyset$.*

Definition 3.13. Let (X,G) be a G-metric space and ρ be a function from $X \times X \times G-CauS(X)$ into $[0,\infty]$. Then X is said to satisfy Condition ρ if for every $\{x_n\} \in G-CauS(X)$, there exists $\omega \in X$ such that $\rho(\omega,\{x_n\},\{x_n\}) + \rho(\{x_n\},\omega,\omega) < \infty$. and

$$\frac{1}{2}\rho(\omega,\{x_n\},\{x_n\}) + \rho(\{x_n\},\omega,\omega) < \rho(x,\{x_n\},\{x_n\}) + \rho(\{x_n\},x,x)$$

for any $x \in X \setminus \{\omega\}$.

Theorem 3.14. *Let (X,G) be a G-metric space and ρ be a function from $X \times X \times G-CauS(X)$ into $[0,\infty]$. Let T be a mapping on X and $\varepsilon > 0$. Suppose the following:*

(i) X satisfies Condition ρ.

(ii) There exists $\omega \in X$ such that $\{T^n\omega\} \in G-CauS(X)$.

(iii) There exists $\alpha \in (0,1)$ such that

(3.1)

$$\rho(Tx,\{T^n\omega\},\{T^n\omega\}) + \rho(\{T^n\omega\},Tx,Tx) \leq \alpha(\rho(x,\{T^n\omega\},\{T^n\omega\}) + \rho(\{T^n\omega\},x,x))$$

for any $x \in X$. Then the following hold:

(a) T has a Banach fixed point z.

(b) $\rho(z,\{T^n\omega\},\{T^n\omega\}) + \rho(\{T^n\omega\},z,z) = 0$ holds.

(c) $lim_{m\to\infty}\rho(T^m x,\{T^n\omega\},\{T^n\omega\}) + \rho(\{T^n\omega\},T^m x,T^m x) = 0$ holds for any $x \in X$.

Proof. Since X satisfies Condition (ρ), there exists $z \in X$ such that

$$\rho(z,\{T^n\omega\},\{T^n\omega\}) + \rho(\{T^n\omega\},z,z) < \infty$$

and

$$\frac{1}{2}(\rho(z,\{T^n\omega\},\{T^n\omega\}) + \rho(\{T^n\omega\},z,z)) < \rho(x,\{T^n\omega\},\{T^n\omega\}) + \rho(\{T^n\omega\},x,x)$$

for any $x \in X \setminus \{z\}$. We consider the following two cases:

* $\rho(z,\{T^n\omega\},\{T^n\omega\}) + \rho(\{T^n\omega\},z,z) = 0$

* $\rho(z,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},z,z)=0$

In the first case, we note $\rho(z,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},z,z)=0$ for any $x\in X\setminus\{z\}$. Since $\rho(Tz,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},Tz,Tz)=0$ holds from 3.1, z is a Banach fixed point T. In the second case, from 3.1 we have

$$lim_{m\to\infty}\rho(T^mz,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},T^mz,T^mz)\leq$$
$$r^m(lim_{m\to\infty}\rho(z,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},z,z)=0.$$

Hence there exists $\lambda\in N$ satisfying

$$\begin{aligned}
& \rho(T^\lambda z,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},T^\lambda z,T^\lambda z) \\
< & \frac{1}{2}(\rho(z,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},z,z)) \\
\leq & min\{inf\{(\rho(x,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},x,x) \\
: & x\in X\setminus\{z\}\},(\rho(z,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},z,z)\} \\
= & inf\{(\rho(x,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},x,x):x\in X\}
\end{aligned}$$

which is a contradiction. So the second case cannot be possible. As above,

$$lim_{m\to\infty}\rho(T^mx,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},T^mx,T^mx)=0$$

holds for any $x\in X$. Therefore the Banach fixed point z is unique. □

Definition 3.15. Let (X,G) be a G-metric space. A mapping $T:X\to X$ is a $G_\alpha-$contraction if there exists $\alpha\in(0,1)$

$$G(Tx,Ty,Ty)+G(Ty,Tx,Tx)\leq\alpha[G(x,y,y)+G(y,x,x)],\forall x,y\in X.$$

By 2.2 , 2.10 we have the following remark:

Remark 3.16. *Let $\{x_n\}\in G-CauS(X)$. Then it is well known that a function θ from X into $[0,\infty)$ defined by*

$$\theta(x)=lim_{n\to\infty}(G(x,x_n,x_n)+G(x_n,x,x)) \tag{3.2}$$

for $x \in X$ is well defined. Also it is well known that

$$|\theta(x)-\theta(y)| \leq (G(x,y,y)+G(y,x,x)) \leq \theta(x)+\theta(y) \tag{3.3}$$

for any $x,y \in X$.

Theorem 3.17. *Let (X,G) be a G-metric space and ρ be a function from $X \times X \times G-CauS(X)$ into $[0,\infty]$. Let T be a $G_\alpha-$contraction on X and $\varepsilon > 0$. Suppose the following:*
(i) X satisfies Condition ρ.
(ii) There exists $\omega \in X$ such that

$$lim_{m\to\infty} G(x,T^m\omega,T^m\omega)+G(\{T^m\omega,x,x) \leq \rho(x,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},x,x\})$$

for any $x \in X$.
(iii) There exists $\alpha \in (0,1)$ satisfying 3.1 for any $x \in X$.
Then T has a unique Banach fixed point z and $\{T^n x\}$ converges to z for any $x \in X$.

Proof. Since T is a $G_\alpha-$contraction, there exists $\beta \in (0,1)$ such that

$$G(Tx,Ty,Ty)+G(Ty,Tx,Tx) \leq \beta[G(x,y,y)+G(y,x,x)],$$

for all $x,y \in X$. Fix $\omega \in X$. Then since

$$\sum_{n=1}^{n=\infty} G(T^n\omega,T^{n+1}\omega,T^{n+1}\omega)+G(T^{n+1}\omega,T^n\omega,T^n\omega) \leq \beta^n[G(\omega,T\omega,T\omega)+G(T\omega,\omega,\omega)] < \infty,$$

we have $\{T^n\omega\} \in G-CauS(X)$. Thus, (ii) of Theorem 3.14 holds. So by Theorem 3.14, (a)–(c) of Theorem 3.14 hold. Thus, T has a unique Banach fixed point z. By 3.3 and (ii), we have

$$\begin{aligned} G(x,y,y)+G(y,x,x) &\leq lim_{n\to\infty}(G(x,T^m\omega,T^m\omega)+G(T^m\omega,x,x)) \qquad (3.4)\\ &+ lim_{n\to\infty}(G(y,T^m\omega,T^m\omega)+G(T^m\omega,y,y))\\ &\leq \rho(x,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},x,x))\\ &+ \rho(y,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},y,y)) \end{aligned}$$

for any $x,y \in X$. Using 3.4 and Theorem 3.14 (b) and (c), we have

$$\begin{aligned} lim_{n\to\infty}(G(T^m x,z,z)+G(z,T^m x,T^m x)) &\leq \rho(\{T^m x,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},T^m x,T^m x) \\ &+ \rho(z,\{T^n\omega\},\{T^n\omega\})+\rho(\{T^n\omega\},z,z)=0. \end{aligned}$$

for any $x\in X$. Therefore, the desired result is achieved. □

Definition 3.18. [12] A metric space X will be said to be η-chainable if for every $a,b\in X$ there exists an η-chain, that is a finite set of points $a=x_0,x_1,\cdots,x_n=b$ (n may depend on both a and b) such that $d(x_{i-1},x_i)<\eta$ $(i=1,2,\cdots,n)$.

Definition 3.19. A G-metric space X will be said to be η-chainable if for every $a,b\in X$ there exists an η-chain, that is a finite set of points $a=x_0,x_1,\cdots,x_n=b$ (n may depend on both a and b) such that $G(x_i,x_{i+1},x_{i+1})+G(x_{i+1},x_i,x_i)<\eta$ $(i=1,2,\cdots,m-1)$.

Definition 3.20. Let (X,G) be a G-metric space, let $x,y\in X$, $\{x_n\}\in CauS(X)$ and $\eta>0$.
$*$ (x,y) is said to be η-chainable if there exists η-chain linking x and y.
$*$ $(x,\{x_n\})$ is said to be η-chainable if there exists $\nu\in\mathbb{N}$ such that $(x,\{x_n\})$ is η-chainable for any $n\geq\nu$.

In this section, we let (X,G) be a G-metric space. For $\eta>0$, we define a function ρ_η from $X\times X\times G-CauS(X)$ into $[0,\infty]$ by

$$\begin{aligned} \rho_\varepsilon(x,\{x_n\},\{x_n\})+\rho_\varepsilon(\{x_n\},x,x) &= lim_{n\to\infty} supinf\{\sum_{j=1}^{j=m}G(x_i,x_{i+1},x_{i+1})+G(x_{i+1},x_i,x_i) \\ &: \{x_1,\cdots,x_m\}\ is\ \eta-chain\ linking\ x\ and\ x_n\}. \end{aligned}$$

where $inf\emptyset=\infty$. We also define a function ρ from $X\times X\times G-CauS(X)$ into $[0,\infty]$ by

$$\rho(x,\{x_n\},\{x_n\})+\rho(\{x_n\},x,x)=sup\{\rho_\varepsilon(x,\{x_n\},\{x_n\})+\rho_\varepsilon(\{x_n\},x,x):\varepsilon>0\} \tag{3.4}$$

for $(x,\{x_n\})\in X\times X\times G-CauS(X)$.

Lemma 3.21. *Let (X,G) be a G-metric space and let ρ be a function defined by 3.4. Let $\{x_n\}\in CauS(X)$ and define a function θ by 3.2. Then the following hold:*

(i) For any $x \in X$ and $\varepsilon, \varepsilon^{'} \in (0,\infty)$ with $\varepsilon < \varepsilon^{'}$,

$$\begin{aligned}\theta(x) &\leq \rho_{\varepsilon'}(x,\{x_n\},\{x_n\}) + \rho_{\varepsilon'}(\{x_n\},x,x)\\ &\leq \rho_{\varepsilon}(x,\{x_n\},\{x_n\}) + \rho_{\varepsilon}(\{x_n\},x,x)\\ &\leq \rho(x,\{x_n\},\{x_n\}) + \rho(\{x_n\},x,x)\}\end{aligned}$$

holds.

(ii) For any $\omega \in X$, $\{x_n\}$ converges to ω if and only if $\rho(\omega,\{x_n\},\{x_n\}) + \rho(\{x_n\},\omega,\omega) = 0$.

(iii) If X satisfies Condition (ρ), then $\{x_n\}$ does not converge if and only if $0 < inf\{\rho(x,\{x_n\},\{x_n\}) + \rho(\{x_n\},x,x) : x \in X\} < \infty$.

Proof. (i) is obvious. In order to show (ii),we fix $\omega \in X$. It follows from (i) that $\rho(\omega,\{x_n\},\{x_n\}) + \rho(\{x_n\},\omega,\omega) = 0$ implies that $\{x_n\}$ converges to ω. In order to show the converse implication, we assume that $\{x_n\}$ converges to ω. For any $\eta > 0$, a two-length sequence $\{\omega, x_n\}$ is $\eta - chain$ linking ω and $\{x_n\}$ for suffciently large $n \in N$. So, $\rho_{\varepsilon}(\omega,\{x_n\},\{x_n\}) + \rho_{\varepsilon}(\{x_n\},\omega,\omega) = 0$ holds and hence $\rho(\omega,\{x_n\},\{x_n\}) + \rho(\{x_n\},\omega,\omega) = 0$ holds. We shall show (iii). We put

$$m = inf\{\rho(x,\{x_n\},\{x_n\}) + \rho(\{x_n\},x,x) : x \in X\}.$$

Since X satisfies Condition (ρ), there exists $\omega \in X$ such that $\{\omega, x_n\} < \infty$ and $(\frac{1}{2})\rho(\omega,\{x_n\},\{x_n\}) + \rho(\{x_n\},\omega,\omega) < \rho(x,\{x_n\},\{x_n\}) + \rho(\{x_n\},x,x)$ for any $x \in X \backslash \omega$. So $t < \infty$ holds. We assume $t = 0$. Then $(\frac{1}{2})\rho(\omega,\{x_n\},\{x_n\}) + \rho(\{x_n\},\omega,\omega) = 0$ holds. So, by (ii), $\{x_n\}$ converges to ω. Conversely, we assume that $\{x_n\}$ converges to some $x \in X$. Then by (ii), we have $\rho(x,\{x_n\},\{x_n\}) + \rho(\{x_n\},x,x) = 0$ and hence $t = 0$. We have shown (iii). □

Theorem 3.22. *Let (X,G) be a complete G-metric space and let T be a mapping on X. Assume there exists $\alpha \in (0,1)$ such that for every $x,y \in X$,*

$$G(Tx,Ty,Ty) \leq \alpha G(x,y,y). \tag{3.5}$$

Then there exists a unique fixed point z of T and $\{T^n x\}$ converges to z for any $x \in X$.

Proof. It follows directly from Theorems 2.2 and 2.8. □

Theorem 3.23. *Let (X,G) be a G-metric space and let ρ be a function defined by 3.4. Assume that X satisfies Condition ρ. Let T be a $G_\alpha-$contraction on X. Then T has a unique fixed point z. Moreover $\{T^n x\}$ converges to z for any $x \in X$.*

Proof. There exists $\alpha \in (0;1)$ satisfying 3.23 for any $x,y \in X$. Fix $\omega \in X$. By Lemma 3.21, (ii) of Theorem 3.14 holds. In order to show (iii) of Theorem 3.14, we fix $x \in X$. We consider the following two cases:

a) $\rho(x,\{T^n\omega\},\{T^n\omega\}) + \rho(\{T^n\omega\},x,x) = \infty$

b) $\rho(x,\{T^n\omega\},\{T^n\omega\}) + \rho(\{T^n\omega\},x,x) < \infty$

In the first case,

$$\rho(x,\{T^n\omega\},\{T^n\omega\}) + \rho(\{T^n\omega\},x,x) \leq \infty = \alpha\rho(x,\{T^n\omega\},\{T^n\omega\}) + \rho(\{T^n\omega\},x,x)$$

holds. In the second case, we fix $\eta > 0$. Then from the definition of ρ, $(x,\{T^n\omega\})$ is "η-chainable. So, there exists $\omega \in \mathbb{N}$ such that $(x,\{T^n\omega\})$ is η-chainable for any $\geq \omega$. Fix $\geq \omega$ and let $\{y_1,y_2,\cdots,y_m\}$ be an arbitrary η-chain linking x and $\{T^n\omega\}$. Then since T is a contraction, $\{Ty_1,Ty_2,\cdots,Ty_m\}$ is $(\eta\varepsilon)$-chain linking Tx and $T^{n+1}\omega$ and hence is η-chain. Also

$$\sum_{j=1}^{j=m-1} G(Ty_j,Ty_{j+1},Ty_{j+1}) + G(Ty_{j+1},Ty_j,Ty_j) \leq \eta \sum_{j=1}^{j=m-1} G(y_j,y_{j+1},y_{j+1}) + G(y_{j+1},y_j,y_j)$$

is obvious. Since $\{y_1,y_2,\cdots,y_m\}$ is arbitrary, we obtain

$$\rho_\varepsilon(Tx,\{T^n\omega\},\{T^n\omega\}) + \rho(\{T^n\omega\},Tx,Tx) \leq \alpha\rho_\varepsilon(x,\{T^n\omega\},\{T^n\omega\}) + \rho(\{T^n\omega\},x,x)$$

Since $\varepsilon > 0$ is arbitrary, we obtain 3.1. We have shown (iii) of Theorem 3.14 holds. So by Theorem 3.14, we obtain the desired result. □

4. NOT POSSESSING BFPP

Theorem 4.1. [8] *Define a subset X of the 2-dimensional Euclidean space $\mathbb{R}^2$ by $X = \{0\} \cup_{k=1}^{\infty} L_k$ where $L_k = \{(t, t2^{-k}) : t \in (0,1]\}$ for $k \in \mathbb{N}$. Then X possesses BFPP.*

In 2007, Xiang proved some splendid results on BFPP. The following is one of them, which includes Theorem 4.1, but does not include Theorem 2.8.

Theorem 4.2. [30] *A locally Lipschitz-connected metric space possesses BFPP if and only if it is Lipschitz-complete*

In this section, we give a sufficient condition on not possessing BFPP. While Theorem 4.2 is of continuous type, the following is of discrete type in some sense.

Theorem 4.3. [30] (X,G) *be a G-metric space and let* $\{x_n\} \in CauS(X)$. *Assume that for any* $x \in X$, *there exists* $\varepsilon > 0$ *such that* $(x,\{x_n\})$ *is not* $\eta - chainable$. *Then* X *does not have* $BFPP$.

Proof. Define a function $\theta : X \to [0,1)$ by 3.2. From the assumption, fxng does not converge. So, $\theta > 0$ for any $x \in X$. Taking a subsequence, without loss of generality, we may assume $\theta(x_{n+1}) < \frac{\theta(x_n)}{3}$ for any $n \in \mathbb{N}$. Define a function $h : X \to [0,1)$ by

$$h(x) = inf\{\eta \in (0,\infty) : (x,\{x_n\})\ is\ \eta - chainable\}.$$

for $x \in X$. From the assumption, $h(x) > 0$ holds. Also, since $(x,\{x_n\})$ is $\eta - chainable$ for any $\eta > \theta(x)$, $h(x) \leq \theta(x)$ holds for any $x \in X$. Define a contraction T on X by

$$Tx = \begin{cases} x_2 & if\ \theta(x) \leq h(x) \\ x_j & if\ \theta(x_{j-1}) \leq h(x) < \theta(x_{j-2})\ for\ some\ j \in \mathbb{N}\ with\ j \geq 3 \end{cases}$$

for $x \in X$. We shall show that T is a contraction. Fix $x, y \in X$ with $x \neq y$ and $h(x) \leq h(y)$. We consider the following two cases:

i)$(G(x,y,y) + G(y,x,x)) \leq h(x)$

ii)$(G(x,y,y) + G(y,x,x)) > h(x)$

In the first case, for any $\eta > h(x)$, from the definition of h, $(x,\{x_n\})$ is $\eta - chainable$. Since $(G(x,y,y)) + G(y,x,x)) < \eta$, $(y,\{x_n\})$ is also $\eta - chainable$. We have $h(y) \leq \varepsilon$ and hence $h(x) = h(y)$. So

$$(G(Tx,Ty,Ty) + G(Ty,Tx,Tx)) = 0 \leq \frac{2}{3}(G(x,y,y) + G(y,x,x))$$

In the second case, let $i, j \geq 2$ satisfy $Tx = x_i$ and $Ty = x_j$. For any $\eta > (G(x,y,y) + G(y,x,x))$, since $h(x) < \eta$, $(x,\{x_n\})$ is $\eta - chainable$. Since $(G(x,y,y) + G(y,x,x)) < \eta$, $(x,\{x_n\})$ is also

$\eta - chainable$. Hence we obtain $h(y) \leq (G(x,y,y) + G(y,x,x))$. We have

$$\begin{aligned}
(G(Tx,Ty,Ty) + G(Ty,Tx,Tx)) &= (G(x_i,x_j,x_j) + G(x_j,x_i,x_i)) \\
&\leq (\theta(x_i) + \theta(x_j)) \\
&\leq \frac{1}{3}(\theta(x_{i-1}) + \theta(x_{j-1})) \\
&\leq \frac{1}{3}(h_(x) + h(y)) \\
&\leq \frac{2}{3}(G(x,y,y) + G(y,x,x))
\end{aligned}$$

We have shown that T is a contraction. It is obvious that T does not have a fixed point. Thus, X does not have BFPP. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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